Modern Robust M-Estimation of the Gamma Distribution with Extreme Observations

By

Dr. Mohamed Abdelsalam Agamy

Assistant Prof. of statistic, Faculty of Commerce, Al-Azhar University

(Boys' Branch), Egypt.

dr.muhammad_alagamy@azhar.edu.eg

Dr. Ezzat Hussein Khmar

Assistant Prof. of statistic, Faculty of Commerce, Al-Azhar University

(Boys' Branch), Egypt.

Ezzat4228@gmail.com

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Modern Robust M-Estimation of the Gamma Distribution with Extreme Observations

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Abstract

In this paper, three objective functions of M-estimates (Huber, Hampel, and Bisquare) are used in order to obtain robust estimates of the gamma distribution parameters, and then compare its estimates with estimates provided by the common conventional methods (moment estimators, maximum-likelihood estimators and method of maximum product spacings), to determine the most appropriate methods for estimating the parameters of gamma distribution, by applying the previous methods to real data as well as to generated data, both of which contain different percentages of outliers. Monte Carlo simulation was used to perform the numerical comparison. The simulation results confirmed that the M-estimates give more accurate and higher efficiency estimations when estimating the gamma distribution parameter. It was concluded that the most suitable method for estimating the gamma distribution parameters is the M- estimation method with its three objective functions (Huber, Hampel, and Bisquare), for small and large samples especially in the presence of outliers, and its estimates, in this case, are characterized by greater accuracy and higher efficiency. It was also concluded that the best estimate of robust M-estimation, in this case, is Bisquare method.

Keywords: Classical Methods, M-estimations, Objective Functions, Outliers, Root Mean Squared Error, Simulation Study, Two-Parameter Gamma Distribution.

1. Introduction

Gamma distribution is one of the most important statistical distributions, as it has useful uses in many fields of real life. For example, as mentioned by Wackerly et al. (2008), that modeling the lags between failures using a gamma distribution is very useful in many devices, such as aircraft engines, and is also useful in some places where everyday transactions such as grocery stores or banks are conducted by modeling the lags between failures in exit lines, etc.

According to Johnson et al. (1995). Laplace introduced the gamma distribution as a constant precision distribution. Wait times are perhaps the most important area of its use, as it is used to model it. Also, one of its areas of use is the life test, where the waiting time to death is a random variable that is distributed probabilistically according to the gamma distribution. In addition, Bayesian statistics is one area of use for the gamma distribution, as it is used with several scale parameters such as parameter θ, provided the mean is known, as a prior conjugated distribution if the variable follows an exponential or a normal distribution. It is also used in many other applications such as sizing of insurance claims see Boland (2007). It has also been used in hydrology research, see Aksoy (2000), as well as in bacterial gene expression research, see Friedman et al. (2006).

There are well-known conventional estimation methods used to estimate the parameters of the gamma distribution, such as (maximum likelihood estimation, moment estimation, maximum product of distances, and approximate maximum likelihood estimation (AMLE), which are commonly used for this purpose, but they are not suitable for estimation using small samples and affected by the presence of outliers. However, it is still widely used. See: Al-Harbi and Kamel (2022). Moment estimations are attributed to Pearson in 1894 and are one of the most widely used estimation methods. The main advantage of this method is its simplicity, which makes it used so far, but it may provide estimators with low efficiency even in the case of large samples, see Kamel and Alqarni (2022). The most well-known, important, and widely used estimation method is the maximum likelihood method, and it often gives estimates with good characteristics, especially in the case of large samples provided that all observations follow the model. This method is attributed to Sir Ronald Fisher, who was the first to point out it and its advantages. However, the estimates generated by the maximum likelihood method cannot be considered robust estimates because it's too sensitive to the presence of outliers in the sample, and its estimations efficiency decrease in the case of small samples see Agostinelli, et al. (2014). Accordingly, there was a need for easy and highly efficient alternative estimation methods, especially when the data contains some outliers, which are known as robust estimation methods.

This paper aims to provide robust M-estimation using the three objective functions (Huber, Hampel, and Bisquare) as an alternative to the common traditional estimation methods and compare them to achieve the most appropriate and efficient estimation methods for estimating the gamma distribution parameters, especially if the data contains outliers and in the case of small samples.

The organization of this paper was as follows: Section 2 is a review of the literature, Section 3 discusses the gamma distribution, Section 4 presents a review of the most common traditional methods for estimating the parameters of the gamma distribution, and Section 5 is dedicated to reviewing the use of the robust M estimate. While in Section 6 we study the effectiveness of all methods for estimating the parameters of the gamma distribution using a Monte Carlo simulation exercise. Section 7 reviews the summary and conclusions.

2. Literature review

Agustinelli et al. (2014) discussed the robust estimates of families of location-scale distributions are proposed, particularly for the generalized log-gamma distribution. For the first time, it took into account distribution models with an additional shape parameter, and in particular, for the generalized log-gamma model - new robust procedures were developed. Kantar and Yildirim (2015) provided estimates of the parameters of the extended Burr Type III distribution by applying them using different robust estimation methods to a set of data containing outliers. Musa (2017) used the M method as a powerful method to estimate the parameters of the expanded Marshal-Olkin burr III distribution by applying it to a set of data with outliers.

On the other hand, Almongy and Almetwally (2020) introduced the robust estimation to estimate the shape and scale parameters of a generalized exponential distribution (GE) by applying it to a complete data set containing outliers with different proportions. Some traditional methods such as maximum likelihood estimation (MLE), least squared (LS) and maximum product spacings (MPS) were used, and the application was carried out using the same data referred to. In order to find the most appropriate method to estimate the parameters of the GE distribution, the traditional estimators were compared with one of the non-robust estimation methods such as Least Absolute Deviations (LAD) and were also compared with some robust estimators such as M-estimation (using M-Huber weight (MH) and M. Bisquare-weight (MB)). The Monte Carlo simulation method was used to make the comparison numerically. The study concluded that the method of estimating M using the Huber object function is the most appropriate method, especially if the data contained different proportions of outliers.

Orabi and Zaidan (2021) estimated the parameter of the Marshall-Olkin Exponential (MOLELE) extended linear distribution is discussed in case the data contains outliers. Where the parameters were estimated using three methods, namely M-estimation, maximum likelihood methods, percentage methods, and testing the performance of estimation methods in case the data contained outliers, a simulation study was conducted. The research concluded that the method of M-estimation is the appropriate method of estimation compared to other methods, especially if the data contained extreme values. A real data set was also used to test these methods, which confirmed the previously reached result. Moreover, Kamel et al. (2022) investigated the distribution of the first-kind Gumbel extreme value (GEV) using Bayesian estimates of the parameters in the presence of outliers.

