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FRACTIONAL STURM-LIOUVILLE PROBLEMS WITH α-ORDINARY AND α-SINGULAR POINTS

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Abstract. In this paper, we verify the solution around an α-ordinary point $x_0 \in [a, b]$ for fractional Sturm-Liouville equation

$$(\mathcal{D}^{2\alpha}y)(x) + p(x)y(x) = \lambda q(x)y(x), \qquad \frac{1}{2} < \alpha < 1. \tag{1}$$

Also, the solutions around an α-singular point $x_0 \in [a, b]$ for fractional differential equation

$$(x - x_0)^{2\alpha}(\mathcal{D}^{2\alpha}y)(x) + p(x)y(x) = (x - x_0)^{2\alpha}\lambda q(x)y(x), \qquad \frac{1}{2} < \alpha < 1, \tag{2}$$

is investigated. Here, $p(x)$ and $q(x)$ are α-analytic functions and $(\mathcal{D}^{2\alpha}y)(x)$ represents fractional sequential derivative of order 2α of function $y(x)$. The fractional derivatives are described in the Caputo sense.

1. Introduction

In the recent decades, we have seen the important role of fractional calculus in many sciences, specially mathematics and engineering sciences. Many natural phenomena can be presented by boundary value problems of fractional differential equations. Many authors in different fields such as chemical physics, fluid flows, electrical networks, viscoelasticity, try to model of these phenomena by boundary value problems of fractional differential equations [[1]-[4]]. To achieve extra information in fractional calculus, specially boundary value problems, reader can refer to more valuable papers or books that are written by authors [[5]-[10]].

The linear sequential fractional differential equation of order $n\alpha$ with constant coefficients has been extensively studied, and there are methods to obtain explicitly the general solution for both equation, homogeneous and non-homogeneous, without using the integral transform [[11]-[13]].

In this article, we apply the series method, based on the expansion of the unknown solution $y(x)$ in a fractional power series to obtain solutions of fractional Sturm-Liouville equation. In singular sense, we present a generalization of the Frobenius theory.

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2. Fractional calculus

In this section, we present some definitions which will be used in the remainder of this paper.

Definition 1 Fractional sequential derivative of function $y(x)$ is defined as:

$$(\mathcal{D}_{a^+}^{\alpha}y)(x) = (D_{a^+}^{\alpha}y)(x),$$

$$(\mathcal{D}_{a^+}^{k\alpha}y)(x) = D_{a^+}^{\alpha}\mathcal{D}_{a^+}^{(k-1)\alpha}y(x), \quad k = 2, 3, ...$$

where $D_{a^+}^{\alpha}$ denote the Caputo fractional derivative of order α.

Definition 2 Let $\alpha \in (0, 1]$, $f(x)$ be a real function defined on the interval $[a, b]$ and $x_0 \in [a, b]$. Then $f(x)$ is said to be α-analytic at x_0, if there exists an interval $N(x_0)$ such that for $x \in N(x_0)$, $f(x)$ can be expressed as $\sum_{n=0}^{\infty} a_n(x-x_0)^{n\alpha}$ $(a_n \in \mathbb{R})$, this series being absolutely convergent for $|x - x_0| < r$, $(r > 0)$.

Definition 3 A point $x_0 \in [a, b]$ is said to be an α-ordinary point of the equation

$$(\mathcal{D}^{n\alpha}y)(x) + \sum_{k=0}^{n-1} a_k(x)(\mathcal{D}^{k\alpha}y)(x) = f(x), \tag{3}$$

if $a_k(x), (k = 0, 1, ..., n - 1)$ are α-analytic in x_0. A point $x_0 \in [a, b]$ which is not α-ordinary will be called α-singular.

Definition 4 Let $x_0 \in [a, b]$ be an α-singular point of the equation (3). Then, x_0 is said to be a regular α-singular point of this equation if the functions

$$(x - x_0)^{(n-k)\alpha}a_k(\alpha), \qquad k = 0, 1, ..., n - 1$$

are α-analytic in x_0. Otherwise, x_0 is said to be an essential α-singular point.

Definition 5 [see [6]-[8]] Let $\alpha \in (0, 1)$, $a \in \mathbb{R}$ and $\beta \in \mathbb{R}\backslash\mathbb{Z}^-$. Then, the derivative of order 2α of $(x - a)^{\beta}$ is defined as:

$$\mathcal{D}_{a^+}^{2\alpha}(x - a)^{\beta} = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - 2\alpha + 1)}(x - a)^{\beta-2\alpha}, \quad x > a. \tag{4}$$

Definition 6 The Mittag-Leffler function $E_{\nu}(z)$ for $\nu > 0$ and $z \in \mathbb{C}$ is defined by the series representation:

$$E_{\nu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\nu + 1)}, \qquad \nu > 0, \qquad z \in \mathbb{C}.$$

3. Solutions of fractional Sturm-Liouville problems with α-ordinary and α-singular points

In first, we consider the solutions around α-ordinary point $x_0 > a$ to the Eq.(1).

