Journal of Fractional Calculus and Applications,
Vol. 5(2) July 2014, pp. 121-132.
ISSN: 2090-5858.
http://fcag-egypt.com/Journals/JFCA/

ON EXISTENCE OF SOLUTION FOR MULTI POINTS BOUNDARY VALUE PROBLEM

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Abstract. In this paper, we study existence and uniqueness of solutions for fractional differential equations with boundary value problem

$$\begin{cases} D^q u(t) = f(t, u(t), D^p u(t)) \quad 2 < q < 3, \quad 0 < p < 1, \\ u(0) = 0, \quad D^p u(1) = \sum_{i=1}^{m-2} \zeta_i D^p u(\eta_i), \quad u''(1) = 0. \end{cases}$$

where D^q, D^p are Caputo's fractional derivative of order q, p respectively. We use Schauder fixed point theorem and Arzela Ascoli theorem. For application of results we present an example.

1. Introduction

Fractional differential equations is gaining much attention and importance in fields like physics, engineering, economics, aerodynamics, and polymer rheology etc. For the basic knowledge we refer the reader to [1, 2, 3, 4, 5, 6]. Much attention has been focused on the existence and uniqueness of solutions of fractional order differential equations of boundary value problems like in [7, 8, 9, 10, 11, 12, 13], and alot of progress has made in this field. In literature we have fractional derivatives such as Riemann-Liouville, Caputo, Hadamard, Weyl and Grunwald-Letnikov, etc. In applied problems we need those definitions of a fractional derivative that allow the utilization of physically interpretable initial and boundary conditions. The Caputo fractional derivative satisfies these demands so we follow Caputo fractional derivative in this paper.

Here we present some differential equations with multi points boundary conditions which have stimulated us for the present work. In [14], Salem study the existence of Pseudo solutions of nonlinear m-point boundary value problem for fractional differential equation

$$\begin{cases} D_{0^+}^{\alpha} u(t) + a(t) f(t, u(t)) = 0, \quad 0 < t < 1 \\ u(0) = u'(0) = ... = u^{(n-2)}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} \zeta_i u(\eta_i), \end{cases} \tag{1}$$

2000 *Mathematics Subject Classification.* 35K55, 45K05, 35K15, 35K20, 92D50.
Key words and phrases. Fractional differential equation; Fixed point theorems.
Submitted Dec. 23, 2013. Revised March 5, 2014.

In [15], El-Shahed and Nieto studied the existence of nontrivial solutions of a multi- point boundary value problem for fractional differential equation given below

$$\begin{cases} D^q u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, n-1 < q < n, n \in N \\ u(0) = u'(0) = ... = u^{(n-2)}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} a_i u(\eta_i), \end{cases} \tag{2}$$

where $n \geq 2$, $\eta_i \in (0,1), a_i > 0$ $(i = 1, 2, 3..., m-2)$. where $\mu > 0$, a and f are continuous functions.

Wang et.al in [23], studied positive solution for the fractional differential equation

$$\begin{cases} D^{\alpha}_{0+} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(o) = 0, \quad u^{\beta}_{0+}(1) = \sum_{i=1}^{m-2} \eta_i D^{\beta}_{0+} u(\xi_i), \end{cases} \tag{3}$$

where $1 < \alpha \leq 2$, $0 < \beta < \alpha - 1$, $0 < \xi_1 < ... < \xi_{m-2} < 1$ with $i = 1, 2, ..., m-2$, $\sum_{i=1}^{m-2} \eta_i \xi_i^{\alpha-\beta-1} < 1$, $f \in C([0,1] \times (0, +\infty) \to [0, +\infty))$.

In this paper we study existence and uniqueness of solution for fractional differential equation

$$D^q u(t) = f(t, u(t), D^p u(t)) \quad 2 < q < 3, \quad 0 < p < 1 \tag{4}$$

$$u(0) = 0, \quad D^p u(1) = \sum_{i=1}^{m-2} \zeta_i D^p u(\eta_i), \quad u''(1) = 0. \tag{5}$$

where $0 < \zeta_i, \eta_i < 1$ and derivative is the Caputo's fractional derivative. We use Schauder fixed point theorem and Arzela Ascoli theorem.

For $q > 0$, choose $n = [q] + 1$ in case q in not an integer and $n = q$ in case q is an integer. In this section we give definitions and fundamental results from references.

