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Research article

On New Three- and Two-Dimensional Ratio-Power Copulas

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Abstract: In recent decades, a great variety of dependence models for data analysis have been elaborated. Among them, those based on copulas have demonstrated a great ability to capture the possible dependence between quantitative measures. The three-dimensional case has received particular interest in several important applications in the last few years. The most commonly used three-dimensional copulas have one or two parameters and are exchangeable; they are often members of the well-known Archimedean family. In this paper, we go beyond this standard framework by proposing a brand-new three-dimensional copula that depends on three parameters, is mainly non-exchangeable, and is constructed from ratio and power functions. It is thus of the ratio-power type. Our findings are purely theoretical. In particular, wide ranges of valid parameter values are determined, the main related functions (density, survival, etc.) are exhibited, and various correlation measures (medial, Spearman rho, etc.) are examined. Subsequently, a unique two-dimensional copula, derived from the three-dimensional one, is discussed and studied. When comparing it to the famous two-dimensional Farlie-Gumbel-Morgenstern copula, some noteworthy advantages are emphasized. As a result, the restrictions imposed by the exchangeable property, which are typical of traditional three-dimensional copulas in the literature, are removed, creating a promising future for cutting-edge methods of dependent modeling.

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1. Introduction

Copulas are powerful tools for building flexible dependence models for multivariate random vectors. Given our current ”era of data analytics”, this ability makes them especially attractive. According to the probabilistic viewpoint, copulas are functions that link the parent joint cumulative distribution function and the marginal cumulative distribution functions. Avoidable references on this subject include [1],

[2], [3], [4], and [5]. The general description of an absolutely continuous copula in multiple dimensions is given below (see [3]).

Definition 1. *Let $n \geq 2$ be an integer. The function $C(x_1, \ldots, x_n)$, $(x_1, \ldots, x_n) \in [0, 1]^n$, is said to be an absolutely continuous n-dimensional copula if and only if*

(I1) $C(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = 0$ *for any* $(x_1, \ldots, x_n) \in [0, 1]^n$ *and* $i = 1, \ldots, n$,

(I2) $C(1, \ldots, 1, x, 1, \ldots, 1) = x$ *for any* $x \in [0, 1]$, *and this, in each of the n vector components,*

(I3) $\partial^n C(x_1, \ldots, x_n)/(\partial x_1 \ldots \partial x_n) \geq 0$ *for any* $(x_1, \ldots, x_n) \in [0, 1]^n$.

In the next, the mention of "absolutely continuous" will be removed for convenience. Naturally, a two-dimensional (2D) copula has the dimension $n = 2$ and a three-dimensional (3D) copula has the dimension $n = 3$. Despite the fact that there are more potential 2D models available today, original copula constructions in three dimensions or more are still uncommon in the literature. This is particularly true for the 3D copulas, which have lately sparked renewed interest in modern applications. To support this claim, we could mention the 3D copula-based models employed in [6], [7], [8], [9], and [10], among others. The limitation on the number of parameters is clearly a disadvantage of the most popular 3D copulas. It is particularly true for those of the Archimedean family, which are often encountered in practice. A significant area of research focuses on creating new varieties of 3D copulas with desirable properties for tractability of computations and estimations as well as an adequate number of parameters. The ultimate goal is to use data to fit the asymmetric tail behavior and various dependence structures as well as possible. For these reasons, new 3D copulas with two or three parameters and non-exchangeability properties have been created. On this topic, the major references are [11], [12], [13], [14], and [15]. However, to the best of our knowledge, 3D copulas with three parameters that are non-exchangeable and based on ratio-power functions are rare. In fact, for a number of 2D copulas, the significance of these ratio-power functionalities has been established (see [16] and [17], among others); however, little is known about 3D copula exploration.

This paper hopes to fill some of this gap by proposing an original three-parameter ratio-power 3D copula of the following form:

$$C(x, y, z) = \frac{\phi(x^a, y^b, z^c)}{\psi(x^a, y^b, z^c)}, \quad (x, y, z) \in [0, 1]^3, \tag{1.1}$$

where $\phi(x, y, z)$ and $\psi(x, y, z)$ are two simple 3D polynomial functions, and a, b, and c are the parameters. We investigate it theoretically, concentrating on the acceptable parameter values, the primary related copula functions (density and survival), medial and Spearman rho correlations, and distribution creation. To illustrate the theory, some numerical work is provided. The proposed 3D copula avoids the problems caused by the exchangeable properties typical of standard 3D copulas in the literature, revealing a bright future for cutting-edge dependence modeling methods. Subsequently, a new three-parameter 2D copula is deduced from our findings. It is unique in that it is of the ratio-power type and can be viewed as a new ratio-power modified 2D Farlie-Gumbel-Morgenstern (FGM) copula (see [18]). Moreover, it has desired properties such as flexible copula density, non-exchangeability, different quadrant and tail dependence, significant copula ordering with an extended version of the FGM copula, and manageable correlation behavior. In particular, for some specific values, we prove that the corresponding Kendall tau and Spearman rho correlation ranges of values can be greater than those of the standard FGM copula. The findings are supported by numerical and graphical work. Last but not least, diverse 3D and 2D inequalities are proven and can be used independently in any multivariate analysis setting.

The content of the rest of the article is as follows: The proposed 3D copula is introduced along with its primary features in Section 2. Section 3 is devoted to the derived 2D copula. Section 4 discusses our findings and future plans.

2. Main 3D ratio-power copula

2.1. Presentation

Our key result is the proposition given below, which specifies a brand-new 3D ratio-power copula.

Proposition 2.1. *Let $(a, b, c) \in \mathbb{R}^3$. Let us consider the following 3D function:*

$$C(x, y, z) = 8xyz\frac{(x^a + y^b)(y^b + z^c)(x^a + z^c)}{(x^a + 1)^2(y^b + 1)^2(z^c + 1)^2}, \quad (x, y, z) \in [0, 1]^3. \tag{2.1}$$

Then, for $(a, b, c) \in (0, 1]^3$ or $(a, b, c) \in [-1, 0)^3$, $C(x, y, z)$ is a 3D copula.

Proof. Let us distinguish the following cases: $(a, b, c) \in (0, 1]^3$ and $(a, b, c) \in [-1, 0)^3$,

- The case $(a, b, c) \in (0, 1]^3$ is first considered. To begin, let us notice that $C(x, y, z)$ can be written as

  $$C(x, y, z) = x^{1-a}y^{1-b}z^{1-c}C_*(x^a, y^b, z^c),$$

  where

  $$C_*(x, y, z) = 8xyz\frac{(x + y)(y + z)(x + z)}{(x + 1)^2(y + 1)^2(z + 1)^2}, \quad (x, y, z) \in [0, 1]^3.$$

  Note that $C_*(x, y, z)$ corresponds to $C(x, y, z)$ defined with $a = b = c = 1$. Therefore, according to Theorem 1 (or Example 1) of [11], since $(a, b, c) \in (0, 1]^3$, it is enough to prove that $C_*(x, y, z)$ is a valid copula. Hence, the goal of the proof is to show that $C_*(x, y, z)$ fulfills the items **(I1)**, **(I2)**, and **(I3)** of Definition 1 (considered with $n = 3$).

