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NUMERICAL SOLUTION OF FREDHOLM-VOLTERRA FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this paper, a numerical method is proposed to solve Fredholm-Volterra fractional integro-differential equation with nonlocal boundary conditions. For this purpose, the Chebyshev wavelets of second kind are used in collocation method. It reduces the given fractional integro-differential equation (FIDE) with nonlocal boundary conditions in a linear system of equations which one can solve easily. The test examples are taken from the literature in order to illustrate the proposed method and different comparisons are also shown. The involved errors are measured with RMS-norm to show the accuracy obtained.

1. Introduction

Qualitative and numerical studies of fractional differential equations have played an important role due to its application [1]. Computation of the solution of fractional differential [2] has proved to be challenging since we do not have separation of variable method and product rules which are readily available for integer order differential equations. Since there is no general method like in case of integer order differential equations. Many researchers in the field of fractional differential equations and fractional integro-differential equations have proposed different methods to compute the solution. A few among these are transform method [3], homotopy analysis [4], Adomian decomposition method [5], variational iteration method [6], homotopy perturbation method [6] and collocation method [7, 8]. But implementation of wavelet based methods [9, 10, 11] for solving fractional differential equations have started recently. Further some researchers have also worked on fractional integro-differential equation by using wavelet based methods [12, 13, 14, 15]. In this paper, we have proposed a numerical method to solve the most general Fredholm-Volterra fractional integro-differential equation (1) with non-local boundary conditions (2) from [16] by using Chebyshev wavelets of second kind.

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TABLE 1. Comparison of recent methods [12, 13] with our proposed method in this paper,

	Method in [12]	Method in [13]	Method in this paper
Type of numerical method	Galerkin	Galerkin	Collocation
Nature of Constraints	Local initial conditions	Local boundary conditions	Non-local boundary conditions
Class of Integral equation	Fredholm	Fredholm	Fredhom-Volterra
Type of Wavelet used	Chebyshev	CAS	Chebyshev
Numerical Stability discussion for noisy data	No	No	Yes

$$D^q u(x) \;=\; g(x) + f(x)u(x) + \int_a^x k_1(x,t)u(t)dt + \int_a^b k_2(x,t)u(t)dt \qquad (1)$$

$$\sum_{j=1}^{m} \Big(\alpha_{ij} u^{(j-1)}(a) + \beta_{ij} u^{(j-1)}(b)\Big) + \mu_i \int_a^b H_i(t)u(t)dt = c_i, i = 1, 2, ..., m. \qquad (2)$$

where $m-1 < q \leq m, a < x, t < b$ and $m \in N, D^q$ denotes a differential operator with fractional order q; $f(x), g(x)$ and $k_i(x,t), i = 1, 2$, are analytic functions, $H_i(t)$ is a continuous function, $\alpha_{ij}, \beta_{ij}, \mu_i$ and $c_i, (i = 1, 2, ..., m)$ are constants and $u(x)$ is a function of class C^m (functions which are continuously differentiable of order m on $I = (0, \infty)$ and integrable on any finite subinterval of $J = [0, \infty)$).
Our work is motivated from very recent work in the field of fractional integro-differential equations involving nonlocal boundary conditions [16, 17], numerical methods based on wavelets [9, 10, 11, 12, 13, 14, 15] and one of an interesting application [1]. To the best of our knowledge, nobody has solved a general problem (1) of Fredholm-Volterra type fractional integro-differetial equation with nonlocal boundary conditions (2) by using any of wavelet based method. Further, in order to conclude the originality of our work, we have compared our method with the recently proposed methods [12, 13] in the Table 1.

2. Basic definition of Chebyshev wavelets

The Chebyshev wavelets of second kind can be defined as

$$\psi_{n,l}(x) \;=\; \begin{cases} \sqrt{\frac{2^{k+1}}{\pi}} T_l(2^k x - (2n-1)), & \frac{n-1}{2^{k-1}} \leq x < \frac{n}{2^{k-1}} \\ 0, & \text{elsewhere} \end{cases} \qquad (3)$$

where $n = 1, 2, ..., 2^{k-1}$; k can assume any positive integer; $T_l(x)$ are second kind Chebyshev polynomials of degree l.