Definition The fractional integral of order $q > 0$ of a function continuous $f(t) : (0, \infty) \to \mathbb{R}$ is defined as under

$$I_0^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds. \tag{6}$$

Definition Caputo fractional derivative of order $q > 0$ is defined as under.

$${}^C D^{\alpha}_{a+} y(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t) dt}{(x-t)^{\alpha-n+1}} \tag{7}$$

Lemma Let $p, q \geq 0$ $f \in L_1[a,b]$. Then $I^p_{0+} I^q_{0+} f(t) = I^{p+q}_{0+} f(t) = I^q_{0+} I^p_{0+} f(t)$ and ${}^c D^q_{0+} I^q_{0+} f(t) = f(t)$, for all $t \in [a,b]$.

Lemma Let $\beta \geq \alpha$. Then the formula ${}^c D^{\alpha}_{0+} I^{\beta}_{0+} f(t) = I^{\beta-\alpha}_{0+} f(t)$, holds almost everywhere on $t \in [a,b]$ for $f \in L_1[a,b]$ and it is valid at any point $x \in [a,b]$ if $f \in C[a,b]$.

Lemma For $a > 0, g(t) \in C(0,1)$,the homogenous fractional differential equation

$${}^c D^{\alpha}_{0+} g(t) = 0$$

has a solution

$$g(t) = c_1 + c_2 t + c_3 t^2 + ... + c_n t^{n-1},$$

where $c_i \in R, i = 1, 2, 3, ..., n$ and $n = [\alpha] + 1$.

Lemma[2] Note that for $\lambda > -1, \lambda \neq \alpha - 1, \alpha - 2, ..., \alpha - n$, we have

$$D^{\alpha} t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha+1)} t^{\lambda-\alpha},$$
$$D^{\alpha} t^{\alpha-i} = 0, \quad i = 1, 2, ..., n.$$

Theorem[13](The Arzela Ascoli Theorem) Let F be an equicontinuous, uniformly bounded family of real valued functions f on an interval I (finite or infinite).Then F contains a uniformly convergent sequence of function in f , converging to a function $f \in C(I)$ where $C(I)$ denotes the space of all continuous bounded functions on I. Thus any sequence in F contains a uniformly bounded convergent subsequence on I and consequently F has a compact closure in $C(I)$.
Theorem[13](Shauder-Tychnoff Fixed Point Theorem) Let B be a locally convex, topological vector space. Let Y be a compact, convex subset of B and T a continuous map of Y into itself. Then T has a fixed point $y \in Y$.
Lemma The unique solution of boundary value problem for fractional differential equation (4) and (5) is given by

$$u(t) = \int_0^1 G(t,s) f(t, u(t), D^p u(t)) dt \tag{8}$$

where

$$G(t,s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{t}{\Delta}(-\frac{1}{\Gamma(q-p)}(1-s)^{q-p-1} + \frac{1}{\Gamma(3-p)\Gamma(q-2)}(1-s)^{q-3} \\ + \sum_{i=1}^{m-2} \frac{\zeta_i}{\Gamma(q-p)}(\eta_i - s)^{q-p-1} - \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-p)}(1-s)^{q-p-1}) \\ -\frac{t^2}{2\Gamma(2-p)}(1-s)^{q-3}, \quad s \le t, \\ \frac{t}{\Delta}(-\frac{1}{\Gamma(q-p)}(1-s)^{q-p-1} + \frac{1}{\Gamma(3-p)\Gamma(q-2)}(1-s)^{q-3} + \sum_{i=1}^{m-2} \frac{\zeta_i(\eta_i - s)^{q-p-1}}{\Gamma(q-p)} \\ -\frac{\sum_{i=1}^{m-2}\zeta_i(\eta)^{2-p}}{\Gamma(3-p)\Gamma(q-p)}(1-s)^{q-p-1}) - \frac{t^2}{2\Gamma(2-p)}(1-s)^{q-3}, \quad t \le s. \end{cases} \tag{9}$$

Proof We assume that $u(t)$ is the solution of fractional differential equation (4) then by lemma 1 we have

$$u(t) = I^q f(t, u(t), D^p u(t)) + c_0 + c_1 t + c_2 t^2 \tag{10}$$

By equation (10) and boundary conditions (5) we have $c_0 = 0$, $c_2 = \frac{-I^{q-2}f(1)}{2}$.

$$c_1 = \frac{1}{\Delta}(-I^{q-p}f(1) + \frac{1}{\Gamma(3-p)}I^{q-2}f(1) + \sum_{i=1}^{m-2}\zeta_i I^{q-p}f(\eta_i) - \sum_{i=1}^{m-2}\frac{\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)}I^{q-2}f(1)),$$