  (I1) For any $(x, y, z) \in [0, 1]^3$, it is clear that

  $$C_*(x, y, 0) = 8xy \times 0 \times \frac{(x + y)(y + 0)(x + 0)}{(x + 1)^2(y + 1)^2(0 + 1)^2} = 0$$

  and, similarly, we have $C_*(0, y, z) = C_*(x, 0, z) = 0$. As a result, the item **(I1)** is proved.

  (I2) For any $x \in [0, 1]$, we have

  $$\begin{aligned} C_*(x, 1, 1) &= 8x \times 1 \times 1 \times \frac{(x + 1)(1 + 1)(x + 1)}{(x + 1)^2(1 + 1)^2(1 + 1)^2} \\ &= 8x\frac{(x + 1) \times 2 \times (x + 1)}{(x + 1)^2 \times 2^2 \times 2^2} = x. \end{aligned}$$

  Similarly, for any $(y, z) \in [0, 1]^2$, we have $C_*(1, y, 1) = y$ and $C_*(1, 1, z) = z$. The item **(I2)** is proved.

(I3) By the differentiation of $C_*(x,y,z)$ with respect to x, y, and z, we obtain

$$\frac{\partial^3}{\partial x \partial y \partial z} C_*(x,y,z) = 16 \frac{P(x,y,z) + Q(x,y,z)}{(1+x)^3(1+y)^3(1+z)^3},$$

where

$$P(x,y,z) = 3x^2(y+z-2yz) + x^3(y+z-2yz)$$

and

$$Q(x,y,z) = yz[y(3+y)+z(3+z)] + x[y^2(3-6z)+y^3(1-2z)+z^2(3+z)+2yz(1-z)(4+z)].$$

Since $(x,y,z) \in [0,1]^3$, it is clear that $(1+x)^3(1+y)^3(1+z)^3 \geq 0$. Therefore, in order to prove the item **(I3)**, i.e., $\partial^3 C_*(x,y,z)/(\partial x \partial y \partial z) \geq 0$, it is enough to prove that $P(x,y,z) \geq 0$ and $Q(x,y,z) \geq 0$.

We can write $P(x,y,z)$ as

$$P(x,y,z) = 3x^2[y(1-z)+z(1-y)] + x^3[y(1-z)+z(1-y)].$$

Since $(x,y,z) \in [0,1]^3$ (and $1-y \geq 0$, and $1-z \geq 0$), we have $P(x,y,z) \geq 0$.

After several nontrivial developments, simplifications and factorizations, a similar work can be done for $Q(x,y,z)$. Precisely, we have

$$\begin{aligned} Q(x,y,z) &= -2xy^3z + xy^3 - 6xy^2z + 3xy^2 - 2xyz^3 - 6xyz^2 \\ &\quad + 8xyz + xz^3 + 3xz^2 + y^3z + 3y^2z + yz^3 + 3yz^2 \\ &= 3xy^2(1-z) + 3y^2z(1-x) + 3xz^2(1-y) + 3yz^2(1-x) + y^3z(1-x) \\ &\quad + y^3x(1-z) + yz^3(1-x) + xz^3(1-y) + 8xyz. \end{aligned}$$

Since $(x,y,z) \in [0,1]^3$ (and $1-x \geq 0$, $1-y \geq 0$, and $1-z \geq 0$), we have $Q(x,y,z) \geq 0$. Thus, the item **(I3)** is proved.

Finally, it can be said that $C_*(x,y,z)$ and $C(x,y,z)$ are valid 3D copulas because the items **(I1)**, **(I2)**, and **(I3)** of Definition 1 with $n = 3$ are proved.

- For the case $(a,b,c) \in [-1,0)^3$, we can remark that

$$\begin{aligned} C(x,y,z) &= 8xyz \frac{x^{2a}y^{2b}z^{2c}}{x^{2a}y^{2b}z^{2c}} \frac{(y^{-b}+x^{-a})(z^{-c}+y^{-b})(z^{-c}+x^{-a})}{(1+x^{-a})^2(1+y^{-b})^2(1+z^{-c})^2} \\ &= 8xyz \frac{(x^{-a}+y^{-b})(y^{-b}+z^{-c})(x^{-a}+z^{-c})}{(x^{-a}+1)^2(y^{-b}+1)^2(z^{-c}+1)^2}. \end{aligned}$$

We obtain the 3D function in Equation (2.1) but with the parameters $a_* = -a \in (0,1]$, $b_* = -b \in (0,1]$, and $c_* = -c \in (0,1]$. Therefore, the developments elaborated in the case $(a,b,c) \in (0,1]^3$ hold under this configuration; it is enough to replace a by a_*, b by b_* and c by c_*.

This ends the proof. □

Remark 2.2. *In Proposition 2.1, we can extend the definition of $C(x, y, z)$ to the case $a = b = c = 0$ by setting $C(x, y, z) = xyz$, which corresponds to the 3D independence copula, i.e., $\Pi(x, y, z) = xyz$.*

Remark 2.3. *In Proposition 2.1, it is not stated that the condition $(a, b, c) \in (0, 1]^3$ is optimal; wider sets of values are perhaps permitted. In light of the functional intricacy of the 3D function under consideration, this does nonetheless necessitate further research.*

We call the 3D copula in Equation (2.1) as the 3D ratio-power (3DRP) copula. To the best of our knowledge, it is one of the rare 3D copula of this ratio-power form. Its basic characteristics are investigated in the next subsection.

2.2. Basic characteristics

To begin, we would like to mention that all the notions mentioned below can be found in [3], [4], and [5]; we recall them at a minimum only.

For $a = b = c$ and any permutation (x_o, y_o, z_o) of (x, y, z), the 3DRP copula satisfies $C(x_o, y_o, z_o) = C(x, y, z)$. It is thus exchangeable. This is not the case for $a \neq b$ or $a \neq c$ or $b \neq c$. To the best of our knowledge, the 3DRP copula is one of the rare 3D ratio-type copulas with such a non-exchangeable property.

