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FRACTIONAL-ORDER PARTIAL DIFFERENTIAL EQUATION FOR PREDATOR-PREY

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Abstract. In this paper we consider the fractional-order partial predator-prey model. The stability of equilibrium points is studied. Numerical solutions of this model are given.

1. Introduction

The use of fractional-order partial differential operator in mathematical models has become increasingly widespread in recent years [12]. Several forms of fractional differential equations and fractional partial differential equations have been proposed in standard models [2, 3, 5, 7, 8, 9].

Partial differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, economics, viscoelasticity, biology, physics and engineering. Recently, a large amount of literatures developed concerning the application of fractional partial differential equations in nonlinear dynamics [12].

The study of population dynamics is nearly as old as population ecology. In the 1920s, Lotka and Volterra independently developed a simple model of interacting species that still bears their joint names. This was a nearly linear model, but the predator-prey version displayed neutrally stable cycles [14]. From then on, the dynamic relationship between predators and their prey has become and will continue to be one of dominant themes in both ecology and mathematical ecology due to its universal existence and importance [11, 14].

In this paper we study the fractional-order partial predator-prey model. The stability of equilibrium points is studied. Numerical solutions of this model are given.

The reason for considering a fractional order system instead of its integer order counterpart is that fractional order partial differential equations are generalizations of integer order partial differential equations.

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we like to argue that fractional order equations are more suitable than integer order ones in modeling biological, economic and social systems (generally complex adaptive systems) where memory effects are important

Now we give the definition of fractional-order partial differentiation:

Definition 1 The fractional partial derivative of order $\alpha \in (0,1]$ of $u(x,t)$ is defined by [12]

$$D^{\alpha}u(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-s)^{-\alpha}\frac{\partial u(x,s)}{\partial s}ds. \tag{1}$$

2. The two dimensional fractional order partial predator prey model

Let $\alpha \in (0,1]$ and consider the system [1]

$$\begin{aligned} D^{\alpha}u_1(x,t) &= f_1(u_1,u_2) + d_1\frac{\partial^2 u_1}{\partial x^2}, \\ D^{\alpha}u_2(x,t) &= f_2(u_1,u_2) + d_2\frac{\partial^2 u_2}{\partial x^2}. \end{aligned} \tag{2}$$

To evaluate the equilibrium points, let

$$D^{\alpha}u_i(x,t) = 0, \qquad i = 1,2$$

then

$$f_i(u_1^{eq}, u_2^{eq}) = 0, \; u_i^{eq} \quad \text{constants},$$

from which we can get the equilibrium points $u_1^{eq}, \; u_2^{eq}$.

To evaluate the asymptotic stability, let

$$u_i(x,t) = u_i^{eq} + \varepsilon_i(x,t),$$

$$\varepsilon_i(x,t) = e^{ikx}\widetilde{\varepsilon}_i(t),$$

then we obtain the system

$$D^{\alpha}\varepsilon = A\varepsilon \tag{3}$$

where

$$\varepsilon = \begin{bmatrix} \widetilde{\varepsilon}_1 \\ \widetilde{\varepsilon}_2 \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} - d_1k^2 & a_{12} \\ a_{21} & a_{22} - d_2k^2 \end{bmatrix}$$

and

$$a_{ij} = \frac{\partial f_i}{\partial u_j} \mid_{eq}, \qquad i,j = 1,2.$$

which implies that

$$D^{\alpha}\eta = C\eta, \;\; \eta = B^{-1}\varepsilon \tag{4}$$

where C is a diagonal matrix of A, B is the eigenvectors of A and

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix},$$

i.e.

$$D^{\alpha}\eta_1 = \lambda_1\eta_1, \tag{5}$$

$$D^{\alpha}\eta_2 = \lambda_2\eta_2, \tag{6}$$

where λ_1 and λ_2 are the eigenvalues of A.

The solutions of Eqs. (5)-(6) are given by Mittag-Leffeler function [6]

$$\begin{aligned} \eta_1(t) &= \sum_{n=0}^{\infty} \frac{(\lambda_1)^n t^{n\alpha}}{\Gamma(n\alpha+1)} \eta_1(0) \\ &= E_\alpha(\lambda_1 t^\alpha)\eta_1(0), \end{aligned} \tag{7}$$

$$\begin{aligned} \eta_2(t) &= \sum_{n=0}^{\infty} \frac{(\lambda_2)^n t^{n\alpha}}{\Gamma(n\alpha+1)} \eta_2(0) \\ &= E_\alpha(\lambda_2 t^\alpha)\eta_2(0). \end{aligned} \tag{8}$$

Using the result of [10] then if

$$|\arg(\lambda_1)| > \alpha\pi/2 \text{ and } |\arg(\lambda_2)| > \alpha\pi/2 \tag{9}$$

then $\eta_1(t)$, $\eta_2(t)$ are decreasing and then $\varepsilon_1(t)$, $\varepsilon_2(t)$ are decreasing.