Where $\Delta = \frac{1-\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{1-p}}{\Gamma(2-p)}$. By (10) and values of c_0, c_1, c_2 we have

$$\begin{aligned} u(t) = I^q f(t) + \frac{t}{\Delta}(-I^{q-p}f(1) + \frac{1}{\Gamma(3-p)}I^{q-2}f(1) + \sum_{i=1}^{m-2}\zeta_i I^{q-p}f(\eta_i) \\ - \frac{1}{\Gamma(3-p)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}I^{q-2}f(1)) - \frac{t^2}{2}I^{q-2}f(1) \end{aligned} \tag{11}$$

Therefore the unique solution of boundary value problem (4), (5) is given by

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1} f(s, u(s), D^p u(s)) ds \\ &+ \frac{t}{\Delta}(\frac{-1}{\Gamma(q-p)}\int_0^1 (1-s)^{q-p-1} f(s, u(s), D^p u(s)) ds \end{aligned} \tag{12}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(3-p)}\frac{1}{\Gamma(q-2)}\int_0^1(1-s)^{q-3}f(s,u(s),D^pu(s))ds\\
&+\frac{1}{\Gamma(q-p)}\sum_{i=1}^{m-2}\zeta_i\int_0^{\eta_i}(\eta_i-s)^{q-p-1}f(s,u(s),D^pu(s))ds\\
&-\frac{1}{\Gamma(3-p)\Gamma(q-2)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}\int_0^1(1-s)^{q-3}f(s,u(s),D^pu(s))ds)\\
&-\frac{t^2}{2\Gamma(q-2)}\int_0^1(1-s)^{q-3}f(s,u(s),D^pu(s))ds
\end{aligned}
\tag{13}
$$

For $0<t<\eta_i$ we have the following estimates

$$
\begin{aligned}
u(t)=&\int_0^t[\frac{1}{\Gamma(q)}(t-s)^{q-1}+\frac{t}{\Delta}(\frac{-1}{\Gamma(q-p)}(1-s)^{q-p-1}+\frac{(1-s)^{q-3}}{\Gamma(3-p)\Gamma(q-2)}\\
&+\frac{1}{\Gamma(q-p)}\sum_{i=1}^{m-2}\zeta_i(\eta_i-s)^{q-p-1}-\frac{1}{\Gamma(3-p)\Gamma(q-2)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}(1-s)^{q-3})\\
&-\frac{t^2}{2\Gamma(q-2)}(1-s)^{q-3}]f(s,u(s),D^pu(s))ds+\int_t^{\eta_1}\{\frac{t}{\Delta}(\frac{-1}{\Gamma(q-p)}(1-s)^{q-p-1}\\
&+\frac{1}{\Gamma(3-p)\Gamma(q-2)}(1-s)^{q-3}+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i-s)^{q-p-1}}{\Gamma(q-p)}\\
&-\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}(1-s)^{q-3}}{\Gamma(3-p)\Gamma(q-2)})-\frac{t^2}{2\Gamma(q-2)}(1-s)^{q-3}\}f(s,u(s),D^pu(s))ds\\
&+\sum_{i=2}^{m-2}\int_{\eta_{i-1}}^{\eta_i}\{\frac{t}{\Delta}(\frac{-1}{\Gamma(q-p)}(1-s)^{q-p-1}+\frac{1}{\Gamma(3-p)\Gamma(q-2)}(1-s)^{q-3}\\
&+\frac{1}{\Gamma(q-p)}\sum_{i=1}^{m-2}\zeta_i(\eta_i-s)^{q-p-1}-\frac{1}{\Gamma(3-p)\Gamma(q-p)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}(1-s)^{q-3})\\
&-\frac{t^2}{2\Gamma(q-2)}(1-s)^{q-3}\}f(s,u(s),D^pu(s))ds+\int_{\eta_{m-2}}^1\{\frac{t}{\Delta}(\frac{-1}{\Gamma(q-2)}(1-s)^{q-p-1}\\
&+\frac{1}{\Gamma(3-p)\Gamma(q-2)}(1-s)^{q-3}-\frac{1}{\Gamma(3-p)\Gamma(q-2)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}(1-s)^{q-3})\\
&-\frac{t^2}{2\Gamma(q-2)}(1-s)^{q-3}\}f(s,u(s),D^pu(s))ds=\int_0^1G(t,s)f(s,u(s),D^pu(s))ds.
\end{aligned}
$$