In full generality, the 3DRP copula density and survival 3DRP copula can not be expressed in condensed forms. In the special case $a = b = c = 1$, the corresponding copula density is relatively manageable; it is given by

$$c(x, y, z) = \frac{\partial^3}{\partial x \partial y \partial z} C(x, y, z) = 16 \frac{P(x, y, z) + Q(x, y, z)}{(1 + x)^3 (1 + y)^3 (1 + z)^3}, \quad (x, y, z) \in [0, 1]^3,$$

where

$$P(x, y, z) = 3x^2(y + z - 2yz) + x^3(y + z - 2yz)$$

and

$$Q(x, y, z) = yz[y(3 + y) + z(3 + z)] + x[y^2(3 - 6z) + y^3(1 - 2z) + z^2(3 + z) + 2yz(1 - z)(4 + z)].$$

The 3DRP copula is not Archimedean because it is not associative. Indeed, for example, when $a = b = c = 1$, we have

$$C\left[\frac{1}{4}, C\left(\frac{1}{2}, \frac{1}{3}, 1\right), 1\right] = 0.06828165 \neq 0.06847826 = C\left[C\left(\frac{1}{4}, \frac{1}{2}, 1\right), \frac{1}{3}, 1\right].$$

As an immediate copula fact, the Fréchet-Hoeffding bounds hold, i.e., for any $(x, y, z) \in [0, 1]^3$, we have

$$\max(x + y + z - 2, 0) \leq C(x, y, z) \leq \min(x, y, z),$$

that is,

$$\max(x + y + z - 2, 0) \leq 8xyz \frac{(x^a + y^b)(y^b + z^c)(x^a + z^c)}{(x^a + 1)^2 (y^b + 1)^2 (z^c + 1)^2} \leq \min(x, y, z),$$

or, with lower and upper bounds for $xyz(x^a + y^b)(y^b + z^c)(x^a + z^c)$,

$$\frac{1}{8} \max(x + y + z - 2, 0)(x^a + 1)^2 (y^b + 1)^2 (z^c + 1)^2 \leq xyz(x^a + y^b)(y^b + z^c)(x^a + z^c)$$

$$\leq \frac{1}{8}\min(x,y,z)(x^a+1)^2(y^b+1)^2(z^c+1)^2.$$

These 3D inequalities are significant on their own; in other contexts, they can be used as multivariate analytical tools.

For any $(x, y, z) \in [0, 1)^3$, the geometric series expansion gives

$$C(x,y,z) = 8\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}(i+1)(j+1)(k+1)x^{ai+1}y^{bj+1}z^{ck+1}(x^a+y^b)(y^b+z^c)(x^a+z^c). \quad (2.2)$$

This series expansion can represent or come close to a number of significant correlation measures, making it useful in certain contexts.

Based on the definition given in [3] and [19], the medial correlation (also called the coefficient of Blomqvist) associated to the 3DRP copula is given by

$$\begin{aligned}\mathcal{M} &= \frac{1}{3}\left[8C\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)-1\right]\\ &= \frac{1}{3}\left[8\frac{(2^{-a}+2^{-b})(2^{-b}+2^{-c})(2^{-a}+2^{-c})}{(2^{-a}+1)^2(2^{-b}+1)^2(2^{-c}+1)^2}-1\right].\end{aligned}$$

It can be calculated for fixed values of a, b and c. For instance, if $a = b = 1$, we have $\mathcal{M} = -0.09922268$, if $a = b = c = 1/2$, we have $\mathcal{M} = -0.0285792$, and if $a = 1/2$, $b = 1/3$ and $c = 1/4$, we have $\mathcal{M} = -0.01302735$.

Based on the definition given in [3] and [19], the Spearman rho associated to the 3DRP copula is given by

$$\begin{aligned}\rho &= 8\int_0^1\int_0^1\int_0^1 C(x,y,z)dxdydz - 1\\ &= 64\int_0^1\int_0^1\int_0^1 xyz\frac{(x^a+y^b)(y^b+z^c)(x^a+z^c)}{(x^a+1)^2(y^b+1)^2(z^c+1)^2}dxdydz - 1.\end{aligned}$$

Eventually, based on Equation (2.2) and upon integrations, it can be expanded as

$$\rho = 64\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}(i+1)(j+1)(k+1)\mathcal{I}_{i,j,k} - 1,$$

where

$$\mathcal{I}_{i,j,k} = \int_0^1\int_0^1\int_0^1 x^{ai+1}y^{bj+1}z^{ck+1}(x^a+y^b)(y^b+z^c)(x^a+z^c)dxdydz.$$

This integral term can be expressed for fixed values of a, b and c, with further integral calculus efforts.

In all circumstances, ρ can be calculated for fixed values of a, b and c. For instance, if $a = b = 1$, we have $\rho = -0.1418688$, if $a = b = c = 1/2$, we have $\rho = -0.0430592$, and if $a = 1/2$, $b = 1/3$ and $c = 1/4$, we have $\rho = -0.0223104$.

Based on the 3DRP copula, 3D distributions or models can be generated. Indeed, based on three unidimensional cumulative distribution functions, say $U(x)$, $V(x)$, and $W(x)$, the following 3D function defines an admissible cumulative distribution function:

$$F(x,y,z) = C[U(x), V(y), W(z)]$$

$$= 8U(x)V(y)W(z)\frac{[U^a(x) + V^b(y)][V^b(y) + W^c(z)][U^a(x) + W^c(z)]}{[U^a(x) + 1]^2[V^b(y) + 1]^2[W^c(z) + 1]^2}, \quad (x, y, z) \in \mathbb{R}^3.$$

For motivated choices of $U(x)$, $V(x)$, and $W(x)$ in a lifetime data analysis scenario, we may refer to the survey of [20]. For instance, in light of the work in [21], we can think of a 3D lifetime distribution with the inverse Weibull distribution as a baseline, specified with the following cumulative distribution function: $U(x) = V(x) = W(x) = e^{-\alpha x^{-\beta}}$ for $x > 0$, and $U(x) = V(x) = W(x) = 0$ for $x \leq 0$, where $\alpha > 0$ and $\beta > 0$. The resulting 3D cumulative distribution function is given as

$$F(x, y, z) = 8e^{-\alpha x^{-\beta} - \alpha y^{-\beta} - \alpha z^{-\beta}} \frac{[e^{-a\alpha x^{-\beta}} + e^{-b\alpha y^{-\beta}}][e^{-b\alpha y^{-\beta}} + e^{-c\alpha z^{-\beta}}][e^{-a\alpha x^{-\beta}} + e^{-c\alpha z^{-\beta}}]}{[e^{-a\alpha x^{-\beta}} + 1]^2[e^{-b\alpha y^{-\beta}} + 1]^2[e^{-c\alpha z^{-\beta}} + 1]^2}, \quad (x, y, z) \in (0, \infty)^3,$$

and $F(x, y, z) = 0$ for $(x, y, z) \notin (0, \infty)^3$. Its use in a data analysis scenario has yet to be tested and presents a challenging perspective.

