Journal of Fractional Calculus and Applications,
Vol. 5(1) Jan. 2014, pp. 146-155.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

IMPULSIVE BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

PRADEEP KUMAR, D. N. PANDEY AND D. BAHUGUNA

Abstract. In this paper, we prove the sufficient conditions for the existence of piecewise continuous ($\mathcal{PC}$) mild solutions to nonlinear impulsive boundary value problems for fractional differential equations with deviating arguments. The results are obtained by using the Banach fixed point and Krasnoselskii's fixed point theorems.

1. Introduction

In our earlier work [4], we have studied the following impulsive fractional integro-differential equations with deviating arguments in a complex Banach space $(X, \|.\|)$

$$\left.\begin{array}{rcl} {}^C D_t^{\eta} u(t) &=& Au(t) + f(t, u(t), u(\psi(t, u(t)))) + \int_0^t a(t,\tau) g(\tau, u(\tau)) d\tau, \\ && t \in J = [0, T],\ t \neq t_j; \\ \Delta u|_{t=t_j} &\equiv& I_j(u(t_j^-)) = u(t_j^+) - u(t_j^-),\ j = 1, 2, \cdots, m, \\ u(0) &=& u_0 \in X, \end{array}\right\} \quad (1)$$

where $u : \mathbb{R}_+ \to X$. The functions $f : \mathbb{R}_+ \times X \times X \to X$, $g : \mathbb{R}_+ \times X \to X$ and $\psi : \mathbb{R}_+ \times X \to \mathbb{R}_+$ are three non-linear functions and satisfy some appropriate conditions. Here we assume that $-A : D(A) \subset X \to X$, for each $t \geq 0$, generates an analytic semigroup of bounded linear operators on X and $a : [0, T] \times [0, T]$ is a complex valued uniformly Hölder continuous function on $[0, T]$ for a fixed $T \in [0, \infty)$.

The objective of this article is to study the impulsive boundary value problems for nonlinear differential equations of fractional order with deviating arguments

$$\left.\begin{array}{r} {}^c D_t^{\alpha} u(t) = J_t^{1-\alpha} f(t, u(t), u(h(t, u(t)))),\ t \in J = [0, T],\ t \neq t_j, \\ \Delta u|_{t=t_j} = I_j(u(t_j)),\ j = 1, 2, \cdots, m, \\ au(0) + bu(T) = c,\ a, b, c \in \mathbb{R},\ \text{such that } a \neq -b, \end{array}\right\} \quad (2)$$

2000 *Mathematics Subject Classification.* 34G20, 34A37, 34K45, 34A60, 35R12, 45J05.

Key words and phrases. Impulsive problems, Equations with deviating arguments, fractional differential equations, boundary value problems, Fixed point methods.

Submitted October 14, 2013.

where $0 < \alpha < 1$, the fractional derivative ${}^cD_t^\alpha$ is to be understood in Caputo sense and J_t^α denote the Riemann-Liouville integral of order α, $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are nonlinear functions and will satisfy some appropriate assumptions. $0 = t_0 < t_1 < t_2 \cdots < t_m < t_{m+1} = T$, $I_j \in C(\mathbb{R}, \mathbb{R})(j = 1, 2, ..., m)$ are bounded functions. $I_j(u(t_j)) = u(t_j^+) - u(t_j^-)$, $u(t_j^+) = \lim_{\epsilon \to 0^+} u(t_j + \epsilon)$ and $u(t_j^-) = \lim_{\epsilon \to 0^-} u(t_j + \epsilon)$ represent the right and left limits of $u(t)$ at $t = t_j$, respectively.

Fractional calculus is a significant and useful branch of mathematics that studies the integration or differentiation of any (i.e., non-integer) order. In the past decades, the theory of fractional differential equations has become an important area of investigation because of its wide applicability in many branches of physics, economics and technical sciences. For general motivations, relevant theory and its applications, we refer the reader to [2, 13, 14, 15, 16, 18, 19] and references cited therein.