The second kind Chebyshev polynomials $T_l(x)$ are defined by

$$T_0(x) = 1, T_1(x) = 2x, T_{l+1}(x) = 2xT_l(x) - T_{l-1}(x) \qquad (4)$$

where $l = 1, 2, ...$

These are orthogonal polynomials with respect to the weight function $w(x) = \sqrt{1-x^2}$ on $[-1,1]$. The dilated and translated weight function can be defined as

$$w_n^k(x) = w(2^k x - 2n + 1) \tag{5}$$

for a given value of k and n appearing in second kind Chebyshev wavelet.
A function $u(x) \in L^2_{w_n^k}[0,1]$ is expanded as

$$u(x) = \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} c_{n,l} \psi_{n,l}(x) \tag{6}$$

where $c_{n,l} = \langle u(x), \psi_{n,l}(x) \rangle$ in which $\langle \cdot, \cdot \rangle$ denotes the inner product in $L^2_{w_n^k}[0,1]$.

The truncated series of (6) can be defined as

$$u(x) = \sum_{n=1}^{2^{k-1}} \sum_{l=0}^{R-1} c_{n,l} \psi_{n,l}(x) = C^T \psi(x), \tag{7}$$

where $C = [c_1, c_2, ..., c_{2^{k-1}}]^T, \psi(x) = [\psi_1, \psi_2, ..., \psi_{2^{k-1}}]^T, c_i = [c_{i,0}, c_{i,1}, ..., c_{i,R-1}]^T, \psi_i = [\psi_{i,0}, \psi_{i,1}, ..., \psi_{i,R-1}]^T$ for $i = 1, 2, ..., 2^{k-1}$.

The definitions and properties of Rieman-Liouville fractional integral operator of a given order and Caputo definition of fractional differential operator have been followed from [2].

3. Methodology for the solution

$$\text{Let } D^q u(x) = C^T \psi(x), m-1 < q \leq m, \tag{8}$$

where C is defined as in (7).

$$\text{This gives } u(x) = C^T P^q \psi(x) + \sum_{k=0}^{m-1} u^{(k)}(0) \frac{x^k}{k!}, m-1 < q \leq m, \tag{9}$$

where $u^{(k)}(x)$ denotes the k^{th} derivative of function $u(x)$ and the matrix P is defined as the operational matrix of the fractional integration [18].

$$\text{Let } f_1(x) \approx {F_1}^T \psi(x), \tag{10}$$

where F_1 is a coefficient matrix defined similar as in (7).

$$f_2(x)u(x) \approx f_2(x) \left[C^T P^q \psi(x) + \sum_{k=0}^{m-1} u^{(k)}(0) \frac{x^k}{k!} \right] \tag{11}$$

$$\begin{aligned} \int_a^x k_1(x,t)u(t)dt &\approx \int_a^x \left[C^T P^q \psi(t) + \sum_{k=0}^{m-1} u^{(k)}(0) \frac{t^k}{k!} \right] \left[\psi(t)^T K_1 \psi(x) \right] dt \\ &= C^T P^q L_x K_1 \psi(x) + \sum_{k=0}^{m-1} \left[u^{(k)}(0) L_x^k K_1 \psi(x) \right] \end{aligned} \tag{12}$$

where

$$\begin{aligned}
\int_a^x \psi(t)\psi(t)^T dt &= L_x \\
\int_a^x \frac{t^0}{0!}\psi(t)^T dt &= L_x^0 \\
\int_a^x \frac{t^1}{1!}\psi(t)^T dt &= L_x^1 \\
&\vdots \\
\int_a^x \frac{t^{m-1}}{(m-1)!}\psi(t)^T dt &= L_x^{m-1}
\end{aligned} \tag{13}$$