For $\eta_{i-1}<t<\eta_i$, $\eta_0=0$, $\eta_{m-1}=1$ where $i=1,2,...,m-1$,

$$
u(t)=\sum_{k=1}^{i-1}\int_{\eta_{k-1}}^{\eta_k}[\frac{1}{\Gamma(q)}(t-s)^{q-1}+\frac{t}{\Delta}(\frac{-1}{\Gamma(q-p)}(1-s)^{q-p-1}
$$

$$+\frac{1}{\Gamma(3-p)\Gamma(q-2)}(1-s)^{q-3}+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i-s)^{q-p-1}}{\Gamma(q-p)}$$
$$-\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}(1-s)^{q-3}}{\Gamma(3-p)\Gamma(q-2)})-\frac{t^2}{2\Gamma(q-2)}(1-s)^{q-3}]f(s,u(s),D^pu(s))ds$$
$$+\int_{\eta_{i-1}}^{t}\{\int_0^t[\frac{(t-s)^{q-1}}{\Gamma(q)}+\frac{t}{\Delta}(\frac{-1}{\Gamma(q-p)}(1-s)^{q-p-1}+\frac{(1-s)^{q-3}}{\Gamma(3-p)\Gamma(q-2)}$$
$$+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i-s)^{q-p-1}}{\Gamma(q-p)}-\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}(1-s)^{q-3}}{\Gamma(3-p)\Gamma(q-2)})$$
$$-\frac{t^2(1-s)^{q-3}}{2\Gamma(q-2)}]\}f(s,u(s),D^pu(s))ds+\int_t^{\eta_i}\{\frac{t}{\Delta}(\frac{-1}{\Gamma(q-p)}(1-s)^{q-p-1}$$
$$+\frac{1}{\Gamma(3-p)\Gamma(q-2)}(1-s)^{q-3}+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i-s)^{q-p-1}}{\Gamma(q-p)}$$
$$-\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}(1-s)^{q-3}}{\Gamma(3-p)\Gamma(q-2)})-\frac{t^2}{2\Gamma(q-2)}(1-s)^{q-3}\}f(s,u(s),D^pu(s))ds$$
$$+\sum_{k=i+1}^{m-1}\int_{\eta_{k-1}}^{\eta_k}\{\frac{t}{\Delta}(\frac{-1}{\Gamma(q-p)}(1-s)^{q-p-1}+\frac{1}{\Gamma(3-p)\Gamma(q-2)}(1-s)^{q-3}$$
$$+\frac{1}{\Gamma(q-p)}\sum_{i=1}^{m-2}\zeta_i(\eta_i-s)^{q-p-1}-\frac{1}{\Gamma(3-p)\Gamma(q-2)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}(1-s)^{q-3})$$
$$-\frac{t^2}{2\Gamma(q-2)}(1-s)^{q-3}\}f(s,u(s),D^pu(s))ds=\int_0^1 G(t,s)f(s,u(s),D^pu(s))ds.$$

2. EXISTENCE AND UNIQUENESS

Let $I=[0,1]$, and $C(I)$ be the space of continuous functions defined on I. The space $\mathbb{E}=\{u(t)\in C(I,\mathbb{R}):D^pu(t)\in C(I,\mathbb{R})\}$ with the norm $\|u(t)\|=\max_{t\in I}|u(t)|+\max_{t\in I}|u^p(t)|$ is a Banach space [21]. We assume the spermium value of $\int_0^1 G(t,s)ds$ as

$$\mathbb{G}=\sup_{t\in I}\int_0^1|G(t,s)|ds.$$

Lemma Assume that

(H1) $f\in C([0,1]\times\mathbb{R}\times\mathbb{R},\mathbb{R})$,

(H2) there exist a constant $k>0$ such that for each $t\in I=[0,1]$ and for all $u=u(t)$, $v=v(t)\in\mathbb{R}$

$$|f(t,u,D^qu)-f(t,v,D^qv)|\le k\{|u-v|+|D^qu-D^qv|\}, \tag{14}$$

if

$$\max\{2k\mathbb{G},2k\{\frac{1}{\Gamma(q-p+1)}+\frac{1}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)}+\frac{1}{\Gamma(3-p)\Gamma(q-1)}$$
$$+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{q-p}}{\Gamma(q-p+1)}+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-1)})+\frac{1}{\Gamma(3-p)\Gamma(q-1)}\}\}=d<1,$$