Impulsive effects are common in the process where the short-term perturbations are to be considered of which duration is negligible in comparison with the total duration of the original process. The governing equations of such phenomena may be modelled as impulsive differential equations. The differential equations with memory effects and impulse effects are modelled as impulsive integro-differential equations.

The study of impulsive nonlinear boundary value problems of fractional differential equations has gained much attention and developed as an interesting branch of research. This is mainly due to a variety of results ranging from theoretical analysis to stability, semistability, asymptotic behavior and numerical methods for fractional equations in the literature. Recently, the impulsive boundary value problems for fractional differential equations has been extensively studied by many authors, see [1, 3, 5, 8, 9, 10, 11, 12, 17, 20, 25, 26, 27, 28, 29] and references cited therein.

On the other hand, for the earlier works on the existence and uniqueness of solutions to functional differential equations with deviating arguments, we refer the reader to [4, 6, 7, 21, 22, 23, 24] and references cited therein.

2. Preliminaries and Assumptions

In this section, we will introduce some basic definitions, notations, lemmas and propositions that are useful throughout this paper.

Definition [2, Def. 1.2] Let $f \in L^1((0,T), X)$ and $\alpha \geq 0$. Then the expression

$$J_t^\alpha f(t) = (f * \Theta_\alpha)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \ t > 0, \ \alpha > 0, \tag{3}$$

where $I_t^0 f(t) = f(t)$ and

$$\Theta_\alpha(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}, & t > 0, \\ 0, & t \leq 0, \end{cases}$$

and $\Theta_0(t) = 0$, is called a Reimann-Liouville integral of order α of f.

Definition [2, Def. 1.3] Let $f \in C^{m-1}((0,T),X)$, $(\Theta_{m-\alpha} * f) \in W^{m,1}((0,T),X)$, $0 \leq m-1 < \alpha < m$, $m \in \mathbb{N}$. Then the expression

$$^cD_t^\alpha f(t) = D_t^m J_t^{m-\alpha}\Big(f(t) - \sum_{0}^{m-1} f^i(0)\Theta_{i+1}(t)\Big), \tag{4}$$

where $D_t^m = \frac{d^m}{dt^m}$, is called the Caputo fractional derivative of order α of f.

Let V_1 and V_2 be open subsets of $\mathbb{R}$. For each $v_1, v_2 \in \mathbb{R}$, there exist balls such that $B_1(v_1, r_1) \subset V_1$ and $B_2(v_2, r_2) \subset V_2$ for any $r_1, r_2 > 0$.

(H1) There exists $L_f = L_f(t, v_1, v_2, r_1, r_2) > 0$ such that the nonlinear map $f : [0,T] \times V_1 \times V_2 \to \mathbb{R}$ satisfies the following condition,

$$|f(t,u,v) - f(t,u_1,v_1)| \leq L_f\{|u-u_1| + |v-v_1|_L\}$$

for all $u, u_1 \in V_1$, $v, v_1 \in V_2$ and $t \in [0,T]$.

(H2) There exists $L_h = L_h(t, v_1, r_1)$ such that $h(0,\cdot) = 0$ and $h : [0,T] \times V_1 \to [0,T]$ satisfies the following condition,

$$|h(t,u) - h(t,v)| \leq L_h \ |u-v|, \tag{5}$$

for all $u, v \in V_1$, $t \in [0,T]$.

(H3) The functions $I_j : \mathbb{R} \to \mathbb{R}$ are continuous and there exist positive constant D_j such that

$$|I_j(u)| \leq D_j, \ j = 1, 2, \cdots, m.$$

(H4) There exist positive constants d_j such that

$$|I_j(u) - I_j(v)| \leq d_j|u-v|, \ j = 1, 2, \cdots, m.$$

Lemma Let $0 < \alpha < 1$ and $f : J \to \mathbb{R}$ be continuous. A function $u \in C(J, \mathbb{R})$ given by

$$u(t) = u_0 + \int_0^t f(s)ds,$$

is the solution of the following Cauchy problems of fractional order

$$\left.\begin{aligned} ^cD_t^\alpha u(t) &= J_t^{1-\alpha} f(t), \ 0 < \alpha < 1, \\ u(0) &= u_0 \in \mathbb{R}. \end{aligned}\right\} \tag{6}$$