Further findings can be obtained from the 3DRP copula, including the one presented in the next section.

3. On a ratio-modified FGM copula

3.1. Presentation

The following 2D copula is immediately derived from the 3DRP copula:

$$C_{\dagger}(x, y) = C(x, y, 1) = 2xy\frac{x^a + y^b}{(x^a + 1)(y^b + 1)}, \quad (x, y) \in [0, 1]^2.$$

It can also be written as

$$C_{\dagger}(x, y) = xy\left(1 - \frac{(1 - x^a)(1 - y^b)}{(x^a + 1)(y^b + 1)}\right), \quad (x, y) \in [0, 1]^2.$$

In this form, this copula looks like a ratio-power modified version of the generalized FGM copula (see [18]). It is, to the best of our knowledge, new in the literature.

As a result, with the presence of the dependence parameter in the construction of the former FGM copula in mind, we can go further by considering the following extended parameter version:

$$C_{\ddagger}(x, y) = xy\left(1 + \lambda\frac{(1 - x^a)(1 - y^b)}{(x^a + 1)(y^b + 1)}\right), \quad (x, y) \in [0, 1]^2,$$

where λ represents a new tuning parameter. This parameter has the features to (i) modulate the ratio-power term, which can be viewed as a perturbation of the 2D independence copula, i.e., $\Pi(x, y) = xy$, and (ii) broaden the range of acceptable values for a and b. As a result, determining broad ranges of values for a, b, and λ is a mathematical challenge. The following proposition takes up this challenge.

Proposition 3.1. *Let $(a, b, \lambda) \in \mathbb{R}^3$. Let us consider the following 2D function:*

$$C_{\ddagger}(x, y) = xy\left(1 + \lambda\frac{(1 - x^a)(1 - y^b)}{(x^a + 1)(y^b + 1)}\right), \quad (x, y) \in [0, 1]^2. \tag{3.1}$$

Then, for $ab > 0$, and

$$|\lambda| \leq \frac{1}{\max(|a|/2, 1)\max(|b|/2, 1)},$$

$C_{\ddagger}(x, y, z)$ is a 2D copula.

Proof. The goal of the proof is to show that $C_{\ddagger}(x, y)$ fulfills the items **(I1)**, **(I2)**, and **(I3)** of Definition 1 (considered with $n = 2$). Let us distinguish the following cases: [$a > 0$ and $b > 0$], and [$a < 0$ and $b < 0$], derived from the condition $ab > 0$.

- The developments below hold for $a > 0$ and $b > 0$.

  (I1) For any $(x, y) \in [0, 1]^2$, since $b > 0$, it is clear that

  $$C_{\ddagger}(x, 0) = x \times 0 \times \left(1 + \lambda \frac{(1 - x^a)(1 - 0^b)}{(x^a + 1)(0^b + 1)}\right) = 0$$

  and, similarly, since $a > 0$, we have $C_{\ddagger}(0, y) = 0$. As a result, the item **(I1)** is proved.

  (I2) For any $x \in [0, 1]$, we have

  $$C_{\ddagger}(x, 1) = x \times 1 \times \left(1 + \lambda \frac{(1 - x^a)(1 - 1^b)}{(x^a + 1)(1^b + 1)}\right) = x.$$

  Similarly, for any $y \in [0, 1]$, we have $C_{\ddagger}(1, y) = y$. The item **(I2)** is proved.

  (I3) By the differentiation of $C_{\ddagger}(x, y)$ with respect to x and y, we obtain

  $$\frac{\partial^2}{\partial x \partial y} C_{\ddagger}(x, y) = 1 + \lambda f(x; a) f(y; b),$$

  where

  $$f(x; a) = \frac{x^{2a} + 2a x^a - 1}{(x^a + 1)^2}$$

  and $f(y; b)$ is defined similarly but with y instead of x, and b instead of a.
  Let us now study this function. Since $x \in [0, 1]$ and $a > 0$, we have

  $$\frac{\partial}{\partial x} f(x; a) = 2a x^{a-1} \frac{x^a + a(1 - x^a) + 1}{(x^a + 1)^3} \geq 0,$$

  implying that $f(x; a)$ is increasing. Therefore, for any $x \in [0, 1]$, we have $f(x; a) \in [f(0; a), f(1; a)] = [-1, a/2]$. With a similar development for $f(y; b)$, we get $f(y; b) \in [f(0; b), f(1; b)] = [-1, b/2]$. Therefore, we have $|f(x; a)| \leq \max(a/2, 1)$ and $|f(x; b)| \leq \max(b/2, 1)$. These results combined with a basic use of the absolute values and $|\lambda| \leq 1/[\max(a/2, 1)\max(b/2, 1)]$ imply that

  $$\begin{aligned} \frac{\partial^2}{\partial x \partial y} C_{\ddagger}(x, y) &\geq 1 - |\lambda| |f(x; a)| |f(y; b)| \\ &\geq 1 - |\lambda| \max\left(\frac{a}{2}, 1\right) \max\left(\frac{b}{2}, 1\right) \geq 0. \end{aligned}$$

  Hence, the item **(I3)** is proved.

Finally, it can be said that $C_{\ddagger}(x, y)$ is a valid 2D copula because the items **(I1)**, **(I2)**, and **(I3)** of Definition 1 with $n = 2$ are proved.

- For the case where $a < 0$ and $b < 0$, we can remark that

$$C_{\ddagger}(x, y) = xy\left(1 + \lambda \frac{x^a y^b}{x^a y^b} \frac{(x^{-a} - 1)(y^{-b} - 1)}{(1 + x^{-a})(1 + y^{-b})}\right) = xy\left(1 + \lambda \frac{(1 - x^{-a})(1 - y^{-b})}{(x^{-a} + 1)(y^{-b} + 1)}\right).$$

We obtain the 2D function in Equation (3.1) but with the parameters $a_* = -a > 0$, $b_* = -b > 0$, and λ. Therefore, the developments elaborated in the ”positive case” hold under this configuration; it is enough to replace a by a_* and b by b_*.

This ends the proof. □

Remark 3.2. *In Proposition 3.1, we can extend the definition of $C_{\ddagger}(x, y)$ to the case $a = b = 0$ by setting $C_{\ddagger}(x, y) = xy = \Pi(x, y)$.*

Remark 3.3. *The conditions $(a, b) \in (0, 1]^2$ or $(a, b) \in [-1, 0)^2$ that can be derived from Proposition 2.1 are significantly improved in Proposition 3.1. One explanation is the presence of λ.*

We call the 2D copula in Equation (3.1) as the 2D ratio-power (2DRP) copula.