3. Two-Parameter Gamma Distribution

The gamma distribution has been extensively used in the areas of reliability, life testing, insurance, meteorology, climatology, and many other physical situations. If a random variable X has a two-parameter gamma distribution, its probability density function (PDF) is of the form;

$$f(x \mid \alpha, \beta) = \{\Gamma(\alpha)\beta^{\alpha}\}^{-1} x^{\alpha-1} e^{-x/\beta},\ x > 0, \alpha, \beta > 0, \qquad (1)$$

where α is a shape parameter and β is a scale parameter and $\Gamma(\alpha)$ represents gamma function, for all positive integers, $\Gamma(\alpha) = (\alpha - 1)!$.The gamma distribution has a single mode at $x = \beta(\alpha - 1)$ if $\alpha \geq 1$. If $\alpha < 1$, the density tends to infinity as x tends to zero. If $\alpha = 1$, the density, called the standard exponential distribution, tends to one as x tends to zero. The cumulative distribution function (CDF) on the support of X is;

$$f(x \mid \alpha, \beta) = P(X \leq x) = \frac{\Gamma\left(\alpha, \frac{x}{\beta}\right)}{\Gamma(\alpha)} \quad ; x > 0, \qquad (2)$$

where;

$$\Gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$$

for $s > 0$ and $x > 0$ is the incomplete gamma function and;

$$\boldsymbol{\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt}$$

for $\mathbf{s > 0}$ is the gamma function. The survivor function on the support of $\boldsymbol{X}$ is

$$\boldsymbol{S(x; \alpha, \beta) = P(X \geq x) = 1 - \frac{\Gamma(\alpha, x/\beta)}{\Gamma(\alpha)}\ x > 0.} \qquad (3)$$

The hazard function on the support of $\boldsymbol{X}$ is;

$$\boldsymbol{h(x;\alpha,\beta)} = \frac{f(x)}{S(x)} = \frac{x^{\alpha-1}e^{-x/\beta}}{\left(\Gamma(\alpha)-\Gamma\left(\alpha,\frac{x}{\beta}\right)\right)\alpha\beta\Gamma(\alpha)}\ \boldsymbol{x>0}. \tag{4}$$

The cumulative hazard function on the support of **X** is;

$$\boldsymbol{H(x;\alpha,\beta) = -ln\ S(x) = -ln\ \left(1-\frac{\Gamma(\alpha,x/\beta)}{\Gamma(\alpha)}\right)\ x>0}. \tag{5}$$

There is no closed-form expression for the inverse distribution function. The moment generating function of $\boldsymbol{X}$ is;

$$\boldsymbol{M(t) = E[e^{tX}] = (1-\beta t)^{-\alpha}\ t < \frac{1}{\beta}}.$$

The characteristic function of **X** is;

$$\boldsymbol{\Phi(t) = E[e^{itX}] = (1-\beta it)^{-\alpha}\ t < \frac{1}{\beta}}.$$

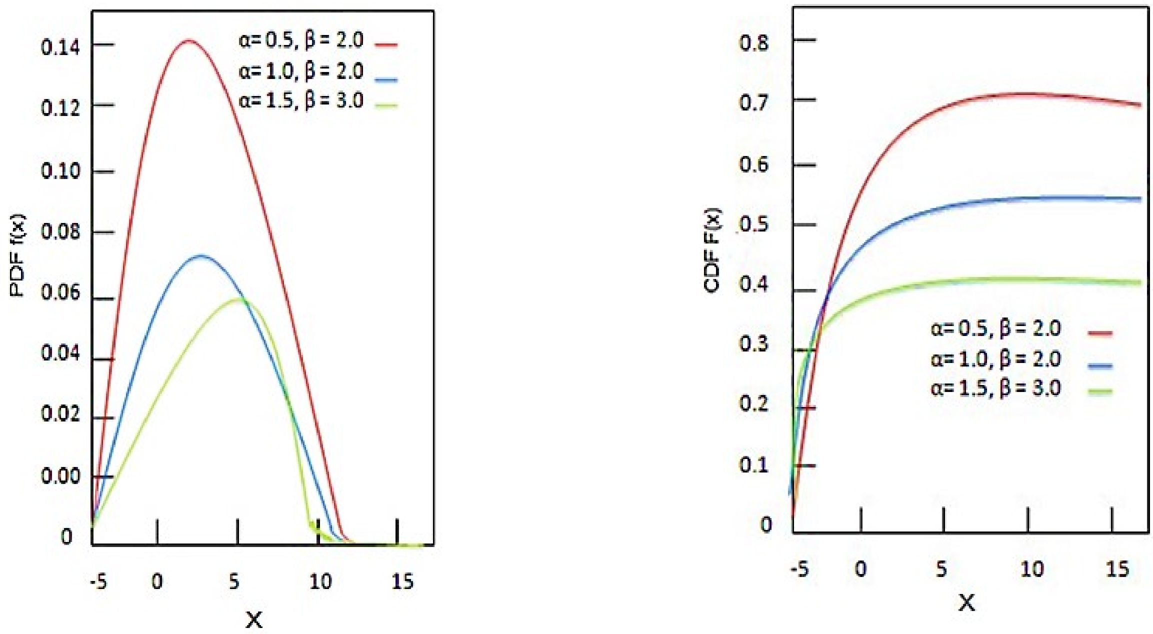

Figure 1: PDF and CDF of the two-parameter Gamma distribution.

4. The Classical Methods of Estimation

In this section, the estimate of the Gamma parameters $(\boldsymbol{\alpha},\boldsymbol{\beta})$ using classical methods is examined using three estimation techniques: method of moment (MOM) estimation, maximum likelihood estimation (MLE) and maximum product spacings (MPS) estimation.