Definition Let $h : [0,T] \to \mathbb{R}$ be a continuous function. A function u is a solution of the fractional integral equation (7)

$$u(t) = \begin{cases} \int_0^t h(s)ds - \frac{1}{a+b}\Big[\sum_{r=1}^m bI_r(u(t_r)) + b\int_0^T h(s) - c\Big], \text{ for } t \in [0,t_1], \\ \\ I_1(u(t_1)) + \int_0^t h(s)ds \\ -\frac{1}{a+b}\Big[\sum_{r=1}^m bI_r(u(t_r)) + b\int_0^T h(s) - c\Big], \text{ for } t \in (t_1,t_2], \\ \vdots \\ \\ \sum_{j=1}^m I_j(u(t_j)) + \int_0^t h(s)ds \\ -\frac{1}{a+b}\Big[\sum_{r=1}^m bI_r(u(t_r)) + b\int_0^T h(s) - c\Big], \text{ for } t \in (t_m,T], \end{cases} \tag{7}$$

if and only if u is a solution of the following impulsive boundary value problem

$$\begin{cases} {}^cD_t^\alpha = J_t^{1-\alpha}h(t), \ t \in [0,T], t \neq t_j, \ 0 < \alpha < 1, \\ \Delta u|_{t=t_j} = I_j(u(t_j)), \ j = 1, \cdots, m, \\ au(0) + bu(T) = c \end{cases} \tag{8}$$

For more details, we refer to [1, Lemma 5.2].

3. The main results

We define the set $\mathcal{PC}(J,\mathbb{R})$ as follows

$$\Upsilon_0 = \mathcal{PC}(J,\mathbb{R}) = \{u : [0,T] \to \mathbb{R} : u(.) \text{ is continuous at } t \neq t_j,$$
$$u(t_j^-) = u(t_j), \ u(t_j^+) \text{ exist, for all } j = 1, 2, \cdots, m\}.$$

$\mathcal{PC}(J,\mathbb{R})$ is a Banach space endowed with the supremum norm

$$|u|_{\mathcal{PC}} := \max\{\sup_{t\in J} |u(t+0)|, \ \sup_{t\in J} u(t-0)\}.$$

Now, we define another space

$$\Upsilon_L = \{u \in \Upsilon_0 : |u(t) - u(s)| \leq L|t-s|, \text{ for all } t \in [t_j, t_{j+1}], \ j = 0, 1, \cdots, m\}.$$

where L is a suitable positive constant to be specified later.

Theorem Let $u_0 \in \Upsilon_0$ and the assumptions (H1)-(H4) hold. Then the problem (2) has a unique mild solution on $[0,T]$ provided that

$$\Big(1 + \frac{|b|}{|a+b|}\Big)(NT + \sum_{r=1}^m D_r) + \frac{|c|}{|a+b|} \leq R, \tag{9}$$

and

$$\Big(1 + \frac{|(b)|}{|a+b|}\Big)\Big[L_f(2 + LL_h)T + \sum_{r=1}^m d_r\Big] < 1. \tag{10}$$

Proof. We define a map $\mathcal{F} : \Omega \to \Omega$, given by

$$(\mathcal{F}u)(t) = \begin{cases} \int_0^t f(s,u(s),u(h(s,u(s))))ds \\ -\frac{1}{a+b}\Big[\sum_{r=1}^m bI_r(u(t_r)) + b\int_0^T f(s,u(s),u(h(s,u(s))))ds - c\Big], \ t \in [0,t_1], \\ \\ I_1(u(t_1)) + \int_0^t f(s,u(s),u(h(s,u(s))))ds \\ -\frac{1}{a+b}\Big[\sum_{r=1}^m bI_r(u(t_r)) + b\int_0^T f(s,u(s),u(h(s,u(s))))ds - c\Big], \ t \in (t_1,t_2], \\ \vdots \\ \\ \sum_{j=1}^m I_j(u(t_j)) + \int_0^t f(s,u(s),u(h(s,u(s))))ds \\ -\frac{1}{a+b}\Big[\sum_{r=1}^m bI_r(u(t_r)) + b\int_0^T f(s,u(s),u(h(s,u(s))))ds - c\Big], \ t \in (t_m,T], \end{cases} \tag{11}$$

where $u \in \Omega = \{u \in \Upsilon_0 \cap \Upsilon_L : u(0) = u_0, |u|_{\mathcal{PC}} \leq R\}$. Then Ω is a closed and bounded subset of Υ_L and also a complete metric space.