Some immediate remarks are below. For $\lambda = 0$, the 2DRP copula is reduced to the 2D independence copula. For $a = b$, it is exchangeable. This is not the case for $a \neq b$. To the best of our knowledge, the 2DRP copula is one of the rare 2D ratio-type copulas with such a non-exchangeable property.

Figures 1, 2 and 3 show some plots of the 2DRP copula under the following arbitrary but admissible parameter configurations: [$a = b = 2$ and $\lambda = 1$], [$a = 1/2$, $b = 1$ and $\lambda = -1/2$], and [$a = 4$, $b = 1$ and $\lambda = 1/2$], respectively.

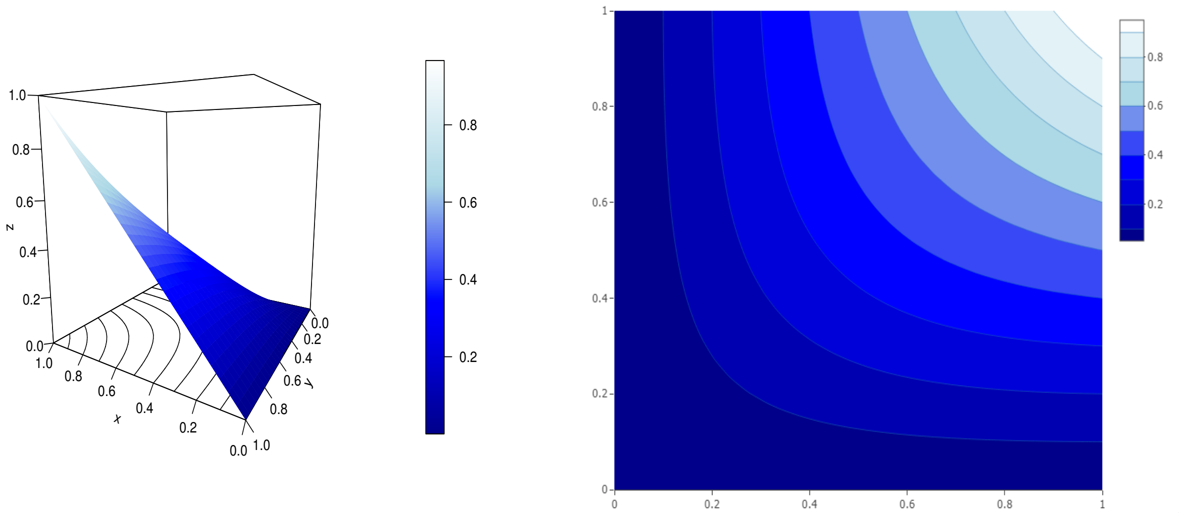

Figure 1. Graphics of the perspective plot (left) and contour plot (right) of the 2DRP copula for $a = b = 2$ and $\lambda = 1$

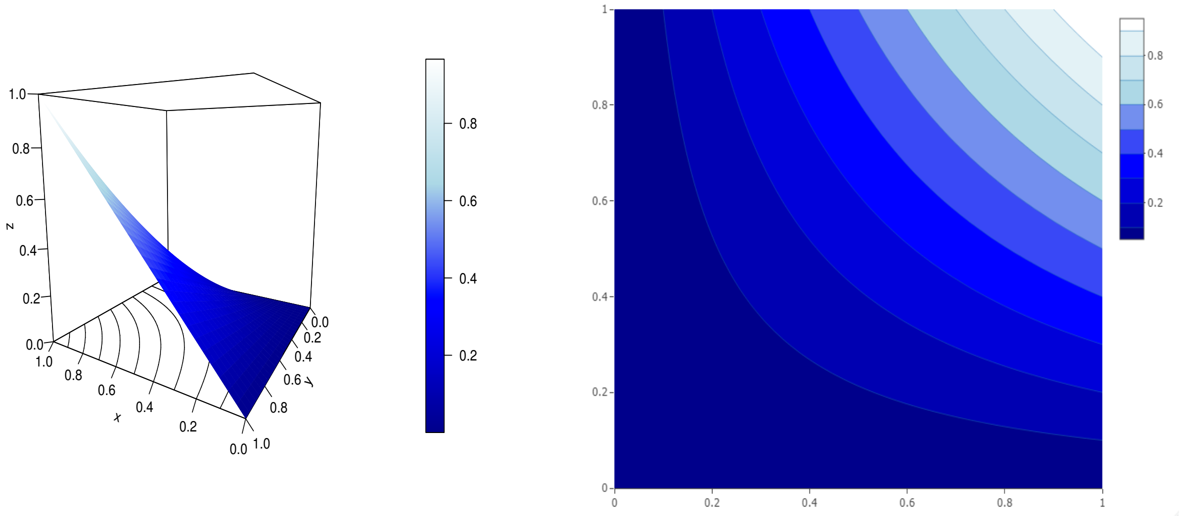

Figure 2. Graphics of the perspective plot (left) and contour plot (right) of the 2DRP copula for $a = 1/2$, $b = 1$ and $\lambda = -1/2$

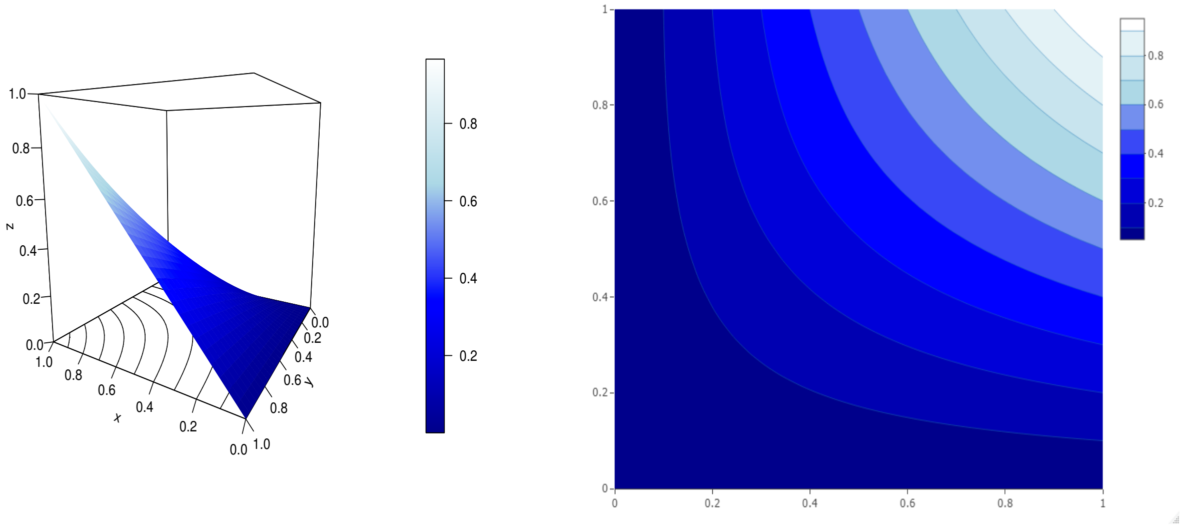

Figure 3. Graphics of the perspective plot (left) and contour plot (right) of the 2DRP copula for $a = 4$, $b = 1$ and $\lambda = 1/2$

Proposition 3.1 is punctually illustrated in these figures; it is clear that the 2DRP copula is valid in the mathematical sense, at least for the selected values of the parameters. Furthermore, versatile shapes can be observed. This versatility is particularly visible in the contour plots.