4.1 MOM Estimation Method

The MOM estimator is derived by equating the population moment to the sample moments for the unknown parameters of a $\boldsymbol{G(\alpha,\beta)}$ distribution. Let $\boldsymbol{X_1,\dots,X_n \overset{IID}{\sim} G\,(\alpha,\beta)}$. *If* $\boldsymbol{X \sim G\,(\alpha,\beta)}$, *t*hen $\mathbb{E}[\boldsymbol{X}] = \frac{\alpha}{\beta}$ and $\mathbb{E}[\boldsymbol{X^2}] = \frac{\alpha+\alpha^2}{\beta^2}$. So the MOM estimators $\boldsymbol{\hat{\alpha},\hat{\beta}}$ solve the equations;

$$\hat{\mu}_1 = \frac{\hat{\alpha}}{\hat{\beta}},$$

$$\hat{\mu}_2 = \frac{\hat{\alpha}+\hat{\alpha}^2}{\hat{\beta}^2}.$$

The first equation is changed into the second equation;

$$\hat{\mu}_2 = \left(\frac{1}{\hat{\alpha}}+1\right)\hat{\mu}_1^2,$$

Then we have;

$$\hat{\alpha}_{MOM} = \frac{1}{\frac{\hat{\mu}_2}{\hat{\mu}_1^2}-1} = \frac{\hat{\mu}_1^2}{\hat{\mu}_2-\hat{\mu}_1^2} = \frac{\bar{X}^2}{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}. \tag{6}$$

The result of the first equation is then;

$$\hat{\beta}_{MOM} = \frac{\hat{\alpha}}{\hat{\mu}_1} = \frac{\bar{X}}{\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2}. \tag{7}$$

Based on the description of a $\boldsymbol{G(\alpha,\beta)}$ distribution as; Hwang and Huang (2002) developed a bias correction of the MOM estimator (denoted MOMB) for small samples.

$$\hat{\alpha}_{MOMB} = \frac{1}{V_X^2}-\frac{1}{n}, \text{ and } \hat{\beta}_{MOMB} = \frac{\bar{X}}{\left(\frac{1}{V_X^2}-\frac{1}{n}\right)}, \tag{8}$$

where $V_X^2 = \frac{S_X^2}{\bar{X}^2}$, $S_X^2 = \frac{\sum_{l=i}^{n}(X_i-\bar{X})^2}{n-1}$.

Unfortunately, the MOM estimators depend on the chosen sample moment and are not unique, see Kamel and Alqarni (2020).

4.2 MLE Estimation Method

Let $(\mathbf{x_1},\dots,\mathbf{x_n})$ be a random sample of size $\mathbf{n}$ from a $\mathbf{G(\alpha,\beta)}$ distribution.

Then the likelihood function is given by:

$$L(\alpha,\beta \mid X) = \{\Gamma(\alpha)\beta^{\alpha}\}^{-n}\prod_{i=1}^{n} X_i^{\alpha-1} exp\,(-\sum_{i=1}^{n} X_i/\beta)$$

$$= n(\alpha-1)\, ln\, x - n\, ln\, \Gamma(\alpha) - n\alpha\, ln\ \beta - n\bar{x}/\beta \qquad (9)$$

By considering the partial derivatives of the log-likelihood function with respect to α and β , respectively, the MLE estimates for the gamma parameters (α,β) are obtained, see Figure 2 for the surface of log-likelihood. The MLE estimators of α and β are then derived by equating the resulting expressions to zero in the manner described below:

$$\frac{\partial}{\partial\alpha}\ell(\alpha,\beta) = -n ln\ \beta - \frac{n}{\Gamma(\alpha)}\left(\frac{\partial}{\partial\alpha}\Gamma(\alpha)\right) + \sum_{i=1}^{n}\ ln\,(x_i) \qquad (10)$$

$$\frac{\partial}{\partial\beta}\ell(\alpha,\beta) = -\frac{n\alpha}{\beta} + \frac{\sum_{i=1}^{n} x_i}{\beta^2}. \qquad (11)$$

Iterative methods are required to create an approximation since Equation (10), in particular, is a non-linear equation without a closed-form solution. Many authors have suggested using these approaches to obtain MLE estimators, such as the Newton-Raphson algorithm. Son and Oh (2006) demonstrated that, in comparison to other approximation methods, the Greenwood and Durand (1960) approximate MLE estimators for $G\ (\alpha,\beta)$ are quite effective. The Greenwood and Durand approximation MLE estimators are produced by if $\vartheta = ln\ \bar{X} - \overline{ln\ X}$ can be stated as;

$$\hat{\alpha}_{MLE} = \begin{cases} \dfrac{0.5000876 + 0.1648852\,\vartheta - 0.0544274\,\vartheta^2}{\vartheta}, & 0 < \vartheta \le 0.5772 \\ \dfrac{8.898919 + 9.05990\,\vartheta - 0.9775373\,\vartheta^2}{\vartheta(17.79728 + 11.968477\,\vartheta + \vartheta^2)}, & 0.5772 < \vartheta < 17, \end{cases}$$

and; $\hat{\beta}_{\mathrm{MLE}} = \dfrac{\bar{X}}{\hat{\alpha}_{\mathrm{MLE}}}$

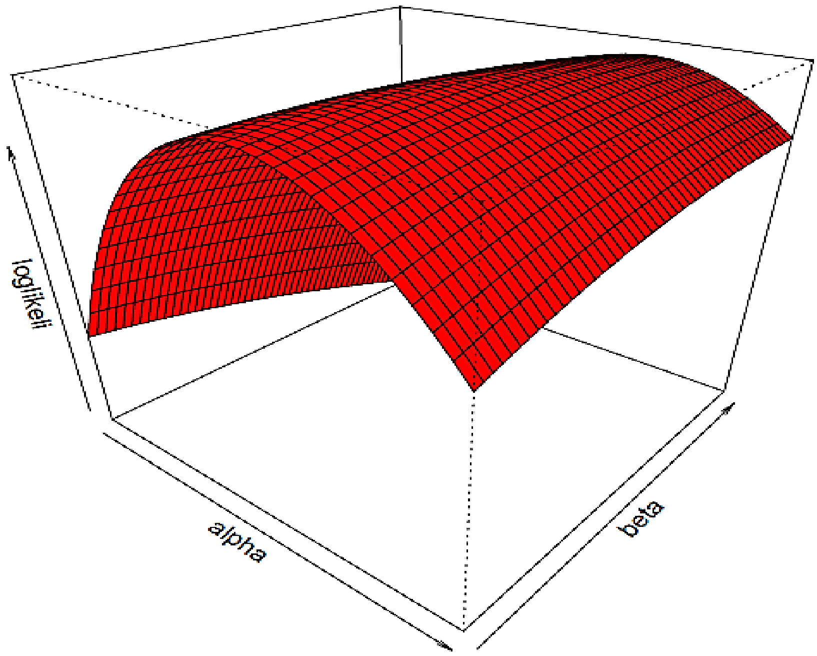

Figure 2: The surface of log-likelihood, the domain is $\mathbf{0.14 \le \alpha \le 0.24}$ and $\mathbf{5 \le \beta \le 7}$.