Clearly, $\mathcal{F}u \in \Upsilon_0$.

Assumptions (H1)-(H2) and $u \in \Upsilon_L$ imply that $f(t,u(t),u(h(t,u(t))))$ is continuous on $[0,T]$. Hence, there exist positive constants

$$N = 2L_f R(1 + LL_h) + N_0 > 0 \text{ and } N_0 = |f(0, u_0, u_0)|,$$

such that

$$|f(t, u(t), u(h(t, u(t)))| \leq N, \text{ for } t \in [0, T]. \tag{12}$$

At first we show that $\mathcal{F}u \in \Upsilon_L$ for a suitable constant L, when $u \in \Upsilon_L$.

Let $u \in \Upsilon_L$, then for each $s_1, s_2 \in [0, t_1], (t_1, t_2], \cdots, (t_m, T]$ with $s_2 > s_1 > 0$, we have

$$\begin{aligned} |(\mathcal{F}u)(s_2) - (\mathcal{F}u)(s_1)| &\leq \int_{s_1}^{s_2} |f(s, u(s), u(h(s, u(s)))|ds \\ &\leq N(s_2 - s_1). \end{aligned} \tag{13}$$

Thus, we have $\mathcal{F}u \in \Upsilon_L$ for a suitable constant

$$L = N.$$

Next, we will prove that $\mathcal{F} : \Omega \to \Omega$. Let $t \in [0, t_1]$ and $u \in \Omega$, we have

$$\begin{aligned} |\mathcal{F}u(t)| &\leq \int_0^t |f(s, u(s), u(h(s, u(s)))|ds \\ &\quad + \frac{1}{|a+b|}\Big[\sum_{r=1}^{m} |b| \; |I_r(u(t_r))| + |b| \int_0^T |f(s, u(s), u(h(s, u(s)))|ds + |c|\Big] \\ &\leq NT + \frac{1}{|a+b|}\Big[\sum_{r=1}^{m} |b| \; D_r + |b| \; NT + |c|\Big]. \end{aligned} \tag{14}$$

Let $t \in (t_j, t_{j+1}]$ for each $j = 1, \cdots, m$ and $u \in \Omega$, we have

$$\begin{aligned} |\mathcal{F}u(t)| &\leq \sum_{r=1}^{j} |I_r(u(t_r))| + \int_0^t |f(s, u(s), u(h(s, u(s)))|ds \\ &\quad + \frac{1}{|a+b|}\Big[\sum_{r=1}^{m} |b| \; |I_r(u(t_r))| + |b| \int_0^T |f(s, u(s), u(h(s, u(s)))|ds + |c|\Big] \\ &\leq NT + \sum_{r=1}^{j} D_r + \frac{1}{|a+b|}\Big[\sum_{r=1}^{m} |b| \; D_r + |b| \; NT + |c|\Big]. \end{aligned} \tag{15}$$

Thus, from (14), (15) and (9), it is clear that

$$|\mathcal{F}u(t)| \leq R,$$

for any $t \in [0, T]$. Therefore, $\mathcal{F} : \Omega \to \Omega$.