Theorem 1 Let $\alpha \in (\frac{1}{2}, 1)$, and $x_0 > a$ be an α-ordinary point of the Sturm-Liouville equation

$$(\mathcal{D}^{2\alpha}y)(x) + p(x)y(x) = \lambda q(x)y(x), \tag{5}$$

where $p(x) = \sum_{n=0}^{\infty} p_n(x - x_0)^{n\alpha}$ and $q(x) = \sum_{n=0}^{\infty} q_n(x - x_0)^{n\alpha}$ are power series expansion of the α-analytic function $p(x)$ and $q(x)$, respectively. Then, there exists function

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{n\alpha}, \tag{6}$$

which is the solution of Eq.(5) for $x \in (x_0, x_0 + r)$, $(r > 0)$. Here, a_0 is a non-zero arbitrary constant, $a_1 = 0$ and the coefficients a_n, $n = 2, 3, ...$, are given by

$$a_{n+2} = \frac{\Gamma(n\alpha + 1)(\lambda c_n - b_n)}{\Gamma((n+2)\alpha + 1)}, \qquad n = 0, 1, 2, ...,$$

where $b_n = \sum\limits_{l=0}^{n} p_l a_{n-l}$ and $c_n = \sum\limits_{l=0}^{n} q_l a_{n-l}$.

Proof. We shall seek a solution of Eq.(5) of the form $\sum\limits_{n=0}^{\infty} a_n (x - x_0)^{n\alpha}$. Substituting (6) in (5) and definition 5, we get

$$\sum_{n=1}^{\infty} \frac{\Gamma(n\alpha + 1)}{\Gamma((n-2)\alpha + 1)} a_n (x - x_0)^{(n-2)\alpha} + \sum_{n=0}^{\infty} (\sum_{l=0}^{n} a_l p_{n-l} - \lambda \sum_{l=0}^{n} a_l q_{n-l})(x - x_0)^{n\alpha} = 0.$$

To add the two series, it is necessary that both summation indices start with the same number and that the powers of x in each series be "in phase" that is, if one series starts with a multiple of, say, x to the first power, then we want the other series to start with the same power. Here, the first series starts with $x^{1-\alpha}$, whereas the second series starts with x^0. By writing the first term of the first series outside the summation notation, we see that both series start with the same power of x-namely x^0

$$\begin{aligned} a_1 \frac{\Gamma(\alpha + 1)}{\Gamma(1 - \alpha)} (x - x_0)^{-\alpha} &+ \sum_{n=2}^{\infty} \frac{\Gamma(n\alpha + 1)}{\Gamma((n-2)\alpha + 1)} a_n (x - x_0)^{(n-2)\alpha} \\ &+ \sum_{n=0}^{\infty} (\sum_{l=0}^{n} a_l p_{n-l} - \lambda \sum_{l=0}^{n} a_l q_{n-l})(x - x_0)^{n\alpha} = 0. \end{aligned} \tag{7}$$

Now, to get the same summation index, we are inspired by the exponents of x. We are in a position to add the series in term by term

$$\begin{aligned} a_1 \frac{\Gamma(\alpha + 1)}{\Gamma(1 - \alpha)} (x - x_0)^{-\alpha} &+ \sum_{n=0}^{\infty} \{ \frac{\Gamma((n+2)\alpha + 1)}{\Gamma(n\alpha + 1)} a_{n+2} \\ &+ (\sum_{l=0}^{n} a_l p_{n-l} - \lambda \sum_{l=0}^{n} a_l q_{n-l}) \} (x - x_0)^{n\alpha} = 0. \end{aligned} \tag{8}$$

Since (8) is identically zero, it is necessary that the coefficient of each power of x be set equal to zero. So, we obtain $a_1 = 0$ and the following recurrence formula

$$a_{n+2} = \frac{\Gamma(n\alpha + 1)(\lambda c_n - b_n)}{\Gamma((n+2)\alpha + 1)}, \qquad n = 0, 1, 2, ..., \tag{9}$$

with

$$b_n = \sum_{l=0}^{n} p_l a_{n-l}, \qquad c_n = \sum_{l=0}^{n} q_l a_{n-l},$$

which allows us to express a_n $(n \geq 2)$, in terms of a_0 and a_1.