4.3 MPS Estimation Methods

In parallel, Cheng and Amin (1983) and Ranneby (1984) proposed the method of maximum product spacings (MPS). Let;

$$\boldsymbol{D_i = \int_{x_{i-1:n}}^{x_{i:n}} f(x;\delta)dx, i = 1, 2, \cdots, n+1,} \tag{12}$$

where $\boldsymbol{x_{0:n}}$ is the lower limit and $\boldsymbol{x_{n+1:n}}$ is the upper limit of the domain of the density function $\boldsymbol{f(x;\delta)}$, and $\boldsymbol{\delta}$ can be vector-valued. Also, $\boldsymbol{X_{1:n}, X_{2:n}, \cdots, X_{n:n}}$ are defined as an ordered random sample from $\boldsymbol{f(x;\delta)}$. Clearly, the spacings sum to unity, that is $\boldsymbol{\sum D_i = 1}$. The MPS estimation is, quite simply, to choose $\boldsymbol{\delta}$ to maximize the geometric mean of the spacings,

$$\boldsymbol{G = \left(\prod_{i=1}^{n+1} D_i\right)^{\frac{1}{n+1}}}$$

or, equivalently, its logarithm

$$\boldsymbol{H = ln\ G}$$

Using the PDF in Equation (1) and the CDF in Equation (2), $\boldsymbol{H}$ can be written as follows:

$$H = \frac{1}{n+1}\left[ln\ F(X_{1:n};\alpha,\beta) + ln\{1 - F(X_{n:n};\alpha,\beta)\}\right] + \frac{1}{n+1}\left[\sum_{i=1}^{n-1} ln\{F(X_{i+1:n};\alpha,\beta) - F(X_{i:n};\alpha,\beta)\}\right] \quad (13)$$

By maximizing Equation (4) for different values of α and β, the MPS estimates can be obtained as $\hat{\alpha}_{MPS}$ and $\hat{\beta}_{MPS}$. The Newton-Raphson method can be used in solving when the two first derivatives are equal to zero. Moreover, the multivariate Newton-Raphson iteration is performed as;

$$\begin{bmatrix} \hat{\beta}_P^{(l+1)} \\ \hat{\alpha}_P^{(l+1)} \end{bmatrix} = \begin{bmatrix} \hat{\beta}_P^{(l)} \\ \hat{\alpha}_P^{(l)} \end{bmatrix} - \begin{bmatrix} H''^{(l)}_{\beta\beta} & H''^{(l)}_{\beta\alpha} \\ H''^{(l)}_{\beta\alpha} & H''^{(l)}_{\alpha\alpha} \end{bmatrix}^{-1} \begin{bmatrix} H'^{(l)}_{\beta} \\ H'^{(l)}_{\alpha} \end{bmatrix},$$

where l is the index for the iterations, for more details about first and second derivatives of H see Rahman et al. (2007).

5. Robust M-Estimation Methods

We are interested in estimating the parameters of a Gamma model when the data contains outliers. Numerous reliable methods to estimate the parameters of mathematical distributions have been put forth in the literature in an effort to lessen the impact of outliers, see Abonazel and Rabie (2019). A strong framework with a wide range of applications is the M-estimation approach. In order to lessen the impact of outliers, we expand the M-estimation method in this study to estimate the Gamma distribution's parameters. The following is where to find the M-estimator; for more details about robust methods in several models, see Youssef et al. (2021, 2022)

$$\varphi(\alpha,\beta) = arg\,min \sum_{i=1}^{n} \rho(x,\alpha,\beta),$$

The objective function ρ is defined as;

$$\rho(x,\alpha,\beta) = -\log\ f(x,\alpha,\beta)$$
$$\psi(x,\varphi) = \frac{\partial\rho(x,\varphi)}{\partial\varphi}$$

where, $\varphi: \alpha, \beta$. Then the M-estimator satisfies $\sum_{i=1}^{n} \psi(x,\varphi) = 0$.

Now, minimize the objective function (ρ) with regard to the parameters and for all invariant errors (ϵ_i) we will use Huber, Hampel's and Tukey's Bisquare weight functions in this rigorous statistical study, see Kamel (2021).

- **Huber objective function is;**

$$\rho(\epsilon_i) = \begin{cases} \frac{1}{2}\epsilon_i^2 & |\epsilon_i| \le c \\ c|\epsilon_i| - \frac{1}{2}a^2 & |\epsilon_i| > c \end{cases}$$

with derivative is;

$$\rho'(\epsilon_i) = \begin{cases} \epsilon_i & |\epsilon_i| \le c \\ c \text{ sign } \epsilon_i & |\epsilon_i| > c \end{cases}$$

where $a = 1.345$, is a measurement's tuning constant affects whether it is an outlier or not.

- **Hampel objective function is;**

$$\rho(\epsilon_i) = \begin{cases} \frac{1}{2}{\epsilon_i}^2 & \text{for } |\epsilon_i| < a \\ a|\epsilon_i| - \frac{1}{2}a^2 & \text{for } a \le |\epsilon_i| \le b \\ a\frac{c|\epsilon_i| - \frac{1}{2}{\epsilon_i}^2}{r-b} - \frac{7c^2}{6} & \text{for } b \le |\epsilon_i| \le r \end{cases}$$

with derivative is;

$$\rho'(\epsilon_i) = \begin{cases} \epsilon_i & \text{for } |\epsilon_i| < a \\ a \text{ sign } \epsilon_i & \text{for } a \le |\epsilon_i| \le b \\ a\frac{\text{sign } \epsilon_i - \epsilon_i}{r-b} & \text{for } b \le |\epsilon_i| \le r \end{cases}$$

where $a = 1.353, b = 3.157, r = 7.216$ give 95% efficiency at the normal.

