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# New View for Volterra Integral Equation

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**ABSTRACT**

In this paper, can introduce transform equations that change integral equations into differential equations that are translated on the fuzzy form (FH). using fuzzy Riemann (FR), fuzzy Aumann (FA), and fuzzy Henstock. The algorithm that could solve a Volterra-type integral equation (VIE) was applied to the earlier forms which call triangular fuzzy function to find the fuzzy solution. We present solution as fuzzy functions such that each function satisfies the initial value problem by some membership degree. Some specific examples were provided to fulfil the effectiveness approach.

***KEYWORDS:*** Volterra fuzzy integral equation. fuzzy differential equation. Triangular fuzzy number.

## 1. Introduction

The fuzzy differential and integral equations are a significant component of the hairy assessment idea, which is essential to the evaluation of numerical results. The ideas of Dubois and Prade [5], Goetschel and Voxman [15], Kaleva [11, 12], and others have added to the idea of integrating fuzzy talents. Wu [4] and Ma's helpful resource was used to modify and supply the initial packages for fuzzy integration. Beginning with Kaleva, Seikkala [17], Mordeson [9], and Newman, the study of fuzzy integral equations is based on the most complex mathematical models of components. The existence and uniqueness, boundedness of solutions, and numerical development are the problems covered in the investigation of fuzzy integral equations methods for estimating the answer. Fuzzy differential equations (FDE) research offers an appropriate framework for the mathematical modelling of real-world issues that are characterized by ambiguity or uncertainty. Fuzzy derivative was defined by Chang and Zadeh. Following suit, Dubois and Pradein applied the extension idea. In 1987, Kandel and Byatt [10] introduced the phrase "fuzzy differential equation." It is challenging to research FDE since fuzzy derivatives have so many distinct definitions. The Hukuhara inferentiality is the original and most popular technique to fuzzy value functions. The solution's uniqueness and existence in this situation FDE In Seikkala, The existence and uniqueness, boundedness of solutions, and numerical development are the issues covered in the study of fuzzy integral equations. Similar to the Hukuhara derivative and fuzzy integral proposed by Dubois and Prade, established the concept of fuzzy derivatives. A general explanation of a fuzzy first-order initial value problem was given by Buckley and Feuring [3]. They search for a crisp solution, fuzzify it, and then determine whether it fulfils the FDE. We are expanding the existing idea of differentiability to the Hukuhara model in order to adapt conventional techniques to the fuzzy environment.

In this work, a new view is defined for Volterra fuzzy integral equation (VFIE) of the form:

$$u\tilde{}(x) = f\tilde{}(x) \oplus (FR)\int_0^x k(x.t)\ \odot f(t))\, dt \tag{1}$$

where f(x) is a fuzzy function of x: $0 \le x \le X$ and k(x.t),u (x) are analytic function on [0, X], therefore x, u: A= [0, X] × [c, d] → Rf are continuous fuzzy-number valued functions and k: A × A→ Rf, f: Rf → Rf are continuous functions on $R_f$. The set Rf is the set of all real fuzzy numbers.

The paper is organized as: In section 2, we present the basic notations of fuzzy numbers, fuzzy functions and fuzzy integrals as Riemann (FR), fuzzy Aumann (FA) and fuzzy Henstock (FH). In section 3, the transform from of VFIE is introduced and then differential equation is applied for solving this equation. We aim to apply the algorithm of the solution in section 4. Finally, in section 5, we illustrate the method by solving example.

## 2. Preliminaries

In this section, can briefly state some definition and results related to fuzzy numbers and Riemann (FR), fuzzy Aumann (FA) and fuzzy Henstock (FH), which will be referred throughout this paper.

*Definition 2.1*

A fuzzy number is a function u: R→ [0,1] satisfying the following properties [15]:

- u is upper semi-continuous on R.
- u(x)=0 outside of some interval [c, d].
- there are the real numbers a and b with $c \leq a \leq b \leq d$ such that u is increasing on [c, a], decreasing [b, d] and u(x)=1for each x ∈ [a, b].
- u is fuzzy convex set i.e. that is u $(\lambda x+(1-\lambda) y \geq \min\{u(x), u(y)\}$ for all x, y ∈ R and λ∈ [0,1].