As another function of interest, the 2DRP copula density is calculated as

$$c_{\ddagger}(x,y) = \frac{\partial^2}{\partial x \partial y} C_{\ddagger}(x,y) = 1 + \lambda \frac{(x^{2a} + 2ax^a - 1)(y^{2b} + 2by^b - 1)}{(x^a + 1)^2 (y^b + 1)^2}, \quad (x,y) \in [0,1]^2.$$

We may examine the shapes of this function to understand the modeling possibilities of the 2DRP copula. To this aim, the 2DRP copula density plots under the previously considered parameter configurations: [$a = b = 2$ and $\lambda = 1$], [$a = 1/2$, $b = 1$ and $\lambda = -1/2$], and [$a = 4$, $b = 1$ and $\lambda = 1/2$], are shown in Figures 4, 5, and 6, respectively.

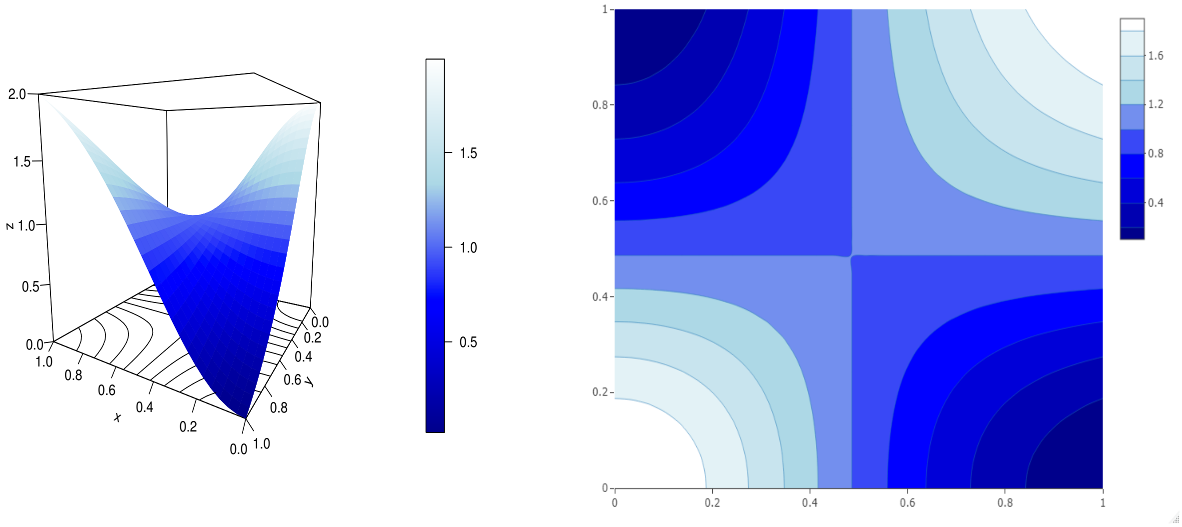

Figure 4. Graphics of the perspective plot (left) and contour plot (right) of the 2DRP copula density for $a = b = 2$ and $\lambda = 1$

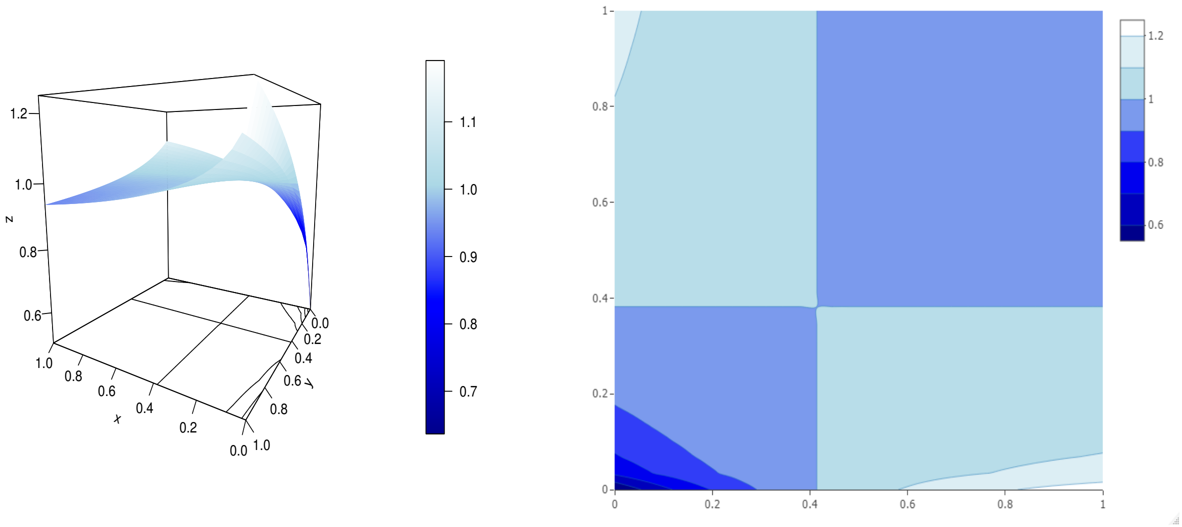

Figure 5. Graphics of the perspective plot (left) and contour plot (right) of the 2DRP copula density for $a = 1/2$, $b = 1$ and $\lambda = -1/2$