Now, we prove the convergence of the series (6). Let $0 < r_1 < r$, since $p(x)$ and $q(x)$ are convergent, there exist constants $M_1 > 0$ and $M_2 > 0$, such that

$$|p_{n-l}| \leq \frac{M_1 r^{l\alpha}}{r^{n\alpha}} \qquad |q_{n-l}| \leq \frac{M_2 r^{l\alpha}}{r^{n\alpha}}.$$

Consequently,

$$|\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}||a_{n+2}| \leq \frac{(M_1+|\lambda|M_2)(\sum\limits_{l=0}^{n} a_l r^{l\alpha})}{r^{n\alpha}}.$$

Now, we define $d_0 = |a_0|$, $d_1 = 0$ and d_n $(n > 0)$ as follows:

$$|\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}|d_{n+2} = \frac{(M_1+|\lambda|M_2)(\sum\limits_{l=0}^{n} d_l r^{l\alpha})}{r^{n\alpha}}. \tag{10}$$

Using the asymptotic representation

$$\frac{\Gamma(w+a)}{\Gamma(w+b)} = w^{a-b}[1+O(\frac{1}{t})], \qquad |arg(w+a)| < \pi, \qquad |w| \to \infty, \tag{11}$$

we have that the series $\sum_{n=0}^{\infty} d_n (x-x_0)^{n\alpha}$, converges for all x, such that $|x-x_0| < r_1$, and from this we conclude that the series (6) converges for $x - x_0 < r$.

Example 1 Consider the regular fractional eigenvalue problem

$$\mathcal{D}^{\frac{3}{2}} y(x) + \lambda y(x) = 0. \tag{12}$$

subject to

$$y'(0) = 0, \qquad y(1) = 0. \tag{13}$$

We shall seek to equation (12) the solution around the α-ordinary point $x_0 = 0$. According with theorem 1, the general solution is $y(x) = \sum\limits_{n=0}^{\infty} a_n x^{n\alpha}$, where a_0 is arbitrary constant, $a_1 = 0$ and

$$a_{n+2} = \frac{(-\lambda)^{\frac{n}{2}}}{\Gamma(\frac{3n}{4}+1)} a_0, \qquad n = 0, 1, 2, ...$$

Thus, general solution is

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{(-\lambda x^{\frac{3}{2}})^n}{\Gamma(\frac{3n}{2}+1)} = a_0 E_{\frac{3}{2}}(-\lambda x^{\frac{3}{2}}),$$

where $E_{\frac{3}{2}}$ denotes the Mittag-Leffler function.
The results are the same as ADM [[14]] and HAM [[15]] when we consider $h = -1$. By using boundary condition (13), we explore the first three eigenvalues ($\lambda_{1,i}$,$\lambda_{2,i}$ and $\lambda_{3,i}$) numerically in following table where represents the number of terms used in the following series, i.e.

$$y(t) \cong \sum_{n=0}^{i} y_n(t).$$

The numerical evidence in table suggests that the first three eigenvalues are

$$\lambda_1 = 2.11027708, \qquad \lambda_2 = 13.76538223, \qquad \lambda_3 = 24.24328676$$

Table : The approximation to the first three eigenvalues

i	$\lambda_{1,i}$	$\lambda_{2,i}$	$\lambda_{3,i}$
17	2.11027708	13.76538387	24.10237991
18	2.11027708	13.76538208	24.26958889
19	2.11027708	13.76538224	24.23941883
20	2.11027708	13.76538223	24.24383027
21	2.11027708	13.76538223	24.24329538
22	2.11027708	13.76538223	24.24328578
23	2.11027708	13.76538223	24.24328687
24	2.11027708	13.76538223	24.24328675
25	2.11027708	13.76538223	24.24328676

Example 2 Consider the regular fractional eigenvalue problem

$$\mathcal{D}^{\frac{3}{2}}y(x) + x^{\frac{3}{4}}y(x) = \lambda x^{\frac{3}{2}}y(x). \tag{14}$$

Using theorem 1, we shall seek to equation (14) the solution around the α-ordinary point $x_0 = 0$. According with theorem 1 the general solution is $y(x) = \sum\limits_{n=0}^{\infty} a_n x^{n\alpha}$, where a_0 is arbitrary constant, $a_1 = a_2 = 0$ and

$$a_n = \frac{\Gamma(\frac{3n}{4} - \frac{1}{2})}{\Gamma(\frac{3n}{4} + 1)}(\lambda a_{n-4} - a_{n-3}), \qquad n = 4, 5, ...$$