then (4), (5) has a unique solution.
Proof For the proof we will have to show that $T:\mathbb{E}\to\mathbb{E}$ is contraction mapping defined by:

$$Tu(t)=\int_0^1 G(t,s)f(s,u(s),D^q u(s))ds \tag{15}$$

Then we have the following estimates

$$\begin{aligned}&|Tu-Tv|=\int_0^1|G(t,s)||f(s,u,D^pu)-f(s,v,D^pv)|ds\\&\leq Sup_{t\in I}G(t,s)\int_0^1|f(s,u,D^pu)-f(s,v,D^pv)|ds\leq\mathbb{G}k\{|u-v|+|D^pu-D^pv|\}\\&\leq\mathbb{G}k(\max_{t\in I}|u-v|+\max_{t\in I}|D^pu-D^pv|)=\mathbb{G}k\|u-v\|.\end{aligned} \tag{16}$$

From (12), we have that

$$\begin{aligned}D^pTu(t)=&\frac{1}{\Gamma(q-p)}\int_0^t(t-s)^{q-p-1}f(s,u(s),D^pu(s))ds\\&+\frac{t^{1-p}}{\Delta\Gamma(2-p)}(\frac{-1}{\Gamma(q-p)}\int_0^1(1-s)^{q-p-1}f(s,u(s),D^pu(s))ds\\&+\frac{1}{\Gamma(3-p)\Gamma(q-2)}\int_0^1(1-s)^{q-3}f(s,u(s),D^pu(s))ds\\&+\frac{1}{\Gamma(q-p)}\sum_{i=1}^{m-2}\zeta_i\int_0^{\eta_i}(\eta_i-s)^{q-p-1}f(s,u(s),D^pu(s))ds\\&-\frac{1}{\Gamma(3-p)\Gamma(q-2)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}\int_0^1(1-s)^{q-3}f(s)ds)\\&-\frac{t^{2-p}}{\Gamma(3-p)\Gamma(q-2)}\int_0^1(1-s)^{q-3}f(s,u(s),D^pu(s))ds\end{aligned} \tag{17}$$

and

$$\begin{aligned}&|D^pTu-D^pTv|\\&\leq\frac{1}{\Gamma(q-p)}\int_0^t(t-s)^{q-p-1}|f(s,u(s),u^p(s))-f(s,v(s),v^p(s))|ds\\&+\frac{t^{1-p}}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p)}\int_0^1(1-s)^{q-p-1}|f(s,u(s),u^p(s))-f(s,v(s),v^p(s))|ds\\&+\frac{1}{\Gamma(3-p)\Gamma(q-2)}\int_0^1(1-s)^{q-3}|f(s,u(s),u^p(s))-f(s,v(s),v^p(s))|ds\\&+\frac{\sum_{i=1}^{m-2}\zeta_i}{\Gamma(q-p)}\int_0^{\eta_i}(\eta_i-s)^{q-p-1}|f(s,u(s),u^p(s))-f(s,v(s),v^p(s))|ds\end{aligned} \tag{18}$$

$$
\begin{aligned}
&+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-2)}\int_0^1(1-s)^{q-3}|f(s,u(s),u^p(s))-f(s,v(s),v^p(s))|ds)\\
&+\frac{t^{2-p}}{\Gamma(3-p)}\int_0^1(1-s)^{q-3}|f(s,u(s),u^p(s))-f(s,v(s),v^p(s))|ds\\
&\le\frac{t^{q-p}}{\Gamma(q-p+1)}k\{|u-v|+|u^p(s)-v^p(s)|\}ds+\frac{t^{1-p}}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)}\{|u-v|\\
&+|u^p(s)-v^p(s)|\}ds+\frac{1}{\Gamma(3-p)\Gamma(q-1)}\{|u-v|+|u^p(s)-v^p(s)|\}\\
&+\frac{\sum_{i=1}^{m-2}(\eta_i)^{q-p}}{\Gamma(q-p+1)}\{|u-v|+|u^p(s)-v^p(s)|\}\\
&+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-1)}\{|u-v|+|u^p(s)-v^p(s)|\})\\
&+\frac{t^{2-p}}{\Gamma(3-p)}\frac{1}{\Gamma(q-1)}\{|u-v|+|u^p(s)-v^p(s)|\}\\
&\le k\{\frac{1}{\Gamma(q-p+1)}+\frac{1}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)}+\frac{1}{\Gamma(3-p)\Gamma(q-1)}\\
&+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{q-p}}{\Gamma(q-p+1)}+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-1)})+\frac{1}{\Gamma(3-p)\Gamma(q-1)}\}\|u-v\|.
\end{aligned}
\tag{19}
$$