Finally, we prove that $\mathcal{F}$ is a contraction on Ω. Let $t \in [0, t_1]$ with $u, v \in \Omega$

$$\begin{aligned} |\mathcal{F}u(t) - \mathcal{F}v(t)| &\leq \int_0^t |f(s, u(s), u(h(s, u(s))) - f(s, v(s), v(h(s, v(s)))|ds \\ &\quad + \frac{1}{|a+b|}\Big[\sum_{r=1}^{m} |b| \; |I_r(u(t_r)) - I_r(v(t_r))| \\ &\quad + |b| \int_0^T |f(s, u(s), u(h(s, u(s))) - f(s, v(s), v(h(s, v(s)))|ds\Big]. \end{aligned} \tag{16}$$

Also, we note that

$$\begin{aligned}
&|f(s,u(s),u(h(s,u(s))) - f(s,v(s),v(h(s,v(s)))| \\
&\le L_f[|u(s)-v(s)| + |u(h(s,u(s))) - v(h(s,v(s)))|_L] \\
&\le L_f[2|u-v|_{\mathcal{PC}} + L|h(s,u(s)) - h(s,v(s))|] \\
&\le L_f[2+LL_h]|u-v|_{\mathcal{PC}}. \qquad (17)
\end{aligned}$$

Now, we use (17) in (16), we get

$$|\mathcal{F}u(t) - \mathcal{F}v(t)| \le \Big[\Big(1+\frac{|b|}{|a+b|}\Big)L_f(2+LL_h)T + \frac{|b|}{|a+b|}\sum_{r=1}^{m} d_r\Big]|u-v|_{\mathcal{PC}}.$$

Similarly, for $t \in (t_j, t_{j+1}], j = 1,2,\cdots,m$, we have

$$\begin{aligned}
&|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \\
&\le \Big[\Big(1+\frac{|b|}{|a+b|}\Big)L_f(2+LL_h)T + \sum_{r=1}^{j} d_r + \frac{|b|}{|a+b|}\sum_{r=1}^{m} d_r\Big]|u-v|_{\mathcal{PC}}.
\end{aligned} \qquad (18)$$

Thus, for each $t \in [0,T]$, we have

$$\begin{aligned}
&|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)| \\
&\le \Big(1+\frac{|b|}{|a+b|}\Big)\Big[L_f(2+LL_h)T + \sum_{r=1}^{m} d_r\Big]|u-v|_{\mathcal{PC}}.
\end{aligned} \qquad (19)$$

Therefore, the map $\mathcal{F}$ is a contraction map on Ω, hence $\mathcal{F}$ has a unique fixed point $u \in \Omega$. That is, problem (2) has a unique mild solution. □

We recall the following classical result due to Krasnoselskii.

Theorem Let $\mathcal{M}$ be a closed convex and nonempty subset of a Banach space B. Let $\mathcal{A}$ and $\mathcal{B}$ be the operators such that

(i). $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$ whenever $x, y \in \mathcal{M}$.
(ii). $\mathcal{A}$ is compact and continuous.
(iii). $\mathcal{B}$ is a contraction mapping.

Then there exists a unique $x_0 \in \mathcal{M}$ such that $x_0 = \mathcal{A}x_0 + \mathcal{B}x_0$.

Theorem Let $f : [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function with $|f(t,u,v)| \le M_f$, for all $(t,u,v) \in [0,T]\times\mathbb{R}\times\mathbb{R}$. Then problem (2) has at least one solution on $[0,T]$.

Proof. Let us choose

$$r_0 \ge M_f T + \sum_{r=1}^{m} D_r + \frac{1}{|a+b|}\Big[\sum_{r=1}^{m} |b|\ D_r + |b|\ M_f T + |c|\Big],$$

and denote

$$B_{r_0} = \{u \in \mathcal{PC}(J,\mathbb{R}) : |u|_{\mathcal{PC}} \le r_0\}.$$

Let $\mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2$, the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ are defined as

$$(\mathcal{F}_1 u)(t) = \int_0^t f(s, u(s), u(h(s, u(s)))ds - \frac{b}{a+b}\int_0^T f(s, u(s), u(h(s, u(s)))ds,$$

$$(\mathcal{F}_2 u)(t) = \sum_{r=1}^{j}(I_r u)(t) - \frac{1}{a+b}\Big[\sum_{r=1}^{m} bI_r(u(t_r))(t) - c\Big],\ j = 1, \cdots, m,$$

where $u \in B_{r_0}$.