*Lemma 2.1*

According to [4] for any $0 < r \leq 1$, can denote the r−level set $[u]^r = \{x \in R_f, u(x) \geq r\}$ that is a closed interval and $[u]^r = [u_-^r . u_+^r]$ for all r ∈ [0,1]. These lead to the usual parametric representation of a fuzzy number, by an ordered pair of functions $(u_-^r . u_+^r)$ which satisfies the following properties: $u_-^r -$ is bounded left continuous nondecreasing function over [0,1], $u_+^r$ is bounded left continuous non-increasing function over [0, 1] and $u_-^r \leq u_+^r$. Any real number a ∈ $R_f$ can be interpreted as a fuzzy number $\tilde{a}$ = x[a] and therefore R ⊂ $R_f$.

For u, v ∈ $R_f$, k ∈ R, the addition, the subtraction and the scalar multiplication are defined by

$$[u\oplus v]^r = [u]^r + [v]^r = [\, u_-^r + v_-^r . u_+^r + v_+^r].$$

$$[u\ominus v]^r = [u]^r - [v]^r = [\, u_-^r - v_-^r . u_+^r - v_+^r].$$

$$[k \odot u]^r = k.\, [u]^r = \left\{\begin{matrix} [ku_-^r . ku_+^r] .if\ k\geq 0 \\ [ku_+^r . ku_-^r] .if\ k<0 \end{matrix}\right\} \text{ for all } r \in [0, 1].$$

The neutral element respect to ⊕ in $R_f$,

denoted by $\tilde{0}$ = x [0]. The algebraic properties of addition and scalar multiplication of fuzzy numbers are given in [4].

*Definition 2.2*

As a distance between fuzzy numbers we use Hausdroff metric [4], defined by

$D(u, v) = \sup_{\in [0,1]} \max \{|u_-^r - v_-^r|, |u_+^r - v_+^r|$ for any u, v ∈ $E^1$.

The Hausdroff metric has the following properties:

- $(E^1, D)$ is a complete metric space.
- $D(u \oplus w, v \oplus w) = D(u, v)$ for all u, v, w ∈ $E^1$.
- $D(u \oplus v, w \oplus e) \leq D(u, w) + D(v, e)$ for all u, v, w and e ∈ $E^1$.
- $D(u \oplus v, 0) \leq D(u, 0) + D(v, 0)$ for all u, v ∈ $E^1$.
- $D(k\odot u, k \odot v) = |k| D(u, v)$ for all u, v ∈ $E^1$, for all k ∈ R, $D(k_1 \odot u, k_2 \odot u) = |k_1 - k_2| D(u, 0)$ for all $k_1, k_2 \in E^1$, with $k_1 k_2 \geq 0$ and for all u ∈ $E^1$.

For any fuzzy-number-valued function $f: A \to E^1$, we can define the functions $f\_(.,.,r)$, $\bar{f}(.,.,r): A \to R$, by $f\_(s,t,r) = (f(s,t))^{r-}$, $\bar{f}(s,t,r) = (f(s,t))^{r+}$, for each $(s,t) \in A$, for each $r \in [0,1]$. These functions are called the left and right r-level functions of $f$.

*Lemma 2.2*

If $\tilde{A}$, $\tilde{B}$ be two FS, then $\tilde{A} \oplus \tilde{B}$ can be formed by α-cut of $\tilde{A} \oplus \tilde{B}$, where $[\tilde{A}]_\alpha$ and $[\tilde{B}]_\alpha$ define as:

$[A \oplus B]_\alpha = [a_1 b_1, a_2 b_2]_\alpha$.

$[A \odot B]_\alpha = [\min(a_1b_1, a_1b_2, a_2b_1, a_2b_2), \max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)]_\alpha$.