$$\begin{aligned}
\int_a^1 k_2(x,t)u(t)dt &\approx \int_a^1 \left[C^T P^q \psi(t) + \sum_{k=0}^{m-1} u^{(k)}(0)\frac{t^k}{k!}\right]\left[\psi(t)^T K_2 \psi(x)\right] dt \\
&= C^T P^q M K_2 \psi(x) + \sum_{k=0}^{m-1}\left[u^{(k)}(0) M^k K_2 \psi(x)\right]
\end{aligned} \tag{14}$$

where

$$\begin{aligned}
\int_a^1 \psi(t)\psi(t)^T dt &= M \\
\int_a^1 \frac{t^0}{0!}\psi(t)^T dt &= M^0 \\
\int_a^1 \frac{t^1}{1!}\psi(t)^T dt &= M^1 \\
&\vdots \\
\int_a^1 \frac{t^{m-1}}{(m-1)!}\psi(t)^T dt &= M^{m-1}
\end{aligned} \tag{15}$$

So using the equations (7) to (14), the equation (1) becomes

$$\begin{aligned}
C^T\psi(x) = F_1^T\psi(x) &+ f_2(x)C^T P^q \psi(x) + f_2(x)\left[\sum_{k=0}^{m-1} u^{(k)}(0)\frac{x^k}{k!}\right] \\
&+ C^T P^q L_x K_1 \psi(x) + \left[\sum_{k=0}^{m-1} u^{(k)}(0) L_x^k K_1 \psi(x)\right] \\
&+ C^T P^q M K_2 \psi(x) + \left[\sum_{k=0}^{m-1} u^{(k)}(0) M^k K_2 \psi(x)\right].
\end{aligned}$$

Choosing $x_i = \frac{2i-1}{2\omega}, i = 1, 2, ..., \omega = 2^{k-1}R$, collocation points between 0 and 1, one gets

$$\begin{aligned} C^T\psi(x_i) &= F_1^T\psi(x_i) + f_2(x_i)C^TP^q\psi(x_i) + f_2(x_i)\left[\sum_{k=0}^{m-1} u^{(k)}(0)\frac{x_i^k}{k!}\right] \\ &+ C^TP^qL_{x_i}K_1\psi(x_i) + \left[\sum_{k=0}^{m-1} u^{(k)}(0)L_{x_i}^kK_1\psi(x_i)\right] \\ &+ C^TP^qMK_2\psi(x_i) + \left[\sum_{k=0}^{m-1} u^{(k)}(0)M^kK_2\psi(x_i)\right], \end{aligned} \tag{16}$$

where $i = 1, 2, ..., \omega$.

Now finding the r^{th} derivative of $u(x)$ as

$$u^{(r)}(x) = D^ru(x) = C^TP^{q-r}\psi(x), r = 0, 1, 2, ..., m-1; m-1 < q \leq m \tag{17}$$

For $i = 1, 2, ..., m$, one can approximate

$$\begin{aligned} \int_a^b H_i(t)u(t)dt &\approx \int_a^b H_i(t)\left[C^TP^q\psi(t) + \sum_{k=0}^{m-1} u^{(k)}(0)\frac{t^k}{k!}\right] \\ &= C^TP^qd_i + \sum_{k=0}^{m-1}\left[u^{(k)}(0)d_i^k\right] \end{aligned} \tag{18}$$

where

$$\begin{aligned} \int_a^b H_i(t)\psi(t)dt &= d_i \\ \int_a^b \frac{t^0}{0!}H_i(t)dt &= d_i^0 \\ \int_a^b \frac{t^1}{1!}H_i(t)dt &= d_i^1 \\ &\vdots \\ \int_a^b \frac{t^{m-1}}{(m-1)!}H_i(t)dt &= d_i^{(m-1)} \end{aligned} \tag{19}$$

So using equations (17) to (19), the equation (2) becomes

$$\sum_{j=1}^{m}\left(\alpha_{ij}[C^TP^{q-j+1}\psi(0)] + \beta_{ij}[C^TP^{q-j+1}\psi(1)]\right) + \mu_i\left(C^TP^qd_i + \sum_{k=0}^{m-1}\left[u^{(k)}(0)d_i^k\right]\right) = c_i \tag{20}$$

where $i = 1, 2, ..., m$.