So the the equilibrium point (u_1^{eq}, u_2^{eq}) is locally asymptotically stable if all the eigenvalues of the matrix A satisfies (9).

Now we study the equilibrium and stability of the fractional-order partial predator-prey model [14].

Let $\alpha \in (0,1]$, the fractional-order partial predator-prey model is given by [14]

$$\begin{aligned} D^\alpha N(x,t) &= R(1-\frac{N}{S})N - \frac{S\,N}{P+S\,N}P + D_1\frac{\partial^2 N}{\partial x^2}, \\ D^\alpha P(x,t) &= \frac{S\,N}{P+S\,N}P - Q\,P + D_2\frac{\partial^2 P}{\partial x^2}, \end{aligned} \tag{10}$$

where N, P stand for prey and predator density, respectively. D_1, D_2 are their respective diffusion coefficients.

The dimensionless model (Eqs. 10) has only three parameters: R, which controls the growth rate of prey; Q, which controls the death rate of the predator; and S, which measures capturing rate.

To evaluate the equilibrium points, let

$$\begin{aligned} D^\alpha N(x,t) &= 0, \\ D^\alpha P(x,t) &= 0, \end{aligned}$$

then the equilibrium points are $(0,0), (S,0), (n^*, p^*)$, where

$$\begin{aligned} n^* &= \frac{S\,(R+(Q-1)S)}{R}, \\ p^* &= \frac{S\,(1-Q)}{Q}n^*. \end{aligned}$$

We have

$$\begin{aligned} D^\alpha N(x,t) &= f(N,P) + D_1\frac{\partial^2 N}{\partial x^2}, \\ D^\alpha P(x,t) &= g(N,P) + D_2\frac{\partial^2 P}{\partial x^2}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} f(N,P) &= R(1-\frac{N}{S})N - \frac{S\,N}{P+S\,N}P, \\ g(N,P) &= \frac{S\,N}{P+S\,N}P - Q\,P. \end{aligned} \tag{12}$$

For the equilibrium point (n^*, p^*) we find that

$$A = \begin{bmatrix} f_N - D_1 k^2 & f_P \\ g_N & g_P - D_2 k^2 \end{bmatrix}_{(n^*,p^*)}$$

its eigenvalues are

$$\lambda_{1,2}(k) = \frac{1}{2}\left(tr_k \pm \sqrt{tr_k^2 - 4\triangle_k} \right), \tag{13}$$

where

$$\begin{aligned} tr_k &= f_N + g_P - k^2(D_1 + D_2) \\ &= tr_0 - k^2(D_1 + D_2), \end{aligned} \tag{14}$$

$$tr_0 = f_N + g_P,$$

$$\begin{aligned} \triangle_k &= f_N\, g_P - f_P\, g_N - k^2(f_N\, D_2 + g_P\, D_1) + k^4 D_1\, D_2 \\ &= \triangle_0 - k^2(f_N\, D_2 + g_P\, D_1) + k^4 D_1\, D_2, \end{aligned} \tag{15}$$

$$\triangle_0 = f_N\, g_P - f_P\, g_N.$$

A sufficient condition for the local asymptotic stability of the equilibrium point (n^*, p^*) is

$$|\arg(\lambda_1)| > \frac{\alpha\pi}{2}, |\arg(\lambda_2)| > \frac{\alpha\pi}{2},$$

i.e.,

$$\left| \frac{\sqrt{4\triangle_k - tr_k^2}}{tr_k} \right| > \tan(\frac{\alpha\pi}{2})$$

$$\left| \begin{array}{c} (((D_1 - D_2)k^2 + (-1+Q)Q + R)^2 + 2(-1+Q)(D_1 k^2(1+Q) - \\ D_2 k^2(1+Q) + R + Q(-(-1+Q)^2 + R))S + \\ (-1+Q^2)^2 S^2))^{0.5}/((D_1 + D_2)k^2 + Q + R + Q^2(-1+S) - S) \end{array} \right| > \tan(\frac{\alpha\pi}{2})$$

and the hopf bifurcation occurs when

$$|\arg(\lambda_1)| = \frac{\alpha\pi}{2}, |\arg(\lambda_2)| = \frac{\alpha\pi}{2},$$