Then, for each $t \in [0, t_1]$ and for any $u, v \in B_{r_0}$, we get

$$|\mathcal{F}_1 u + \mathcal{F}_2 v|_{\mathcal{PC}} \le M_f T + \frac{1}{|a+b|}\Big[\sum_{r=1}^{m} |b|\ D_r + |b|\ M_f T + |c|\Big] \le r_0.$$

Similarly, for each $t \in (t_j, t_{j+1}],\ j = 1, 2, \cdots, m$, we get

$$|\mathcal{F}_1 u + \mathcal{F}_2 v|_{\mathcal{PC}} \le M_f T + \sum_{r=1}^{j} D_r + \frac{1}{|a+b|}\Big[\sum_{r=1}^{m} |b|\ D_r + |b|\ M_f T + |c|\Big] \le r_0.$$

Therefore, for each $t \in [0, T]$ and for any $u, v \in B_{r_0}$, we get

$$\begin{aligned} |\mathcal{F}_1 u + \mathcal{F}_2 v|_{\mathcal{PC}} &\le M_f T + \sum_{r=1}^{m} D_r + \frac{1}{|a+b|}\Big[\sum_{r=1}^{m} |b|\ D_r + |b|\ M_f T + |c|\Big] \\ &\le r_0. \end{aligned}$$

Thus, $\mathcal{F}_1 u + \mathcal{F}_2 v \in B_{r_0}$.

$\mathcal{F}_2$ is a constant map and hence compact. Also, $\mathcal{F}_2$ is a contraction on B_{r_0} with the constant zero.

Our next aim is to prove that the operator $\mathcal{F}_1$ is compact.

The continuity of f implies that $\mathcal{F}_1$ is continuous and

$$|(\mathcal{F}_1 u)(t)| \le \Big(1 + \frac{|b|}{|a+b|}\Big)M_f T, \text{ for any } u \in B_{r_0},$$

i.e., $\mathcal{F}_1$ is uniformaly bounded on B_{r_0}.

Let $t_1, t_2 \in (t_j, t_{j+1}]$ with $t_1 < t_2,\ j = 0, 1, \cdots, m$ and $u \in B_{r_0}$, we get

$$\begin{aligned} |(\mathcal{F}_1 u)(t_2) - (\mathcal{F}_1 u)(t_1)| &= |\int_0^{t_2} f(s, u(s), u(h(s, u(s)))ds \\ &\quad - \int_0^{t_1} f(s, u(s), u(h(s, u(s)))ds| \\ &\le \int_{t_1}^{t_2} |f(s, u(s), u(h(s, u(s)))|ds \\ &\le M_f(t_2 - t_1), \end{aligned} \tag{20}$$

Thus, $|(\mathcal{F}_1 u)(t_2) - (\mathcal{F}_1 u)(t_1)|$ tends to zero as $t_2 \to t_1$, for any $u \in B_{r_0}$. This gives that $\mathcal{F}_1$ is equicontinuous on the interval $(t_j, t_{j+1}],\ j = 0, 1, \cdots, m$. Therefore $\mathcal{F}_1$ is relatively compact on B_{r_0}.

Hence, by Arzela-ascoli's theorem, $\mathcal{F}_1$ is compact on B_{r_0}. Therefore, Krasnoselskii's fixed point theorem ensures that $\mathcal{F}$ has a fixed point, which ensures the existence and uniqueness of a mild solution.