Hence (16) and (20) form a system of $\omega + m$ equations in $\omega + m$ unknowns which one can easily solve.

In the next section, the proposed method is implemented for the test examples corresponding to fractional integro-differential equations (1) with nonlocal boundary conditions (2). The comparisons are also shown with the results in [16].

4. Illustrative examples

In equation (1), the exact data function is denoted by $g(x)$. And, one can obtain a noisy data function by adding a random error $\epsilon\theta_i$ to $g(x)$, i.e. $g^{\epsilon}(x_i) = g(x_i) + \epsilon\theta_i$ where $x_i = ih; i = 0, 1, 2, ..., N; Nh = 1$ and θ_i is a uniform random variable with values in $[-1, 1]$ such that $\max\limits_{1\leq i\leq N} |g_{\epsilon}(x_i) - g(x_i)| \leq \epsilon$. The accuracy of the proposed method is further measured with the help of absolute errors and discrete l_2-norm of absolute errors. The discrete l_2-norm of absolute errors can be defined as $\left(\frac{1}{N+1}\sum_{i=0}^{N} |u_{exact}(x_i) - u(x_i)|^2\right)^{1/2}$. This discrete l_2-norm of absolute errors is also known as root mean square error norm (i.e. RMS-norm). Further E_0, E_1 and E_2 are defined as the random fluctuations $u_{\epsilon}(x_i) - u_{exact}(x_i)$, of the solution $u(x)$. These occur when random noises are introduced in the given data function $g(x)$. The random noises are $\epsilon = 0, 0.001$ and 0.01, respectively for E_0, E_1 and E_2.

Theorem 1(a) Assume η be the number of vanishing moments for a wavelet $\psi_{n,l}(x)$ and let $f(x) \in C^{\eta}[0, 1]$. Then the wavelet coefficient $c_{n,l}$ decays as follows:
$|c_{n,l}| \leq C_{\eta} 2^{-n(\eta+\frac{1}{2})} \max\limits_{\xi\in[0,1]} |f^{\eta}(\xi)|$
where C_{η} is an independent constant from n, l and $f(x)$.

(b) Suppose $f(x) \in C^{\eta}[0, 1]$ and $f(x) \approx \sum_{n=1}^{2^k-1} \sum_{l=0}^{R-1} c_{n,l}\psi_{n,l}(x) = C^T\psi(x)$ is the approximate solution by using Chebyshev wavelets method. Then the error bound is obtained as follows:
$\|error(f(x))\| \leq \frac{1}{\eta! 2^{\eta(k-1)}} \max\limits_{\xi\in[0,1]} |f^{\eta}(\xi)|,$
where $error(f(x)) = f(x) - C^T\psi(x)$.

Proof: For details, one can refer [19].

Remark: Importance of Theorem 1.
Theorem 1(a) implies that wavelet coefficients are exponentially decayed with respect to η and by increasing η the decay increases while Theorem 1(b) provides an error bound in a function approximation by Chebyshev wavelet of second kind. One can notice that $f(x)$ and $g(x)$ (with reference to equation (1)) in all the examples, belong to a class of infinitely differentiable continuous functions $C^{\infty}[0, 1]$. So the above theorem concludes that the wavelet coefficients are going to be very small and a good approximation can be obtained easily.