- **Bisquare objective function is;**

$$\rho(\epsilon_i) = \begin{cases} 1 - \left(1 - \left(\frac{\epsilon_i}{c}\right)^2\right)^3 & |\epsilon_i| \le c \\ 1 & |\epsilon_i| > c \end{cases}$$

with derivative is;

$$\rho'(\epsilon_i) = \begin{cases} \frac{6\epsilon_i}{c^2}\left(1 - \left(\frac{\epsilon_i}{c}\right)^2\right)^2 & |\epsilon_i| \le c \\ 0 & |\epsilon_i| > c \end{cases}$$

where $c = 4.685$, is a measurement's tuning constant affects whether it is an outlier or not. Since ρ is differentiable, M-estimates can be obtained for the two selected objective function by minimize $\sum_{i=1}^{n}\rho(\epsilon_i)$ with respect to α, β and equating to zero as following:

$$\psi(x,\alpha) = -n\ln\beta - \frac{n}{\Gamma(\alpha)}\left(\frac{\partial}{\partial\alpha}\Gamma(\alpha)\right) + \sum_{i=1}^{n} w_i \ln(x_i)$$

$$\psi(x,\beta) = \frac{n\alpha}{\beta} + \frac{\sum_{i=1}^{n} w_i x_i}{\beta^2}$$

where, $\mathbf{w_i}$ is the weighted and can be defined as follows:

$$w_i = m\left\{1, \frac{\psi(x,\varphi)}{f(x,\varphi)}\right\}.$$

Numerical method is used because we can't find an explicit solution to compute the estimate of these parameters. Up until the convergence condition is met, iteration continues, see Almetwally and Almongy (2019).

6. Simulation Study

To evaluate the performance of well-known classical methods for estimating MOM, MLE, MPS, and a robust M estimator, a Monte Carlo simulation approach was used. The gamma distribution is used to generate complete data with outliers with the specified values for the parameters α and . Here are the sample creation instructions:
Step (1): We generate random samples $x_1, \ldots, x_n$ of sizes $n = 25, 70, 150, 200$ and 300 from the Gamma distribution. The parameters have taken values $(\alpha,\beta) = (0.8, 0.3)$, $(1, 1.5)$, and $(2.5, 3)$.

Step (2): For each random sample, the outliers are generated from the uniform distribution as $U(\bar{x} + 4s, \bar{x} + 7s)$, where $\bar{x}$ is the sample mean and s is the standard deviation of x, see Wei and Fung (1999). With different percentages of outliers ϕ = 5%, 10%, 20% and 25%.

Step (4): To evaluate the effectiveness of the various approaches, compute the root mean square error (RMSE) for the estimators. They come from;

$$\text{RMSE}(\hat{\alpha}) = \sqrt{\left(\frac{1}{n}\right)\sum_{i=1}^{n}(\hat{\alpha}_i - \alpha)^2},$$

$$\text{RMSE}(\hat{\beta}) = \sqrt{\left(\frac{1}{n}\right)\sum_{i=1}^{n}\left(\hat{\beta}_i - \beta\right)^2}.$$

Additionally, 1000 replications were used to create each simulation condition. Using R software, the simulation program was executed, see Abonazel, (2018). The values of the RMSE for the MOM, MLE, MPS, and robust M-estimators (Huber, Hampel, and Bisquare) under various values of and and various sample sizes in the presence of outliers are shown in the Tables from 1 to 5. Tables 1 through 5 shows that the RMSE for each technique decreases as sample size **n** increases.

On the other hand, it has been found that for all sample sizes, the M-estimators (Huber, Hampel, and Bisquare) are less sensitive to outliers than the classical estimators (MOM, MLE, MPS), which are more susceptible to outliers. The robust M-estimator built on the Bisquare function has the least RMSE overall.

Table 1: RMSE value for Gamma parameters, with different percentages of outliers when n =25

n	Estimate	MOM	MLE	MPS	Huber	Hampel	Bisquare
	$\phi = 5\%$						
25	$\alpha = 0.8$	0.8483	0.7361	0.6412	0.3711	0.3555	0.2619
	$\beta = 0.3$	0.4457	0.3394	0.2836	0.1677	0.2073	0.1476
	$\alpha = 1$	0.8777	0.7615	0.6633	0.3678	0.3840	0.2710
	$\beta = 1.5$	0.4612	0.3511	0.2934	0.1735	0.2145	0.1527
	$\alpha = 2.5$	0.9080	0.7879	0.6863	0.3973	0.3805	0.2804
	$\beta = 3$	0.4771	0.3632	0.3036	0.2219	0.1580	0.1795
	$\phi = 10\%$						
	$\alpha = 0.8$	0.8985	0.7796	0.6791	0.3931	0.2774	0.3765
	$\beta = 0.3$	0.4721	0.3594	0.3003	0.2195	0.1563	0.1776
	$\alpha = 1$	0.9295	0.8065	0.7026	0.3895	0.4067	0.2870
	$\beta = 1.5$	0.4884	0.3718	0.3107	0.1837	0.2271	0.1617
	$\alpha = 2.5$	0.9617	0.8344	0.7269	0.4208	0.2970	0.4030
	$\beta = 3$	0.5053	0.3847	0.3215	0.2350	0.1901	0.1673
	$\phi = 20\%$						
	$\alpha = 0.8$	0.9516	0.8256	0.7192	0.4163	0.3988	0.2938
	$\beta = 0.3$	0.5261	0.3806	0.3181	0.2325	0.1881	0.1656
	$\alpha = 1$	0.9845	0.8542	0.7441	0.4307	0.4126	0.3040
	$\beta = 1.5$	0.5173	0.3938	0.3291	0.1713	0.2405	0.1946
	$\alpha = 2.5$	1.0185	0.8838	0.7698	0.3145	0.4456	0.4268
	$\beta = 3$	0.5352	0.4074	0.3405	0.2489	0.2013	0.1772
	$\phi = 25\%$						
	$\alpha = 0.8$	1.0171	0.8825	0.7687	0.4262	0.4450	0.3141
	$\beta = 0.3$	0.5344	0.4069	0.3400	0.2010	0.2485	0.1770
	$\alpha = 1$	1.0523	0.9131	0.7953	0.4604	0.4410	0.3249
	$\beta = 1.5$	0.5529	0.4210	0.3518	0.2571	0.1831	0.2080
	$\alpha = 2.5$	1.0887	0.9447	0.8229	0.4763	0.3362	0.4562
	$\beta = 3$	0.5721	0.4355	0.3640	0.2660	0.2152	0.1894