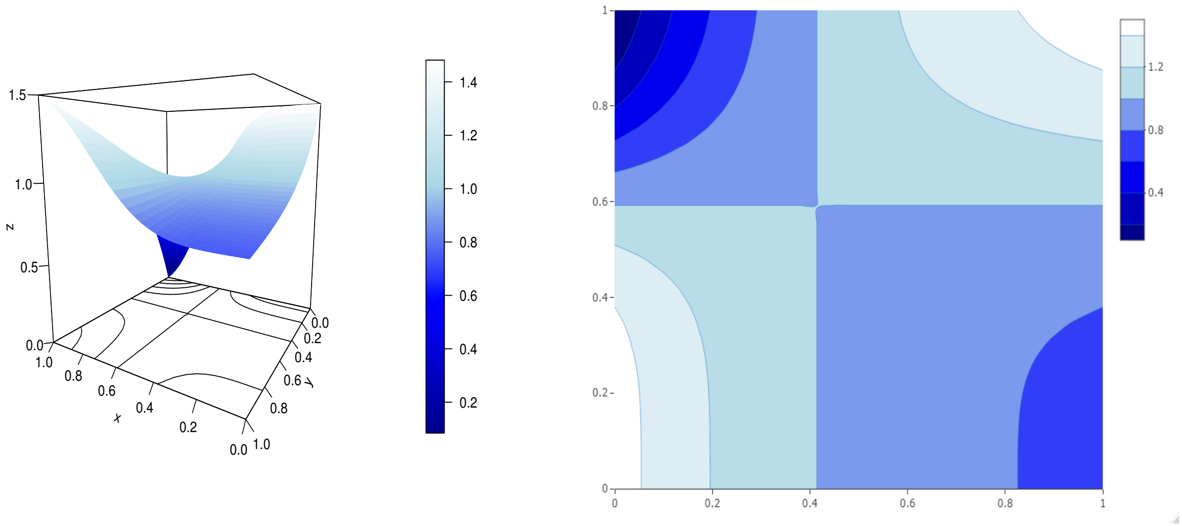

Figure 6. Graphics of the perspective plot (left) and contour plot (right) of the 2DRP copula density for $a = 4$, $b = 1$ and $\lambda = 1/2$

From these figures, the overall shapes of the 2DRP copula density are completely different. The effects of a, b, and λ on these shapes are strong, especially in the region around the extreme points of the unit square. These facts demonstrate a real flexibility of the 2DRP copula in terms of dependence.

To end this part, let us specify the 2DRP survival copula. It is given by

$$\begin{aligned}\hat{C}_{\ddagger}(x,y) &= x + y - 1 + C_{\ddagger}(1-x, 1-y)\\ &= xy + \lambda(1-x)(1-y)\frac{[1-(1-x)^a][1-(1-y)^b]}{[(1-x)^a+1][(1-y)^b+1]}, \quad (x,y) \in [0,1]^2.\end{aligned}$$

This survival copula is a brand-new three-parameter 2D copula to be added to the body of existing literature.

3.2. *Basic characteristics*

This part establishes some basic characteristics of the 2DRP copula in order to understand its modeling possibilities. We recall that all of the information regarding the upcoming concepts is contained in [3], [4], and [5].

First of all, for $a = b$, since $C_{\ddagger}(x,y) = C_{\ddagger}(y,x)$ for any $(x,y) \in [0,1]^2$, the 2DRP copula is diagonally symmetric. It is not the case for $a \neq b$.

As its parent 3DRP copula, the 2DRP copula is not Archimedean because it is not associative. Indeed, for example, when $a = b = \lambda = 1$, we have

$$C_{\ddagger}\left[\frac{1}{4}, C_{\ddagger}\left(\frac{1}{2}, \frac{1}{3}\right)\right] = 0.06828165 \neq 0.06847826 = C_{\ddagger}\left[C_{\ddagger}\left(\frac{1}{4}, \frac{1}{2}\right), \frac{1}{3}\right].$$

For $\lambda \neq 0$, the 2DRP copula is clearly not radially symmetric because there exists (x,y) such that $\hat{C}_{\ddagger}(x,y) \neq C_{\ddagger}(x,y)$. In the case $\lambda = 0$, it is obviously radially symmetric.

Of course, as for any copula, the Fréchet-Hoeffding bounds are satisfied: For any $(x, y) \in [0, 1]^2$, we have $\max(x + y - 1, 0) \leq C_{\ddagger}(x, y) \leq \min(x, y)$, which can also be expressed as

$$\max(x + y - 1, 0) \leq xy\left(1 + \lambda\frac{(1 - x^a)(1 - y^b)}{(x^a + 1)(y^b + 1)}\right) \leq \min(x, y)$$

or, with lower and upper bounds for $\lambda xy(1 - x^a)(1 - y^b)$,

$$\begin{aligned}&[\max(x + y - 1, 0) - xy]\,(x^a + 1)(y^b + 1) \leq \lambda xy(1 - x^a)(1 - y^b)\\&\leq [\min(x, y) - xy]\,(x^a + 1)(y^b + 1).\end{aligned}$$

These inequalities can be of independent interest, for purposes out of the scope of copulas.

For $\lambda \geq 0$, the 2DRP copula is positively quadrant dependent; it is immediate that $C_{\ddagger}(x, y) \geq xy$ for any $(x, y) \in [0, 1]^2$. For $\lambda < 0$, it is negatively quadrant dependent.

In addition, interesting first-order copula orders are satisfied. Because $(x^a+1)(y^b+1) \geq 1$, for $\lambda \geq 0$, we have $C_{\ddagger}(x, y) \leq C_{\star}(x, y)$ for any $(x, y) \in [0, 1]^2$, where

$$C_{\star}(x, y) = xy\left[1 + \lambda(1 - x^a)(1 - y^b)\right], \quad (x, y) \in [0, 1]^2, \tag{3.2}$$

can be viewed as an extended version of the FGM copula (see [18]). For $\lambda < 0$, the reversed inequality holds; we have $C_{\ddagger}(x, y) \geq C_{\star}(x, y)$ for any $(x, y) \in [0, 1]^2$.

For any $(x, y) \in [0, 1)^2$, the geometric series expansion gives

$$C_{\ddagger}(x, y) = xy + \lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}[x^{ai+1}(1 - x^a)][y^{bj+1}(1 - y^b)]. \tag{3.3}$$

This series expansion is helpful in specific circumstances since it may represent or approximate a variety of important correlation measures.

Let us now study the tail dependence of the 2DRP copula. We have

$$\lambda_+ = \lim_{x\to 0}\frac{C_{\ddagger}(x, x)}{x} = \lim_{x\to 0} x\left(1 + \lambda\frac{(1 - x^a)(1 - x^b)}{(x^a + 1)(x^b + 1)}\right) = 0.$$

Furthermore, we have

$$\begin{aligned}\lambda_- &= \lim_{x\to 1}\frac{1 - 2x + C_{\ddagger}(x, x)}{1 - x} = \lim_{x\to 1}\frac{1 - 2x + x^2\left\{1 + \lambda(1 - x^a)(1 - x^b)/[(x^a + 1)(x^b + 1)]\right\}}{1 - x}\\&= \lim_{x\to 1}\left(1 - x + \frac{\lambda ab}{4}(1 - x)\right) = 0.\end{aligned}$$

Since $\lambda_+ = \lambda_- = 0$, we conclude that the 2DRP copula has no tail dependence.