i.e.,

$$\left| \frac{\sqrt{4\triangle_k - tr_k^2}}{tr_k} \right| = \tan(\frac{\alpha\pi}{2})$$

$$\left| \begin{array}{c} (((D_1 - D_2)k^2 + (-1+Q)Q + R)^2 + 2(-1+Q)(D_1 k^2(1+Q) - \\ D_2 k^2(1+Q) + R + Q(-(-1+Q)^2 + R))S + \\ (-1+Q^2)^2 S^2))^{0.5}/((D_1 + D_2)k^2 + Q + R + Q^2(-1+S) - S) \end{array} \right| = \tan(\frac{\alpha\pi}{2}).$$

3. Numerical solutions

In Fig. 1 we take $S = 1.2, Q = 0.6, R = 0.5, D_2 = 0.2$,and (1) $D_1 = 0.15$; (2) $D_1 = 0.12$; (3) $D_1 = 0.10$; (4) $D_1 = 0.07$; (5) $D_1 = 0.04$; (6) $D_1 = 0.02$.

Now we approximate Eqs. (10) by using an explicit finite-difference approximation [4, 13]. The approximate solution displayed in Figs. 2-6 for the step sizes $\tau = 0.01, h = 0.01$ and different $0 < \alpha \leq 1$. In Figs. 2-6 we take $T = 1, L = 1, S = 0.6, Q = 0.6, R = 0.5, D_1 = 1, D_2 = 1, N(x, 0) = 0.15, P(x, 0) = 0.33, N(0, t) = t$ and $P(0, t) = 2$.

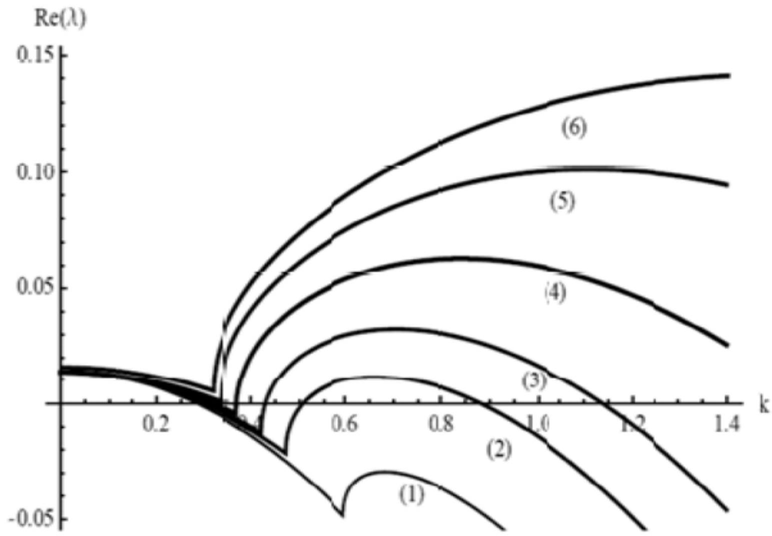


Fig. 1.
$S = 1.2$, $R = 0.5$, $Q = 0.6$, $D_2 = 0.2$ and (1) $D_1 = 0.15$; (2) $D_1 = 0.12$; (3) $D_1 = 0.10$; (4) $D_1 = 0.07$; (5) $D_1 = 0.04$; (6) $D_1 = 0.02$.

Fig. 2.