Thus, general solution is

$$y(x) = a_0\big(1 - \frac{\Gamma(\frac{7}{4})}{\Gamma(\frac{13}{4})}x^{\frac{9}{2}}\big) + \sum_{n=4}^{\infty} \frac{\Gamma(\frac{3n}{4} - \frac{1}{2})}{\Gamma(\frac{3n}{4} + 1)}(\lambda a_{n-4} - a_{n-3})x^{\frac{3n}{4}}.$$

Now, we consider the solutions around regular α-singular point $x_0 > a$ to the equation

$$(x - x_0)^{2\alpha}\mathcal{D}^{2\alpha}y(x) + p(x)y(x) = (x - x_0)^{2\alpha}\lambda q(x)y(x), \qquad \frac{1}{2} < \alpha < 1, \tag{15}$$

where $p(x) = \sum\limits_{n=0}^{\infty} p_n(x - x_0)^{n\alpha}$ and $q(x) = \sum\limits_{n=0}^{\infty} q_n(x - x_0)^{n\alpha}$ are power series expansion of the α-analytic function $p(x)$ and $q(x)$, respectively.

Theorem 2 Let $\alpha \in (\frac{1}{2}, 1)$, and $x_0 > 0$ be a singular point of the Eq.(10). Then, there exists a unique solution on semi interval $(x_0, x_0 + r)$, for some $(r > 0)$ of Eq.(13) given by

$$y(x) = (x - x_0)^s \sum_{n=0}^{\infty} a_n(x - x_0)^{n\alpha}, \tag{16}$$

where $a_0 \neq 0$ and s being a number to be determined.

Proof. We shall seek a solution of Eq.(15) of the form (16). Substituting (16) in (15) and using definition 5, similar to theorem (3.1), we get

$$a_0(\frac{\Gamma(s + 1)}{\Gamma(s - 2\alpha + 1)} + p_0)(x - x_0)^s + (a_1\frac{\Gamma(s + \alpha + 1)}{\Gamma(s - \alpha + 1)} + p_0a_1 + p_1a_0)(x - x_0)^{s+\alpha}$$
$$+ \sum_{n=2}^{\infty} a_n(\frac{\Gamma(n\alpha + s + 1)}{\Gamma((n - 2)\alpha + s + 1)} + b_n - \lambda c_{n-2})(x - x_0)^{n\alpha+s} = 0$$

where

$$b_n = \sum_{l=0}^{n} p_l a_{n-l}, \qquad c_n = \sum_{l=0}^{n} q_l a_{n-l}.$$

So, since $a_0 \neq 0$, we obtain

$$\frac{\Gamma(s+1)}{\Gamma(s-2\alpha+1)} + p_0 = 0, \quad a_1 = \frac{-p_1 a_0}{\frac{\Gamma(s+\alpha+1)}{\Gamma(s-\alpha+1)} + p_0},$$

$$a_n = \frac{\lambda c_{n-2} - b_{n-1}}{\frac{\Gamma(n\alpha+s+1)}{\Gamma((n-2)\alpha+s+1)} + p_0}, \qquad n = 2, 3, \ldots.$$

The proof of convergence for $x - x_0 < r$ is analogous to that used for theorem 1, if we take into a account the above mentioned asymptotic relation (11).

Example 3 Consider the following fractional equation

$$x^{\frac{3}{2}} \mathcal{D}^{\frac{3}{2}} y(x) + y(x) = \lambda x^{\frac{3}{2}} y(x). \tag{17}$$

According to theorem 2 we find to equation (17), the solution around singular point $x = 0$ in the form (13). From (16), we obtain $s = -.25$ $a_0 \neq 0$ (arbitrary constant), $a_1 = 0$ and

$$a_n = \frac{\lambda a_{n-2} - a_{n-1}}{\frac{\Gamma(\frac{3}{4}(n+1))}{\Gamma(\frac{3}{4}(n-1))} + 1}, \qquad n = 2, 3, \ldots$$

Thus, the general solution of (17) is

$$y(x) = a_0 x^{\frac{-1}{4}} + \sum_{n=2}^{\infty} \frac{\lambda a_{n-2} - a_{n-1}}{\frac{\Gamma(\frac{3}{4}(n+1))}{\Gamma(\frac{3}{4}(n-1))} + 1} x^{\frac{1}{4}(3n-1)}.$$

Remark For the case when the Caputo derivatives is replaced by the Riemann-Liouville derivative the results coincide exactly with those in the Caputo sense.

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