Example 4.1 Consider the following fractional integro-differential equation from [16] :

$$D^{1/2}u(x) = \frac{\sqrt{x}}{\Gamma(3/2)} - \frac{x^2}{2} - x^2\frac{e^x}{3}u(x) + e^x\int_0^x tu(t)dt + \int_0^1 x^2u(t)dt.$$

The nonlocal boundary condition is given by

$$u(0) + u(1) - 3\int_0^1 tu(t)dt = 0.$$

The Example 4.1 has an exact solution $u(x) = x$. For the problem defined in Example 4.1, the absolute errors are compared by using different basis functions. Like the basis functions in matrix form for $k = 2, R = 2$, are given by $\psi(x) = [\psi_{1,0}, \psi_{1,1}, \psi_{2,0}, \psi_{2,1}]^T$. The numerical results are shown in Figure 1. It is observed that as the value of k or R or both increases, the absolute error decreases. From the Figure 1, it can be noticed that the maximum absolute error occurs for $k = 2, R = 2$,

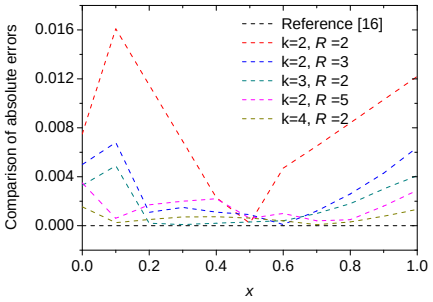

FIGURE 1. compares the absolute errors by using different basis functions for Example 4.1.

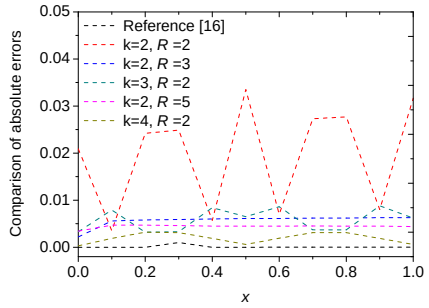

FIGURE 2. compares the absolute errors by using different basis functions for Example 4.2.

while the least error occurs for $k = 4, R = 2$. And, the absolute errors are also compared with [16].

Example 4.2 Consider the following fractional integro-differential equation from [16] :

$$D^{5/4}u(x) = \frac{8}{3}\frac{x^{3/4}}{\Gamma(3/4)} + 2 - 2\cos x + x^2 \sin x - 2x \sin x + [\cos x - \sin x]u(x) + \int_0^x \sin t u(t)dt.$$

The nonlocal boundary conditions are given as follows:

$$u(0) + u(1) + \left(\frac{e+1}{e+2}\right)u'(0) + \frac{1}{2}u'(1) - \int_0^1 tu(t)dt = 0,$$

$$2u(0) + 2u(1) + \left(\frac{e}{e+1}\right)u'(0) - u'(1) = 0.$$

Further, the Example 4.2 has an exact solution $u(x) = x^2$. And, the numerical results for absolute errors by using different pair of values of k and R (like $k =$

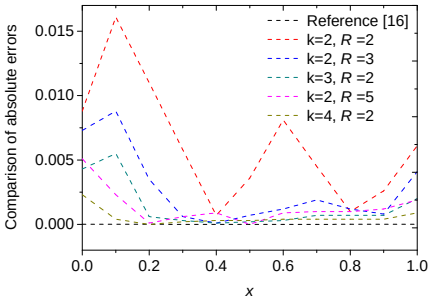

FIGURE 3. compares the absolute errors by using different basis functions in our method with the method in [16] for Example 4.3.

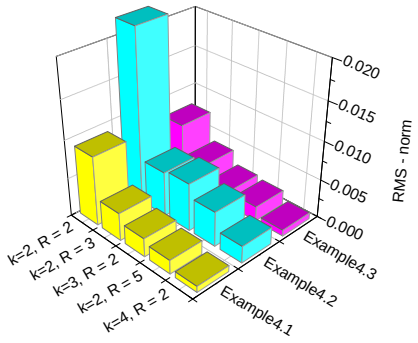

FIGURE 4. compares the errors measured with RMS-norm for all the examples by using different basis functions.

$2, R = 2; k = 2, R = 3; k = 3, R = 2; k = 2, R = 5$ and $k = 4, R = 2$) are shown in Figure 2. We have obtained a similar behaviour as in Example 4.1, but magnitudes of errors are increased.