From (16), (18) we have the following estimates

$$
\begin{aligned}
&\|Tu-Tv\|=\max_{t\in I}|Tu-Tv|+\max_{t\in I}|D^pTu-D^pTv|\\
&\le\{k\mathbb{G}\|u-v\|+k\{\frac{1}{\Gamma(q-p+1)}+\frac{1}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)}\\
&+\frac{1}{\Gamma(3-p)\Gamma(q-1)}+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{q-p}}{\Gamma(q-p+1)}+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-1)})\\
&+\frac{1}{\Gamma(3-p)\Gamma(q-1)}\}\|u-v\|\\
&\le\max\{2k\mathbb{G},2k\{\frac{1}{\Gamma(q-p+1)}+\frac{1}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)}+\frac{1}{\Gamma(3-p)\Gamma(q-1)}\\
&+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{q-p}}{\Gamma(q-p+1)}+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-1)})+\frac{1}{\Gamma(3-p)\Gamma(q-1)}\}\}\|u-v\|=d\|u-v\|.
\end{aligned}
\tag{20}
$$

Thus T is contraction mapping and hence by theorem (1) the boundary value problem (4), (5) has a unique fixed point u.
Theorem (A1) Assume that $f\in C(I\times\mathbb{R}\times\mathbb{R})$.
(A2) There exist $h=h(t)\in C(I,\mathbb{R})$ and $\phi:[0,\infty]\to(0,\infty)$ is continuous and non-decreasing function such that

$$
|f(t,u,D^pu)|\le h\phi|(D^pu)| \quad \forall \quad t\in I,u,v\in\mathbb{R}. \tag{21}
$$

(A3) There exists constant $r > 0$ such that

$$\max\{2\mathbb{G}h^*\phi|D^pu(t)|, 2h^*\phi|D^pu(t)|\{\frac{1}{\Gamma(q-p+1)} + \frac{1}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)} + \frac{1}{\Gamma(3-p)\Gamma(q-1)} + \frac{1}{\Gamma(q-p+1)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{q-p} + \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-1)}) + \frac{1}{\Gamma(3-p)\Gamma(q-1)}\}\} = r < 1$$

where $h^* = \sup\{h(t), t \in I\}$, then (4), (5) has at least one solution with $|u| < r$ for each $t \in I$.

Proof Step 1: Assume that we have a convergent sequence $\{u_n(t)\}$ such that $\{u_n(t) \to u(t)\}$ as $n \to \infty$ then it is easy to show that $T(t, u_n, D^pu_n) \to T(t, u, D^pu)$. Thus $|T(t, u_n, D^pu_n) - T(t, u, D^pu)| \to 0$ or $T(t, u_n, D^pu_n) \to T(t, u, D^pu)$ as $n \to \infty$.

Step 2: Let $\mathbb{D} \subseteq C(I, \mathbb{R})$ is bounded set, to show that $T(\mathbb{D})$ is also a bounded set. We consider $u(t) \in \mathbb{D}$ then we have

$$\begin{aligned}|Tu| &\le |\int_0^1 G(t,s)f(s,u(s),D^pu(s))ds| \le \mathbb{G}\int_0^1 |f(s,u(s),D^pu(s))|ds \\ &\le \mathbb{G}h^*\phi|D^pu(t)|\end{aligned} \tag{22}$$