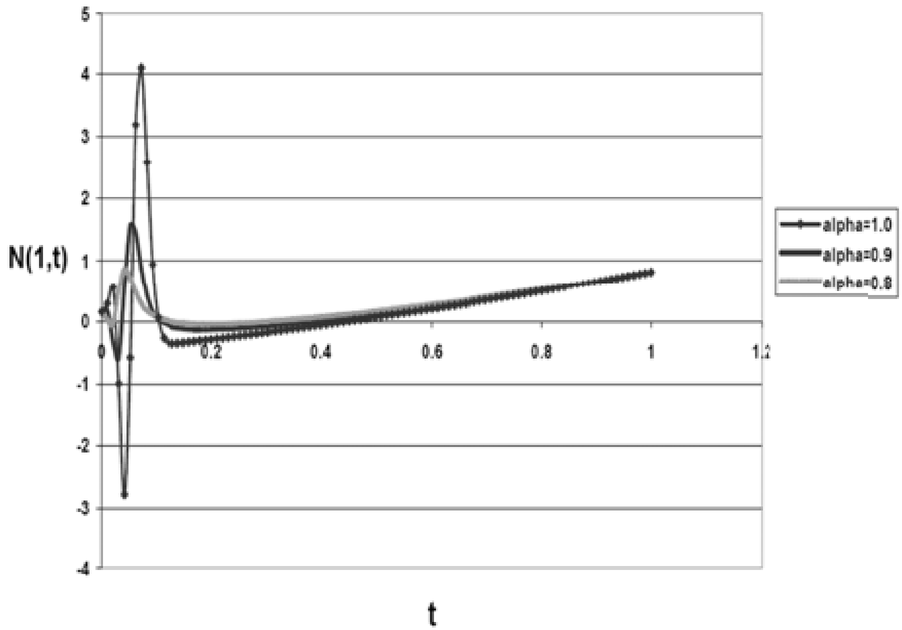


Fig. 3.

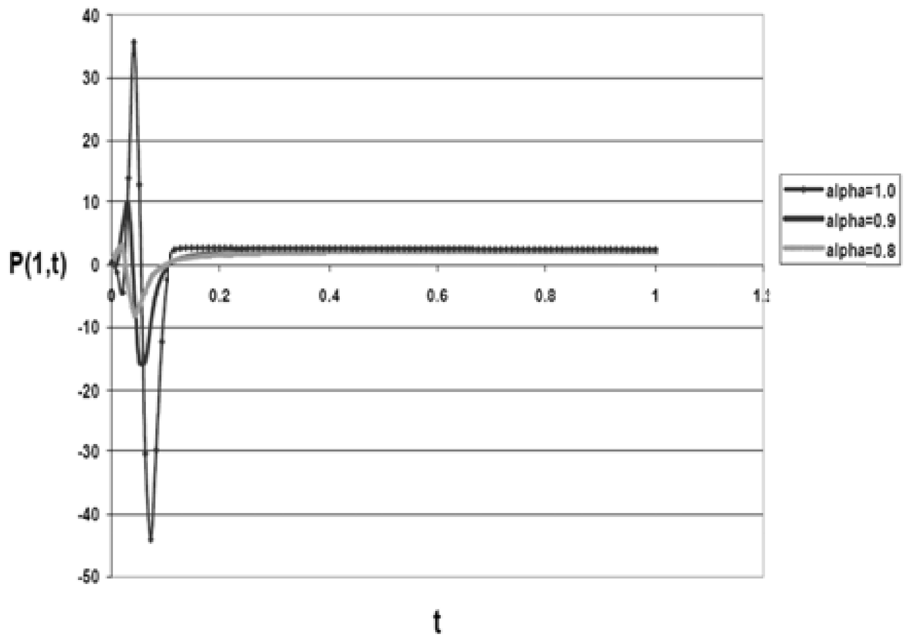
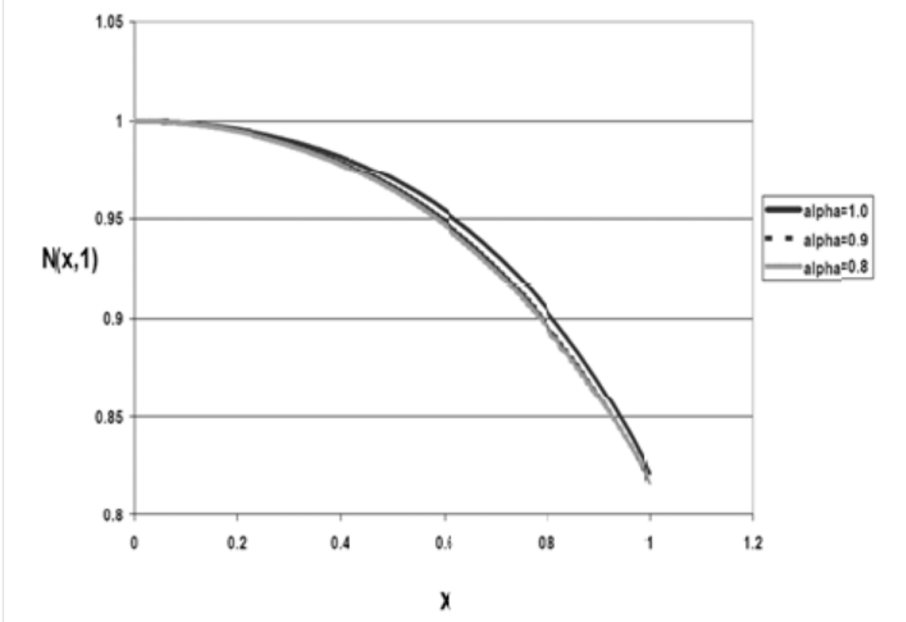

P(1,t)
t
alpha=1.0
alpha=0.9
alpha=0.8

Fig 4.