*Definition 2.3*

Let $f: A \to E^1$, then f is (FH)-integrable if and only if $f\_$ and $\bar{f}$ are Henstock integrable for any $r \in [0,1]$. Moreover

$$[(FH)\int_c^d (FH)\int_a^b f(s.t)\,dsdt]^r = [\int_c^d (H)\int_a^b (H) f\_(s.t.r))\,dsdt.(H)\int_c^d (H)\int_a^b \bar{f}(s.t.r))\,dsdt)]$$

Also, if f is continuous then $f\_(.,.,r)$ and $\bar{f}(.,.,r)$ are continuous for any $r \in [0,1]$ and consequently, they are Henstock integrable, we deduce that f is (FH)-integrable [16].

*Definition 2.4*

If f and g are fuzzy Henstock integrable functions on A and if the function given by D (f (s, t), g (s, t)) is Lebesgue integrable [16], then

$$D((FH)\int_c^d (FH)\int_a^b f(s.t))\,dsdt.(FH)\int_c^d (FH)\int_a^b g(s.t))\,dsdt) \le (L)\int_c^d (L)\int_a^b D\big(f(s.t)\big).g(s.t)dsdt)$$

## 3. Integration of Fuzzy-Number-Valued Functions

It is simple to see that there are no significant issues with the definition of the integral of a fuzzy-number valued function. Integrals of the Aumann, Riemann, and Henstock types will be covered in the sections that follow.

An Aumann-type integral, which has been used in several studies, is introduced first.

*Definition 3.1*

A mapping $f: [0, X] \to R_f$ is said to be strongly measurable if the level set mapping $[f(x)]_\alpha$ are measurable for all $\alpha \in [0, 1]$. Here measurable means Borel measurable [11]. A fuzzy-valued mapping $f: [0, X] \to R_f$ is called integrable bounded if there exists an integrable function $h: [0, X] \to R$, such that

$$\|f(t)\|_{F_0} \le h(t).\forall t \in [0.x]$$

A strongly measurable and integrably bounded fuzzy-valued function is called integrable. The fuzzy Aumann integral of

$f: [0, X] \to R_f$ is defined level wise by the equation:

$$[(FA)\int_0^x f(x))\,dx]^r = \int_0^x [f(x)]^r dx. r \in [0.1]$$

The following Riemann type integral presents an alternative to Aumann-type definition**.**

*Definition 3.2*

A function $f: [0, X] \to R_f$, $[0, x] \subset R$ is called Riemann integrable on [0, x] [17], if there exists $I \in R_f$, with the property: $\forall \varepsilon \succ 0$, $\exists \delta \succ 0$, such that for any division of [0, x], $d: 0 < \ldots < x_n = X$ of norm $\nu(d) \prec \delta$, and for any points $\xi_i \in [x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, we have

$$I \approx \sum_{i=0}^{n-1} f(\xi_i) \cdot h_i$$

Then we denote $I = (FA) \int_0^x k(x.t) \odot f(t))\, dt$ and it is called fuzzy Riemann integral. In Wu-Gong [5] the Henstock integral of a fuzzy-valued function is introduced. Surely as a particular case the Riemann-type integral of a fuzzy-valued function can be re-obtained.

*Definition 3.3*

Let $f$: [a, b] → $R_f$ a fuzzy-number-valued function [1, 2] and △n: $a = 0 < x_1 < ... < x_{n-1} < x_n = X$ a partition of the interval [a, b], $\xi_i \in [x_i, x_{i+1}]$, i = 0, 1, ..., n - 1, a sequence of points of the partition △n and $\delta(x) \succ 0$ a real-valued function over [a, b]. The division P = (△n, ξ) is said to be δ-fine if $[x_i, x_{i+1}] \subseteq (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i))$.

The function f is said to be Henstock (or (FH)-) integrable having the integral $I \in R_f$ if for any $\varepsilon \succ 0$ there exists a real-valued function δ, such that for any δ -fine division P we have

$$I \approx \sum_{i=0}^{n-1} f(\xi_i) \cdot h_i$$

where $h_i = x_{i+1} - x_i$. Then I called the fuzzy Henstock integral of f and it is denoted by $I = (FH) \int_0^x k(x.t) \odot f(t)\, dt$ .