The medial correlation of the 2DRP copula is expressed as

$$\mathcal{M} = 4C_{\ddagger}\left(\frac{1}{2}, \frac{1}{2}\right) - 1 = \lambda\frac{(1 - 2^{-a})(1 - 2^{-b})}{(2^{-a} + 1)(2^{-b} + 1)}.$$

Table 1 determines numerical values of $\mathcal{M}$ under the following configuration: $a = b = 2$ and $\lambda \in \{-1, -0.8, \ldots, 0.8, 1\}$.

Table 1. Numerical values of the medial correlation of the 2DRP copula for $a = b = 2$ and $\lambda \in \{-1, -0.8, \ldots, 0.8, 1\}$.

λ	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
$\mathcal{M}$	-0.36	-0.288	-0.216	-0.144	-0.072	0	0.072	0.144	0.216	0.288	0.36

According to this table, under the special configuration considered, we have $\mathcal{M} \in [-0.36, 0.36]$, which is quite acceptable for this measure.

The Kendall tau of the 2DRP copula is specified as

$$\tau = 4\int_0^1\int_0^1 C_\ddagger(x,y)c_\ddagger(x,y)dxdy - 1$$
$$= 4\int_0^1\int_0^1 \left[xy\left(1+\lambda\frac{(1-x^a)(1-y^b)}{(x^a+1)(y^b+1)}\right)\right]\left[1+\lambda\frac{(x^{2a}+2ax^a-1)(y^{2b}+2by^b-1)}{(x^a+1)^2(y^b+1)^2}\right]dxdy - 1.$$

Unlike the medial correlation, this measure does not have a straightforward expression. In Table 2, numerical values of τ under the following configuration: $a = b = 2$ and $\lambda \in \{-1, -0.8, \ldots, 0.8, 1\}$, are presented.

Table 2. Numerical values of the Kendall tau of the 2DRP copula for $a = b = 2$ and $\lambda \in \{-1, -0.8, \ldots, 0.8, 1\}$.

λ	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
τ	-0.2984	-0.2388	-0.1791	-0.1194	-0.0597	0	0.0597	0.1194	0.1791	0.2388	0.2984

According to this table, under the special configuration considered, we have $\tau \in [-0.2984, 0.2984]$. This set of values is larger, in particular, than the τ values for the classical FGM copula, which satisfy $\tau \in [-0.2222, 0.2222]$ in a similar setting. In this sense, the 2DRP copula is competitive for $a = b = 2$ (among others).

The Spearman rho of the 2DRP copula is specified as

$$\rho = 12\int_0^1\int_0^1 C_\ddagger(x,y)dxdy - 3 = 12\lambda\int_0^1\int_0^1 xy\frac{(1-x^a)(1-y^b)}{(x^a+1)(y^b+1)}dxdy.$$

This measure does not have a straightforward expression, but Equation (3.3) can be used to get the following expansion:

$$\rho = 12\lambda\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\left[\int_0^1 x^{aj+1}(1-x^a)dx\right]\left[\int_0^1 y^{bj+1}(1-y^b)dy\right]$$

$$= 12\lambda ab \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}}{(ai+2)[a(i+1)+2](bj+2)[b(j+1)+2]}.$$

Table 3 determines numerical values of ρ under the following configuration: $a = b = 2$ and $\lambda \in \{-1, -0.8, \ldots, 0.8, 1\}$.

Table 3. Numerical values of the Spearman rho of the 2DRP copula for $a = b = 2$ and $\lambda \in \{-1, -0.8, \ldots, 0.8, 1\}$.

λ	-1.0	-0.8	-0.6	-0.4	-0.2	0.0	0.2	0.4	0.6	0.8	1.0
ρ	-0.4477	-0.3581	-0.2686	-0.1791	-0.0895	0	0.0895	0.1791	0.2686	0.3581	0.4477

According to this table, under the special configuration considered, we have $\rho \in [-0.4477, 0.4477]$. Once again, this set of values is larger than the ρ values for the classical FGM copula, which satisfy $\rho \in [-0.333, 0.3333]$ in a comparable setting. In this sense, the 2DRP copula, configured with $a = b = 2$ but not exclusively, can be considered an acceptable alternative.

Like with all other 2D copulas, the 2DRP copula can define new parametric distributional models. In fact, we can create a new 2D cumulative distribution function by combining two uni-dimensional cumulative distribution functions, say $U(x)$ and $V(x)$, as follows:

$$F(x, y) = C_{\ddagger}(U(x), V(y)) = U(x)V(y)\left\{1 + \lambda \frac{[1 - U^a(x)][1 - V^b(y)]}{[U^a(x) + 1][V^b(y) + 1]}\right\}, \quad (x, y) \in \mathbb{R}^2.$$

This result can be used in a number of scenarios for data analysis. For instance, in light of the 2D distributional work in [21], we can consider a 2D lifetime distribution using the inverse Weibull distribution specified with the following cumulative distribution function: $U(x) = V(x) = e^{-\alpha x^{-\beta}}$ for $x > 0$, and $U(x) = V(x) = 0$ for $x \leq 0$, where $\alpha > 0$ and $\beta > 0$. The resulting five-parameter cumulative distribution function is given as

$$F(x, y) = e^{-\alpha x^{-\beta} - \alpha y^{-\beta}}\left\{1 + \lambda \frac{[1 - e^{-a\alpha x^{-\beta}}][1 - e^{-b\alpha y^{-\beta}}]}{[e^{-a\alpha x^{-\beta}} + 1][e^{-b\alpha y^{-\beta}} + 1]}\right\}, \quad (x, y) \in (0, \infty)^2,$$

and $F(x, y) = 0$ for $(x, y) \notin (0, \infty)^2$. This distribution is an interesting competitor to the absolutely continuous bivariate inverse Weibull (ACBIW) distribution as introduced in [21], among others. Its use in a data analysis scenario has yet to be tested, which we will leave to future investigations.

4. Conclusion

The paper can be divided into two parts. In the first part, a brand-new 3D copula was introduced. It stands out from the other 3D copulas because of the following facts: (i) it is only defined using ratio and power functions, and depends on three tuning parameters; (ii) broad ranges for the acceptable values of the parameters are identified; and (iii) it is primarily not exchangeable, which, in addition to its simplicity, continues to be a desired property in the applications. In the second part, we use this 3D copula to derive a new 2D copula with similar characteristics. In particular, it can be viewed as a modified version of the FGM copula with some enhanced properties in terms of correlation. The findings were illustrated numerically and graphically, when possible. For other multidimensional analysis uses, a number of 2D and 3D inequalities were demonstrated. This work has many perspectives, including the creation of brand-new, straightforward 3D dependence models for the analysis of actual 3D data, the expansion of the proposed copula to the fourth dimension, and the incorporation of the proposed copulas into a "free to share" R package.

Conflict of interest

The authors state that they have no financial or other conflicts of interest to disclose with connection to this research.

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