From equation (17) we have

$$\begin{aligned}
&|D^pTu| = |\frac{1}{\Gamma(q-p)}\int_0^t (t-s)^{q-p-1}f(s,u(s),D^pu(s))ds \\
&+ \frac{t^{1-p}}{\Delta\Gamma(2-p)}(\frac{-1}{\Gamma(q-p)}\int_0^1 (1-s)^{q-p-1}f(s,u(s),D^pu(s))ds \\
&+ \frac{1}{\Gamma(3-p)\Gamma(q-2)}\int_0^1 (1-s)^{q-3}f(s,u(s),D^pu(s))ds \\
&+ \frac{1}{\Gamma(q-p)}\sum_{i=1}^{m-2}\zeta_i\int_0^{\eta_i}(\eta_i-s)^{q-p-1}f(s,u(s),D^pu(s))ds \\
&- \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-2)}\int_0^1 (1-s)^{q-3}f(s,u(s),D^pu(s))ds) \\
&- \frac{t^{2-p}}{\Gamma(3-p)\Gamma(q-2)}\int_0^1 (1-s)^{q-3}f(s,u(s),D^pu(s))ds| \\
&\le h^*\phi|D^pu(t)|\{\frac{t^{q-p}}{\Gamma(q-p+1)} + \frac{t^{1-p}}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)} + \frac{1}{\Gamma(3-p)\Gamma(q-1)} \\
&+ \frac{\sum_{i=1}^{m-2}(\eta_i)^{q-p}}{\Gamma(q-p+1)} + \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-p+1)}) + \frac{t^{2-p}}{\Gamma(3-p)\Gamma(q-1)}\} \\
&\le h^*\phi|D^pu(t)|\{\frac{1}{\Gamma(q-p+1)} + \frac{1}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-1)} + \frac{1}{\Gamma(3-p)\Gamma(q-1)} \\
&+ \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{q-p}}{\Gamma(q-p+1)} + \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-1)}) + \frac{1}{\Gamma(3-p)\Gamma(q-1)}\}
\end{aligned} \tag{23}$$

Thus we have that

$$\begin{aligned}
&\|T(t,u,D^pTu)\| = \max_{t\in I}|Tu| + \max_{t\in I}|D^pTu|\\
&\leq \mathbb{G}h^*\phi|D^pu(t)| + h^*\phi|D^pu(t)|\{\frac{1}{\Gamma(q-p+1)} + \frac{1}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)}\\
&+\frac{1}{\Gamma(3-p)\Gamma(q-1)} + \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{q-p}}{\Gamma(q-p+1)} + \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-1)}) + \frac{1}{\Gamma(3-p)\Gamma(q-1)}\}\\
&\leq \max\{2\mathbb{G}h^*\phi|D^pu(t)|, 2h^*\phi|D^pu(t)|\{\frac{1}{\Gamma(q-p+1)} + \frac{1}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)}\\
&+\frac{1}{\Gamma(3-p)\Gamma(q-1)} + \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{q-p}}{\Gamma(q-p+1)} + \frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-1)}) + \frac{1}{\Gamma(3-p)\Gamma(q-1)}\}\}\\
&= r < 1.
\end{aligned}$$

Hence we have that $Tu \in \mathbb{D}\ \forall u(t) \in \mathbb{D}$, this implies that $T(\mathbb{D})\subseteq\mathbb{D}$.
Step 3: Now in order to show that T maps $\mathbb{D}$ into equicontinuous set of $C(I,\mathbb{R})$. For this we assume that $t_1,\ t_2 \in I$ such that $t_1 < t_2$ and $u(t) \in \mathbb{D}$, we have:

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &\leq \int_0^1 |G_2(t,s) - G_1(t,s)||f(s,u(s),D^pu(s))|ds\\
&\leq h^*\phi|D^pu(s)|\int_0^1 |G(t_2,s) - G(t_1,s)|ds.
\end{aligned}$$

This implies that

$$|Tu(t_2) - Tu(t_1)| \to 0 \text{ as } t_1 \to t_2 \tag{24}$$

Equation (17) and proposed conditions (A_1), (A_2) and (A_3) implies that

$$\begin{aligned}
&|D^pTu(t_2) - D^pTu(t_1)|\\
&\leq |\frac{1}{\Gamma(q-p)}(\int_0^{t_2}(t_2-s)^{q-p-1}f(s)ds - \int_0^{t_1}(t_1-s)^{q-p-1}f(s,u(s),D^pu(s))ds)\\
&+\frac{t_2^{1-p}-t_1^{1-p}}{\Delta\Gamma(2-p)}(\frac{-1}{\Gamma(q-p)}\int_0^1(1-s)^{q-p-1}f(s,u(s),D^pu(s))ds\\
&+\frac{1}{\Gamma(3-p)\Gamma(q-2)}\int_0^1(1-s)^{q-3}f(s,u(s),D^pu(s))ds\\
&+\frac{1}{\Gamma(q-p)}\sum_{i=1}^{m-2}\zeta_i\int_0^{\eta_i}(\eta_i-s)^{q-p-1}f(s,u(s),D^pu(s))ds\\
&-\frac{1}{\Gamma(3-p)\Gamma(q-2)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}\int_0^1(1-s)^{q-3}f(s,u(s),D^pu(s))ds)
\end{aligned}$$