*Proposition 3.1*

A continuous fuzzy-number-valued function is fuzzy Aumann integrable, fuzzy Riemann integrable and fuzzy Henstock integrable too, and moreover

$$(FA) \int_0^x k(x.t) \odot f(t))\, dt = (FH) \int_0^x k(x.t) \odot f(t))\, dt = (FR) \int_0^x k(x.t) \odot f(t))\, dt$$

*Proof.* It is immediate to observe that

$$[(FA) \int_0^x k(x.t) \odot f(t))\, dt]^r = [\int_0^x k(x.t) \odot f_-^r(t))\, dt \,.\int_0^x k(x.t) \odot f_+^r(t))\, dt\,] \; \forall\, r \in [0.1]$$

If f is Riemann integrable then it is also Henstock integrable. Indeed, if the function δ is constant in the Henstock definition, it will generate the Riemann case. The Riemann sum can be written level-wise

$$I \approx \sum_{i=0}^{n-1} f(\xi_i) \cdot h_i$$

Equicontinuity implies integrability of the functions $f_-^r$ and $f_+^r$ uniformly w.r.t. r ∈ [0, 1]. Then we obtain

$$[(FR) \int_0^x k(x.t) \odot f(t))\, dt]^r = [\int_0^x k\_(x.t) \odot f_-^r(t))\, dt \,.\int_0^x k + (x.t) \odot f_+^r(t))\, dt\,]$$

The common value of these integrals for a continuous function $f$: [0, X] → $R_f$ will be denoted by $I = \int_0^x k(x.t) \odot f(t)\, dt$ . For general $f$: [0, X] → $R_f$ the above assertion does not hold.

The properties of the integrals for fuzzy functions are similar to the properties of their classical counterparts

*Proposition 3.2*

The fuzzy integral has the following properties [17]:

(i) If f, g: [0, X] → $R_f$ are integrable where $\alpha_1, \beta_1 \in R_f$ and positive we have

$$\int_0^x \left[\left(\alpha_1 f(x) + \beta_1 g(x)\right) \odot k(x.t)\right] dx = \alpha_1 \int_0^x k(x.t) \odot f(x)\, dx + \beta_1 \int_0^x k(x.t) \odot g(x)\, dx$$

(ii) If f: [0, X] → $R_f$ is integrable and z ∈ [0, X], then

$$\int_0^z k(x.t) \odot f(x)\, dx + \int_z^x k(x.t) \odot f(x) dx = \int_0^x k(x.t) \odot f(x)\, dx$$

(iii) If c ∈ $R_f$ and f: [0, X] → $R_f$ has constant sign on [0, X], then

$$\int_0^x [C \odot k(x.t) \odot f(x)] dx = C \int_0^x k(x.t) \odot f(x)\, dx$$

**4. Differentiability of Fuzzy-Number-Valued Functions**

Puri-Ralescu [13] presented the Hukuhara derivative of a fuzzy-number-valued function, which takes its cue from the Hukuhara derivative of multivalued functions. The Hukuhara derivative-based method has the drawback of a differentiable function having a growing support interval (Diamond [6, 7]). This isn't necessarily a reasonable assumption. Bede-Gal [1] introduces and studies strongly generalized differentiability of fuzzy-number-valued functions. A differentiable function in this situation might have a diminishing length of support.

*Definition 4.1*

A function $f$: (a, b) → $R_f$ is called Hukuhara differentiable [8, 14], if for h≻ 0 sufficiently small the H-Differences

f (x + h) ⊖f(x) and f(x)⊖ f (x - h) exist and if there exist an element $f'$ (x) ∈ $R_f$ such that

$$\lim_{h \searrow 0} \frac{f(x+h) \ominus f(x)}{h} = \lim_{h \searrow 0} \frac{f(x) \ominus f(x-h)}{h} = f'(x)$$

The fuzzy number $f'(x)$ is called the Hukuhara derivative of $f$ at x.