N(x,1)
x
alpha=1.0
alpha=0.9
alpha=0.8

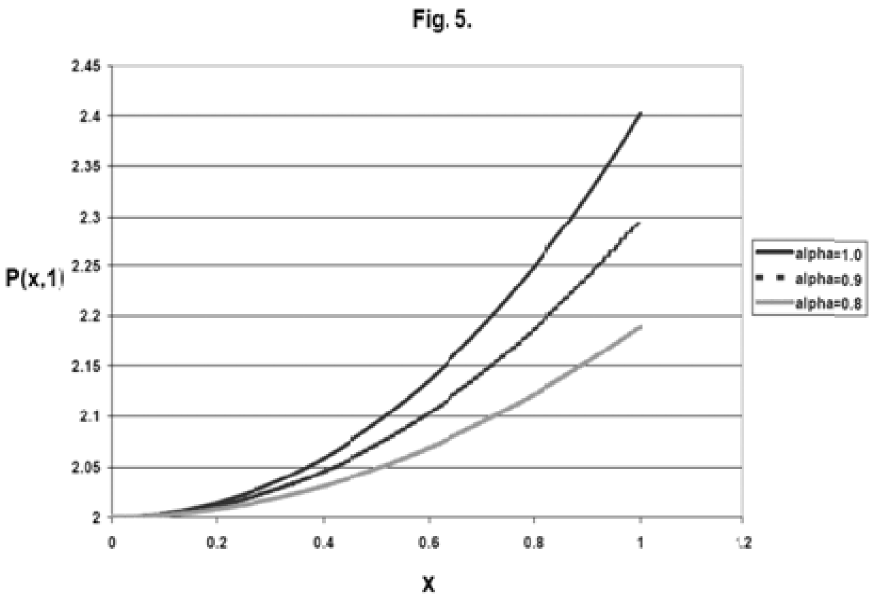


Fig. 5.
2.45
2.4
2.35
2.3
2.25
2.2
2.15
2.1
2.05
2
0
0.2
0.4
0.6
0.8
1
1.2
P(x,1)
X
alpha=1.0
alpha=0.9
alpha=0.8

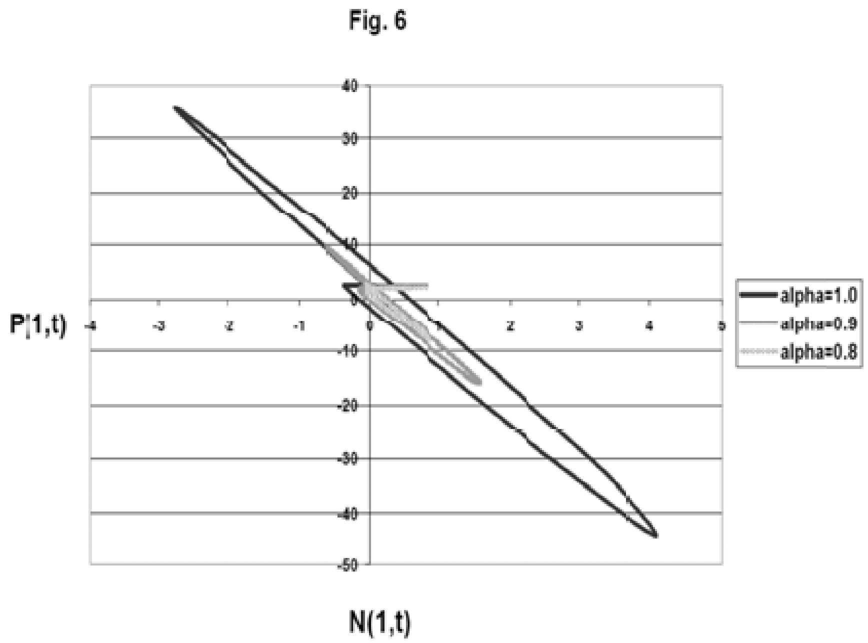


Fig. 6
40
30
20
10
-10
-20
-30
-40
-50
-4
-3
-2
-1
0
1
2
3
4
5
P(1,t)
N(1,t)
alpha=1.0
alpha=0.9
alpha=0.8

4. Conclusions

In this paper we have studied the fractional-order partial predator-prey model. The stability of equilibrium points is studied. Numerical solutions of this model are given.

The reason for considering a fractional order system instead of its integer order counterpart is that fractional order partial differential equations are generalizations of integer order partial differential equations.

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