□

4. Applications

Let us consider the following nonlinear Cauchy problems for impulsive boundary value problems of fractional order

$$\begin{cases} {}^cD_t^\alpha u(t) &=& \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha} f(s,u(s),u(h(s,u(s))))ds, \ t\in(0,1], \ t\neq\frac{1}{2} \\ \Delta u|_{\frac{1}{2}} &=& \frac{|(u(\frac{1}{2})|}{(2+t)(1+|(u(\frac{1}{2})|)}, \\ u(0) &=& -u(1). \end{cases} \quad (21)$$

Case 1. we define

$$f(t,u(t),u(h(t,u(t)))=\frac{1}{(3+t^3)}\Big[\frac{|u(t)|}{6(1+|u(t))}+\frac{|u(\frac{1}{3}u(t)))|}{(1+|u(\frac{1}{3}u(t)))}\Big], \quad (22)$$

where $u\in C^1([0,1],[0,1])$. Then, clearly $u\in C_L([0,1],[0,1])$, where

$$C_L([0,1],[0,1])=\{u:|u(t)-u(s)|_L\leq L|t-s|, \text{ for all } t,s\in[0,1]\}.$$

Hence, f will satisfy the condition

$$|f(t,u_1,v_1)-f(t,u_2,v_2)|\leq L_f\{|u_1-u_2|+|v_1-v_2|_L\},$$

with $L_f=\frac{1}{9}$.

Case 2. we may define

$$f(t,u(t),u(h(t,u(t))))=\frac{e^{-t}(\cos(u(t))+\sin(u(\frac{1}{2}u(t)))}{(2+t^2)(e^t+e^{-t})}+e^{-t}, \ t\in[0,1],$$
$$u\in\mathbb{R}.$$

Clearly,

$$|f(t,u,v)|\leq\frac{2e^{-t}}{(2+t^2)(e^t+e^{-t})}+e^{-t}=m(t),$$

with $m(t)\in L^\infty([0,1];\mathbb{R}_+)$.

Also, we have

$$\begin{aligned} |I(u_1\Big(\frac{1}{2}\Big))-I(u_2\Big(\frac{1}{2}\Big))| &\leq \frac{1}{2}|u_1\Big(\frac{1}{2}\Big)-u_2\Big(\frac{1}{2}\Big)|, \\ |I(u\Big(\frac{1}{2}\Big))| &\leq \frac{1}{2}|u\Big(\frac{1}{2}\Big)|. \end{aligned}$$

Thus all the assumptions of Theorem (3) and Theorem (3) are satisfied. Therefore, system (21) has at least one $\mathcal{PC}$- mild solution.

5. Acknowledgement

The authors would like to thank the referees for their positive comments and reviews. The third author would like to acknowledge the financial aid from the Department of Science and Technology, New Delhi, under its research project SR/S4/MS:796.

References

[1] J. Rong Wang, Y. Zhou, M. Fečkan, Nonlinear impulsive problems for fractional differential equations and Ulam stability. Comput. Math. Appl. 64 (2012), no. 10, 3389-3405.

[2] N. Rong Wang, D. Han Chen, T. Jun Xiao, Abstract fractional Cauchy problems with almost sectorial operators. J. Differential Equations 252 (2012), no. 1, 202-235.

[3] J. Rong Wang, M. Fečkan, Y. Zhou, Fractional order iterative functional differential equations with parameter. Appl. Math. Model. 37 (2013), no. 8, 6055-6067.

[4] R. Haloi, P. Kumar, D. N. Pandey, Sufficient conditions for the existence and uniqueness of solutions to impulsive fractional integro-differential equations with deviating arguments. Journal of Fractional Calculus and Applications 5(1) (2014), no. 7, 73-84.

[5] J. Xin Cao, H. Chen, Impulsive fractional differential equations with nonlinear boundary conditions. Math. Comput. Modelling 55 (2012), no. 3-4, 303-311.

[6] C. G. Gal, Nonlinear abstract differential equations with deviated argument. J. Math. Anal. Appl. 333 (2007), no. 2, 971983.

[7] R. Haloi, D. N. Pandey, D. Bahuguna, Existence and uniqueness of solutions for quasi-linear differential equations with deviating arguments. Electron. J. Differential Equations 2012), No. 13, 1-10.

[8] Z. Yang, The asymptotic behavior for a class of impulsive delay differential equations. Abstr. Appl. Anal. (2013), 1-7.

[9] T. Faria, M. C. Gadotti, J. Oliveira, Stability results for impulsive functional differential equations with infinite delay. Nonlinear Anal. 75 (2012), no. 18, 6570-6587.