Example 4.3 Let the fractional integro-differential equation from [16] be:

$$D^{1/3}u(x) = \frac{3}{2}\frac{x^{2/3}}{\Gamma(2/3)} - 1 + e^{x^2} - x^2e^{x^2} + \int_0^x x^2e^{xt}u(t)dt.$$

The nonlocal boundary condition is

$$u(0) + 2u(1) + 3\int_0^1 tu(t)dt = 3.$$

Example 4.3 has an exact solution $u(x) = x$. Figure 3 compares the absolute errors when different basis functions are used in Example 4.3. The results are also

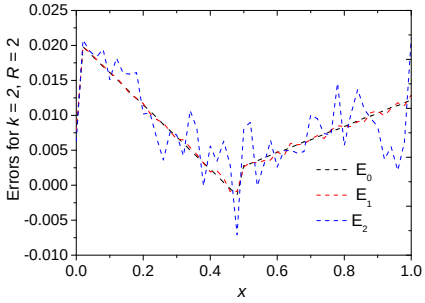

FIGURE 5. compares the errors E_0, E_1 and E_2 for Example 4.1

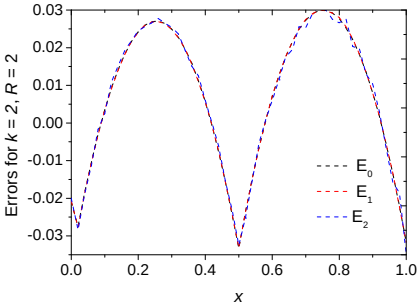

FIGURE 6. compares the errors E_0, E_1 and E_2 for Example 4.2

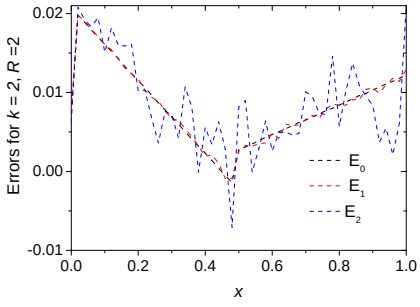

FIGURE 7. compares the errors E_0, E_1 and E_2 for Example 4.3.

compared with those obtained in [16]. And, one can easily observe that a desired

TABLE 2. It measures errors with the help of RMS-norm for $k = 2, R = 2; N = 50$.

	E_0	E_1	E_2
Example 4.1	0.0096	0.0096	0.0104
Example 4.2	0.0203	0.0203	0.0204
Example 4.3	0.0088	0.0087	0.0087

accuracy can be obtained by choosing higher values of k and R. Figure 4 compares the errors measured with RMS-norm. It also shows the comparison among all the three examples by using different basis functions. Figures 5, 6 and 7 compare the random fluctuations E_0, E_1 and E_2 for Example 4.1, Example 4.2 and Example 4.3, respectively. It shows that on introducing a small random noise in the given data, the random fluctuations also remain small. It concludes that the proposed numerical solutions are numerically stable with respect to noisy data. In practical situations, it may not be possible all the times to find the exact data or sometimes it may not be economical to obtain it. Then the numerical stability result is of great importance.

In Table 2, the errors E_0, E_1, E_2 are measured by using RMS-norm for $N = 50$. A particular case of basis functions (i.e. $k = 2, R = 2$), is used and the results are compared for all the three examples. It is observed from Table 2 that the least random fluctuation occurs in Example 4.3. This shows that the problem described in Example 4.3 is more stable than the problems described in other two examples.

5. Conclusions

A numerical method is proposed for solving the most general Fredholm-Volterra fractional integro-differential equation with non-local boundary conditions by using Chebyshev wavelet of second kind. The accuracy of proposed algorithm is illustrated by calculating the parameters like absolute error and RMS-error as in [20]. Figure 2(b) clearly shows that as the basis functions increase, the RMS-errors decrease very fast and a given accuracy can be obtained by choosing the suitable number of basis functions. Table 2 confirms the numerical stability of the proposed algorithm even when given data function is very noisy.

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