Table 2: RMSE value for Gamma parameters, with different percentages of outliers when n =70

n	Estimate	MOM	MLE	MPS	Huber	Hampel	Bisquare
		$\phi = 5\%$					
70	$\alpha = 0.8$	0.5199	0.4511	0.3929	0.2275	0.2179	0.1605
	$\beta = 0.3$	0.2732	0.2080	0.1738	0.1270	0.0905	0.1028
	$\alpha = 1$	0.5379	0.4667	0.4065	0.2254	0.1661	0.2353
	$\beta = 1.5$	0.2826	0.2152	0.1798	0.1063	0.0936	0.1314
	$\alpha = 2.5$	0.5565	0.4829	0.4206	0.2332	0.2435	0.1718
	$\beta = 3$	0.2924	0.2226	0.1860	0.1360	0.1182	0.0968
	$\phi = 10\%$						
	$\alpha = 0.8$	0.5506	0.4778	0.4162	0.1700	0.2307	0.2409
	$\beta = 0.3$	0.2893	0.2203	0.1841	0.0958	0.1088	0.1345
	$\alpha = 1$	0.5697	0.4943	0.4306	0.1759	0.2387	0.2492
	$\beta = 1.5$	0.2993	0.2279	0.1904	0.1392	0.0991	0.1126
	$\alpha = 2.5$	0.5894	0.5114	0.4455	0.2470	0.2579	0.1820
	$\beta = 3$	0.3097	0.2358	0.1970	0.1165	0.1440	0.1026
	$\phi = 20\%$						
	$\alpha = 0.8$	0.6403	0.5555	0.4839	0.2801	0.2683	0.1977
	$\beta = 0.3$	0.3364	0.2561	0.2140	0.1564	0.1114	0.1265
	$\alpha = 1$	0.6624	0.5748	0.5007	0.2898	0.2045	0.2776
	$\beta = 1.5$	0.3481	0.2650	0.2214	0.1619	0.1153	0.1309
	$\alpha = 2.5$	0.6853	0.5946	0.5180	0.2998	0.2872	0.2116
	$\beta = 3$	0.3601	0.2742	0.2291	0.1675	0.1354	0.1192
	$\phi = 25\%$						
	$\alpha = 0.8$	0.7153	0.6207	0.5406	0.3130	0.2998	0.2209
	$\beta = 0.3$	0.3759	0.2862	0.2391	0.1748	0.1414	0.1245
	$\alpha = 1$	0.7401	0.6421	0.5594	0.3238	0.3101	0.2285
	$\beta = 1.5$	0.3889	0.2961	0.2474	0.1808	0.1463	0.1288
	$\alpha = 2.5$	0.7657	0.6644	0.5787	0.3350	0.3209	0.2364
	$\beta = 3$	0.4023	0.3063	0.2560	0.1871	0.1513	0.1332

Table 3: RMSE value for Gamma parameters, with different percentages of outliers when n =150

n	Estimate	MOM	MLE	MPS	Huber	Hampel	Bisquare
		$\phi = 5\%$					
150	$\alpha = 0.8$	0.0936	0.0812	0.0707	0.0409	0.0392	0.0289
	$\beta = 0.3$	0.0492	0.0374	0.0313	0.0229	0.0163	0.0185
	$\alpha = 1$	0.0968	0.0840	0.0732	0.0424	0.0299	0.0406
	$\beta = 1.5$	0.0509	0.0387	0.0324	0.0237	0.0191	0.0168
	$\alpha = 2.5$	0.1002	0.0869	0.0757	0.0309	0.0420	0.0438
	$\beta = 3$	0.0526	0.0401	0.0335	0.0174	0.0198	0.0245
	$\phi = 10\%$						
	$\alpha = 0.8$	0.1035	0.0898	0.0782	0.0453	0.0434	0.0320
	$\beta = 0.3$	0.0544	0.0414	0.0346	0.0205	0.0253	0.0180
	$\alpha = 1$	0.1071	0.0929	0.0809	0.0449	0.0469	0.0331
	$\beta = 1.5$	0.0563	0.0428	0.0358	0.0212	0.0262	0.0186
	$\alpha = 2.5$	0.1108	0.0961	0.0837	0.0464	0.0485	0.0342
	$\beta = 3$	0.0582	0.0443	0.0370	0.0271	0.0219	0.0193
	$\phi = 20\%$						
	$\alpha = 0.8$	0.1204	0.1044	0.0910	0.0527	0.0504	0.0372
	$\beta = 0.3$	0.0632	0.0482	0.0402	0.0294	0.0238	0.0209
	$\alpha = 1$	0.1245	0.1081	0.0941	0.0545	0.0522	0.0385
	$\beta = 1.5$	0.0654	0.0498	0.0416	0.0304	0.0246	0.0217
	$\alpha = 2.5$	0.1288	0.1118	0.0974	0.0564	0.0540	0.0398
	$\beta = 3$	0.0677	0.0515	0.0431	0.0315	0.0255	0.0224
	$\phi = 25\%$						
	$\alpha = 0.8$	0.1345	0.1167	0.1016	0.0588	0.0564	0.0415
	$\beta = 0.3$	0.0707	0.0538	0.0450	0.0266	0.0329	0.0234
	$\alpha = 1$	0.1391	0.1207	0.1052	0.0583	0.0609	0.0430
	$\beta = 1.5$	0.0731	0.0557	0.0465	0.0275	0.0340	0.0242
	$\alpha = 2.5$	0.1439	0.1249	0.1088	0.0603	0.0630	0.0444
	$\beta = 3$	0.0756	0.0576	0.0481	0.0352	0.0284	0.0250