$$
\begin{aligned}
&-\frac{t_2^{2-p}-t_1^{2-p}}{\Gamma(3-p)\Gamma(q-2)}\int_0^1(1-s)^{q-3}f(s,u(s),D^pu(s))ds|\\
\leq& h^*\phi|D^pu(t)|\{\frac{1}{\Gamma(q-p+1)}(t_2^{q-p}-t_1^{q-p})+\frac{t_2^{1-p}-t_1^{1-p}}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)}\\
&+\frac{1}{\Gamma(3-p)\Gamma(q-1)}+\frac{1}{\Gamma(q-p+1)}\sum_{i=1}^{m-2}(\eta_i)^{q-p}+\frac{\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}}{\Gamma(3-p)\Gamma(q-1)})\\
&+\frac{t_2^{2-p}-t_1^{2-p}}{\Gamma(3-p)}\frac{1}{\Gamma(q-1)}\}
\end{aligned}
$$

This implies that

$$|D^pTu(t_2)-D^pTu(t_1)|\to 0 \text{ as } t_1\to t_2 \tag{25}$$

Thus, by (24) and (25) we have:

$$
\begin{aligned}
&\|Tu(t_2)-Tu(t_1)\|=\max_{t\in I}|Tu(t_2)-Tu(t_1)|+\max_{t\in I}|D^pTu(t_2)-D^pTu(t_1)|\\
&\leq h^*\phi|D^pu(s)|\int_0^1|G(t_2,s)-G(t_1,s)|ds+h^*\phi|D^pu(t)|\{\frac{t_2^{q-p}-t_1^{q-p}}{\Gamma(q-p+1)}\\
&+\frac{t_2^{1-p}-t_1^{1-p}}{\Delta\Gamma(2-p)}(\frac{1}{\Gamma(q-p+1)}+\frac{1}{\Gamma(3-p)\Gamma(q-1)}+\frac{\sum_{i=1}^{m-2}(\eta_i)^{q-p}}{\Gamma(q-p+1)}\\
&+\frac{1}{\Gamma(3-p)\Gamma(q-1)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p})+\frac{(t_2^{2-p}-t_1^{2-p})}{\Gamma(3-p)\Gamma(q-1)}\}
\end{aligned}
$$

This implies that $\|Tu(t_2)-Tu(t_1)\|\to 0$ as $t_1\to t_2$. Thus, by Arzela Ascoli theorem, T is completely continuous operator. And as a result, by Shauder fixed point theorem T has a fixed point $u(t)\in\mathbb{D}$ which is the solution of boundary value problem (4) and (5).

Example

$$
\begin{aligned}
&u^{(5/2)}(t)=f(t,u(t),u^{(1/2)}(t))\\
&u(0)=0,\quad u^{(1/2)}(1)=\sum_{i=1}^{m-2}\zeta_iu^{(1/2)}(\eta_i)=1/5,\quad u''(1)=0.
\end{aligned}
\tag{26}
$$

where $\zeta_i=1/2,\eta_i=1/2$, $m=3$ and $f(t,u(t),u^{(1/2)}(t))=\frac{1}{40e^t(1+3|u(t)|+5|u^{1/2}(t)|)}$. Where

$$
\begin{aligned}
|f(t,u,u^{(p)})-f(t,v,v^{(p)})|&\leq\frac{1}{40e^t}(\frac{3|u|-3|u^p|+5|v|-5|v^p|}{(1+3|u|+5|u^p|)(1+3|v|+5|v^p|)})\\
&\leq\frac{1}{40}(5|u-u^p|+5|v-v^p|)=\frac{1}{8}(|u-u^p|+|v-v^p|).
\end{aligned}
\tag{27}
$$

This implies that $k = \frac{1}{8}$, $\mathbb{G} = 2.55$, $2 * k * \mathbb{G} = 0.64 < 1$ and

$$
\begin{aligned}
2 * k * \{ & \frac{1}{\Gamma(q-p+1)} + \frac{1}{\Delta\Gamma(2-p)}\Big(\frac{1}{\Gamma(q-p+1)} + \frac{1}{\Gamma(3-p)\Gamma(q-1)} \\
& + \frac{1}{\Gamma(q-p+1)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{q-p} + \frac{1}{\Gamma(3-p)\Gamma(q-1)}\sum_{i=1}^{m-2}\zeta_i(\eta_i)^{2-p}\Big) \\
& + \frac{1}{\Gamma(3-p)\Gamma(q-1)}\} < 0.4752 < 1.
\end{aligned} \tag{28}
$$

Thus (H1), (H2) are satisfied and hence by lemma (2) the boundary value problem (26) has a unique solution.

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