[10] F. Jiang, J. Shen, Asymptotic behavior of solutions for a nonlinear differential equation with constant impulsive jumps. Acta Math. Hungar. 138 (2013), no. 1-2, 1-14.

[11] G. L. Zhang, M. H. Song and M. Z. Liu, Asymptotic stability of a class of impulsive delay differential equations. J. Appl. Math. (s2012), 1-9.

[12] X. Mu, Y. Liang Gao, Semistability of nonlinear impulsive systems with delays. Math. Probl. Eng. (2012), 1-10.

[13] Podlubny I.; Fractional differential equations. Academic Press; 1999.

[14] V. Lakshmikantham, S. Leela and J. Devi Vasundhara, Theory of fractional dynamic systems. Cambridge Scientific Publishers; 2009.

[15] K. S. Miller, B. Ross, An introduction to the fractional calculus and differential equations. John Wiley; 1993.

[16] MW. Michalski, Derivatives of non-integer order and their applications. Dissertationes mathematicae, Inst Math Polish Acad Sci; 1993.

[17] J. Dabas, A. Chauhan, Existence and uniqueness of mild solution for an impulsive neutral fractional integro-differential equation with infinite delay. Math. Comput. Modelling 57 (2013), no. 3-4, 754-763.

[18] E. Bazhlekova, Fractional evolution equations in Banach spaces. Ph. D. Thesis, Eindhoven University of Technology, (2001).

[19] M. H. Drbajan, A. B. Nersesjan, Fractional derivatives and the Cauchy problem for differential equations of fractional order. Izv. Akad. Nauk Armjan. SSR Ser. Mat. 3 (1968) no. 1, 3-29.

[20] Y. Tian, Z. Bai, Impulsive boundary value problem for differential equations with fractional order. Differ. Equ. Dyn. Syst. 21 (2013), no. 3, 253-260.

[21] R. W. Ibrahim, Existence of deviating fractional differential equation. Cubo 14 (2012), no. 3, 129-142.

[22] Z. Liu, J. Liang, A class of boundary value problems for first-order impulsive integro-differential equations with deviating arguments. J. Comput. Appl. Math. 237 (2013), no. 1, 477-486.

[23] W. Wang, J. Wang and P. Shi, Initial value problems for nonlinear first-order impulsive integro-differential equations with deviating arguments. Int. J. Nonlinear Sci. 12 (2011), no. 3, 331-338.

[24] G. Wang, L. Zhang and G. Xing Song, Systems of first order impulsive functional differential equations with deviating arguments and nonlinear boundary conditions. Nonlinear Anal. 74 (2011), no. 3, 974-982.

[25] R. P. Agarwal, M. Benchohra and S. Hamani, A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions. Acta Appl. Math. 109 (2010), no. 3, 973-1033.

[26] A. Boucherif, A. S. Al-Qahtani, Chanane, Bilal Existence of solutions for second-order impulsive boundary-value problems. Electron. J. Differential Equations 2012, no. 24, 1-10.
[27] X. Wang, Impulsive boundary value problem for nonlinear differential equations of fractional order. Comput. Math. Appl. 62 (2011), no. 5, 2383-2391.
[28] X. Jing Wang, C. Bai, Impulsive boundary-value problems for first-order integro-differential equations. Electron. J. Differential Equations 2010, no. 166, 1-10.
[29] J. Hua Shen, W. Wang, Impulsive boundary value problems with nonlinear boundary conditions. Nonlinear Anal. 69 (2008), no. 11, 4055-4062.

Pradeep Kumar
Dept. of Mathematics and Statistics, Indian Institute of Technology, Kanpur, India
E-mail address: prdipk@gmail.com

Dwijendra N. Pandey
Department of Mathematics, Indian Institute of Technology, Roorkee, India
E-mail address: dwij.iitk@gmail.com

Dhirendra Bahuguna
Dept. of Mathematics and Statistics, Indian Institute of Technology, Kanpur
E-mail address: dhiren@iitk.ac.in, Tel. +91-512-2597053, Fax +91-512-2597500.