Table 4: RMSE value for Gamma parameters, with different percentages of outliers when n =200

n	Estimate	MOM	MLE	MPS	Huber	Hampel	Bisquare
		$\phi = 5\%$					
200	$\alpha = 0.8$	0.0168	0.0146	0.0127	0.0074	0.0071	0.0052
	$\beta = 0.3$	0.0089	0.0067	0.0056	0.0041	0.0033	0.0029
	$\alpha = 1$	0.0174	0.0151	0.0132	0.0076	0.0073	0.0054
	$\beta = 1.5$	0.0092	0.0070	0.0058	0.0043	0.0034	0.0030
	$\alpha = 2.5$	0.0180	0.0156	0.0136	0.0079	0.0076	0.0056
	$\beta = 3$	0.0095	0.0072	0.0060	0.0044	0.0036	0.0031
	$\phi = 10\%$						
	$\alpha = 0.8$	0.0195	0.0169	0.0147	0.0085	0.0082	0.0060
	$\beta = 0.3$	0.0102	0.0078	0.0065	0.0048	0.0038	0.0034
	$\alpha = 1$	0.0201	0.0175	0.0152	0.0088	0.0084	0.0062
	$\beta = 1.5$	0.0106	0.0081	0.0067	0.0049	0.0040	0.0035
	$\alpha = 2.5$	0.0208	0.0181	0.0157	0.0091	0.0087	0.0064
	$\beta = 3$	0.0109	0.0083	0.0070	0.0051	0.0041	0.0036
	$\phi = 20\%$						
	$\alpha = 0.8$	0.0226	0.0196	0.0171	0.0099	0.0095	0.0070
	$\beta = 0.3$	0.0119	0.0091	0.0076	0.0055	0.0045	0.0039
	$\alpha = 1$	0.0234	0.0203	0.0177	0.0102	0.0098	0.0072
	$\beta = 1.5$	0.0123	0.0094	0.0078	0.0057	0.0046	0.0041
	$\alpha = 2.5$	0.0242	0.0210	0.0183	0.0106	0.0102	0.0075
	$\beta = 3$	0.0127	0.0097	0.0081	0.0059	0.0048	0.0042
	$\phi = 25\%$						
	$\alpha = 0.8$	0.0253	0.0219	0.0191	0.0111	0.0106	0.0078
	$\beta = 0.3$	0.0133	0.0101	0.0085	0.0062	0.0050	0.0044
	$\alpha = 1$	0.0262	0.0227	0.0198	0.0114	0.0110	0.0081
	$\beta = 1.5$	0.0137	0.0105	0.0087	0.0064	0.0052	0.0046
	$\alpha = 2.5$	0.0271	0.0235	0.0205	0.0118	0.0113	0.0084
	$\beta = 3$	0.0142	0.0108	0.0090	0.0066	0.0053	0.0047

Table 5: RMSE value for Gamma parameters, with different percentages of outliers when n =300

n	Estimate	MOM	MLE	MPS	Huber	Hampel	Bisquare
		$\phi = 5\%$					
300	$\alpha = 0.8$	0.0030	0.0026	0.0023	0.0013	0.0013	0.0009
	$\beta = 0.3$	0.0016	0.0012	0.0010	0.0007	0.0006	0.0005
	$\alpha = 1$	0.0031	0.0027	0.0024	0.0014	0.0013	0.0010
	$\beta = 1.5$	0.0016	0.0013	0.0010	0.0008	0.0006	0.0005
	$\alpha = 2.5$	0.0032	0.0028	0.0025	0.0014	0.0014	0.0010
	$\beta = 3$	0.0017	0.0013	0.0011	0.0008	0.0006	0.0006
	$\phi = 10\%$						
	$\alpha = 0.8$	0.0037	0.0032	0.0028	0.0016	0.0015	0.0011
	$\beta = 0.3$	0.0019	0.0015	0.0012	0.0009	0.0007	0.0006
	$\alpha = 1$	0.0038	0.0033	0.0029	0.0017	0.0016	0.0012
	$\beta = 1.5$	0.0020	0.0015	0.0013	0.0009	0.0007	0.0007
	$\alpha = 2.5$	0.0039	0.0034	0.0030	0.0017	0.0016	0.0012
	$\beta = 3$	0.0021	0.0016	0.0013	0.0010	0.0008	0.0007
	$\phi = 20\%$						
	$\alpha = 0.8$	0.0043	0.0037	0.0032	0.0019	0.0018	0.0013
	$\beta = 0.3$	0.0022	0.0017	0.0014	0.0010	0.0008	0.0007
	$\alpha = 1$	0.0044	0.0038	0.0033	0.0019	0.0018	0.0014
	$\beta = 1.5$	0.0023	0.0018	0.0015	0.0011	0.0009	0.0008
	$\alpha = 2.5$	0.0046	0.0040	0.0034	0.0020	0.0019	0.0014
	$\beta = 3$	0.0024	0.0018	0.0015	0.0011	0.0009	0.0008
	$\phi = 25\%$						
	$\alpha = 0.8$	0.0048	0.0041	0.0036	0.0021	0.0020	0.0015
	$\beta = 0.3$	0.0025	0.0019	0.0016	0.0012	0.0009	0.0008
	$\alpha = 1$	0.0049	0.0043	0.0037	0.0022	0.0021	0.0015
	$\beta = 1.5$	0.0026	0.0020	0.0016	0.0012	0.0010	0.0009
	$\alpha = 2.5$	0.0051	0.0044	0.0038	0.0022	0.0021	0.0016
	$\beta = 3$	0.0027	0.0020	0.0017	0.0012	0.0010	0.0009

7. CONCLUSIONS

This paper aimed to compare the estimations of the gamma distribution parameters by the classical methods, MOM, MLE and MPS, with estimates using robust estimation methods using three objective functions for M estimates (Huber, Hampel and Bisquare) in small and large samples, especially when the data contain outliers. It was proved by using simulation that the most suitable method for estimating the gamma parameters is the M method, and that the M estimators (Huber, Hampel and Bisquare) are significantly less sensitive to outliers than the classical estimators (MOM, MLE, MPS), and that the classical methods are very susceptible to outliers. Also, the robust M estimator based on the Bisquare function has the lowest RMSE for small and large samples.

8. References

1. Abonazel, M. R. (2018). A practical guide for creating Monte Carlo simulation studies using R. *International Journal of Mathematics and Computational Science*, 4(1), 18-33.
2. Abonazel, M., & Rabie, A. (2019). The impact of using robust estimations in regression models: An application on the Egyptian economy. *Journal of Advanced Research in Applied Mathematics and Statistics*, 4(2), 8-16.
3. Agostinelli, C., Marazzi, A., & Yohai, V. J. (2014). Robust estimators of the generalized log-gamma distribution. *Technometrics*, *56*(1), 92-101.
4. Aksoy, H. (2000). Use of gamma distribution in hydrological analysis. *Turkish Journal of Engineering and Environmental Sciences*, 24(6), 419-428.
5. Alharbi, Y. S., & Kamel, A. R. (2022). Fuzzy System Reliability Analysis for Kumaraswamy Distribution: Bayesian and Non-Bayesian Estimation with Simulation and an Application on Cancer Data Set. *WSEAS Transactions on Biology and Biomedicine*, 19, 118-139.
6. Almetwally, E. M., & Almongy, H. M. (2019). Estimation Methods for the new Weibull-Pareto distribution: Simulation and application. *Journal of Data Science*, 17(3), 610-630.
7. Almongy, H. M., & Almetwally, E. M. (2020). Robust estimation methods of generalized exponential distribution with outliers. *Pakistan Journal of Statistics and Operation Research*, p545-559.
8. Boland, P. J. (2007). *Statistical and probabilistic methods in actuarial science*. Chapman and Hall/CRC, New York.

9. Cheng, R. C. H., & Amin, N. A. K. (1983). Estimating parameters in continuous univariate distributions with a shifted origin. *Journal of the Royal Statistical Society: Series B (Methodological)*, 45(3), 394-403.
10. Friedman, N., Cai, L., & Xie, X. S. (2006). Linking stochastic dynamics to population distribution: an analytical framework of gene expression. *Physical review letters*, 97(16), p1-5.
11. Greenwood, J. A., & Durand, D. (1960). Aids for fitting the gamma distribution by maximum likelihood. *Technometrics*, 2(1), 55-65.
12. Hwang, T. Y., & Huang, P. H. (2002). On new moment estimation of parameters of the gamma distribution using its characterization. *Annals of the Institute of Statistical Mathematics*, 54(4), 840-847.
13. Johnson, N. L., Kotz, S., & Balakrishnan, N. (1995). *Continuous univariate distributions, volume* 2 (Vol. 289). John wiley & sons.
14. Kamel, A. R. (2021). *Handling outliers in seemingly unrelated regression equations model*, MSc thesis, Faculty of graduate studies for statistical research (FGSSR), Cairo University, Egypt.
15. Kamel, A. R., & Alqarni, A. A. (2020). A new characterization of exponential distribution through minimum chi-squared divergence principle. *Journal of Advanced Research in Applied Mathematics and Statistics*, 5(1&2), 14-26.
16. Kamel, A. R., & Alqarni, A. A. (2022). A New Approach for Model Selection with Two Qualitative Regressors. *Computational Journal of Mathematical and Statistical Sciences*, 1(1), 63-79.

17. Kamel, A. R., Alqarni, A. A., & Ahmed, M. A. (2022). On the Performance Robustness of Artificial Neural Network Approaches and Gumbel Extreme Value Distribution for Prediction of Wind Speed. *Int. J. Sci. Res. in Mathematical and Statistical Sciences*.Vol, 9(4).
18. Kantar, Y. M., & Yildirim, V. (2015). Robust estimation for parameters of the extended burr type III distribution. *Communications in Statistics-Simulation and Computation*, 44(7), 1901-1930.
19. Mousa, S. A. (2017). Robust estimators for marshall-olkin extended burr III distribution. *International Journal of Research in Applied Natural and Social Sciences*, 5, 69-79.
20. Orabi, A., & Ziedan, D. (2021). Robust Estimators for Marshal-Olkin Extended Linear Exponential Distribution. International Journal of Statistics and Applications, 11(1): 12-18.

21. Rahman, M., Pearson, L. M., & Martinovic, U. R. (2007). Method of product of spacings in the two parameter gamma distribution. *Journal of Statistical Research*, 41(1), 51-58.
22. Ranneby, B. (1984). The maximum spacing method. An estimation method related to the maximum likelihood method. *Scandinavian Journal of Statistics*, 93-112.
23. Son, Y. S., & Oh, M. (2006). Bayesian estimation of the two-parameter Gamma distribution. *Communications in Statistics-Simulation and Computation*, 35(2), 285-293.
24. Wackerly, D., Mendenhall, W., & Scheaffer, R. L. (2014). *Mathematical statistics with applications*. Cengage Learning.
25. Youssef, A. H., Abonazel, M. R., & Kamel, A. R. (2022). Efficiency Comparisons of Robust and Non-Robust Estimators for Seemingly Unrelated Regressions Model. *WSEAS Transactions on Mathematics*, 21, pp 218-244.
26. Youssef, A. H., Kamel, A. R., & Abonazel, M. R. (2021). Robust SURE estimates of profitability in the Egyptian insurance market. *Statistical journal of the IAOS*, 37(4), 1275-1287.

طريقة حديثة في استخدام اسلوب التقديرات الحصينة M لتقدير معالم توزيع جاما في وجود القيم متطرفة

د. محمد عبد السلام عجمي، د. عزت حسين خمار

الملخص:

في هذا البحث، تم استخدام ثلاث دوال هدف objective functions لتقديرات M-Estimation الحصينة وهي (Huber و Hampel و Bisquare) للحصول على تقديرات حصينة لمعلمات توزيع جاما Gamma distribution ، ثم تتم مقارنة تقديراتها مع التقديرات التي تقدمها الطرق التقليدية الشائعة مثل (مقدرات العزوم moment estimators ، مقدرات الامكان الأعظم-maximum likelihood estimators ، وطريقة الحد الأقصى maximum product spacings)، بهدف تحديد أفضل الطرق لتقدير معالم توزيع جاما في وجود القيم المتطرفة outliers ، وذلك بتطبيق الطرق السابقة على بيانات تم توليدها من خلال دراسة المحاكاة simulation study ، وذلك في وجود نسب مختلفة من القيم المتطرفة outliers. وقد أكدت نتائج المحاكاة أن تقديرات M-Estimation تعطي تقديرات أكثر دقة وأعلى كفاءة عند تقدير معالم توزيع جاما. أظهرت النتائج إلى أن الطريقة الأنسب لتقدير معاملات توزيع جاما Gamma distribution ، هي طريقة تقدير M-Estimation بدوال الهدف الثلاث (Huber وHampel وBisquare) للعينات الصغيرة والكبيرة خاصة اذا احتوت البيانات على القيم المتطرفة. وتقديراتها، في هذه الحالة، تتميز بدقة أكثر وكفاءة أفضل. كما استنتجنا أيضًا أن أفضل طرق تقدير M-Estimation الحصينة، هي طريقة تقدير Bisquare.

الكلمات المفتاحية: الطرق الكلاسيكية، مقدر Mالحصين، دالة الهدف، القيم المتطرفة، دراسة المحاكاة، مقدر الإمكان الأعظم، توزيع جاما ثنائي المعالم.