*Definition 4.2*

The Seikkala derivative of a fuzzy number-valued function [16] $f$ : (a, b) → $R_f$ is defined by

$$f'(x)r = [(f_-^r(x)').(f_+^r(x)')]$$

$0 \le r \le 1$, provided that it defines a fuzzy number $f'(x) \in R_f$.

*Definition 4.3*

Let $f$: (a, b) → $R_f$ and $x_0 \in$ (a, b). We say that f is strongly generalized differentiable at $x_0$, if there exists an element f ($x_0$) ∈ $R_f$ [3], such that

(i) for all h ≻ 0 sufficiently small,

∃ $f$ ($x_0$ + h) ⊖ $f$ ($x_0$), $f$ ($x_0$) ⊖ $f$ ($x_0$ - h)

and the limits (in the metric D) element $f$ ($x_0$) ∈$R_f$ such that

$$\lim_{h \searrow 0} \frac{f(x_0+h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0-h)}{h} = f'(x_0)$$

(ii) for all h≻ 0 sufficiently small,

$\exists f(x_0) \ominus f(x_0 + h), f(x_0 - h) \ominus f(x_0)$

and the limits

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{h} = f'(x_0)$$

(iii) for all h ≻ 0 sufficiently small,

$\exists f(x_0) \ominus f(x_0 + h)$,

$f(x_0) \ominus f$ f $(x_0 - h)$ and the limits

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{(-h)} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

(h) and (-h) at denominators mean (1/h) and -(1/h), respectively.

## 5. Transform form for Volterra integral equation

*5.1 Converting Volterra Equation to an ODE*

In this section can present the technique that converts Volterra integral equations of the second kind to equivalent differential equations. This may be easily achieved by applying the important Leibniz Rule for differentiating an integral.

*Differentiating Any Integral: Leibniz Rule*

To differentiate the integral

$$\int_{\alpha(X)}^{\beta(X)} K(x.t) \odot f(t)\, dt$$

with respect to x, using the Leibniz rule given by:

$$d/dx \int_{\alpha(X)}^{\beta(X)} K(x.t) \odot f(t)dt = \left[K(x.\beta(X)t) \odot f\big(\beta(X)\big)\right] d\beta/dx - [K(x.\beta(X)t) \odot f\big(\beta(X)\big)]d\alpha/dx + \int_{\alpha(X)}^{\beta(X)} \frac{\partial K}{\partial x} \odot f(t)dt \qquad (2)$$

where K (x, t) and are continuous functions in the D in the xt-plane that contains the rectangular region R, $a \leq x \leq b$, $t_0 \leq t \leq t_1$, and the limits of integration α(x) and β(x) are defined functions having continuous derivatives for $a < x < b$.

The next examples are discussed in the approach that will be used to convert Volterra integral equations into differential equations.

*Example 1* Find

$$d/dx \int_0^x (x - t) \odot f(t)dt$$

In this example, α(x) = 0, β(x) = x, hence α′(x) = 0, β′(x) = 1.

Using Leibniz rule (2), get

$$d/dx \int_0^x (x-t) \odot f(t)\, dt = \int_0^x f(t)\, dt$$

View our main goal to transform a Volterra integral equation into equivalent differential equation. This achieve by differentiating both sides of the integral equation, the Leibniz rule used in differentiating the integral. The differentiating process should be continued a lot times as needed until we get a differential equation with the integral sign removed. The initial conditions can calculate by substituting x = 0 in the integral equation.

Now, give the following illustrative example.

*Example 2* Transform the Volterra integral equation

$$u(x) = 5 + \int_0^x u(t)\, dt \tag{3}$$

Differentiating both sides of (3), using Leibniz rule we get

$$u'(x) = u(x) \tag{4}$$

The initial condition can calculate by substituting x = 0 into both sides of the integral equation, find u (0) = 1.

Consequently, the corresponding (IVP)

$$u'(x) - u(x) = 0. u(0) = 1 \tag{5}$$

*5.2 Converting IVP to Volterra Equation*

In this section, we will study the method that transforms (IVP) into equivalent Volterra integral equation. Before declaring the method used, we recall the useful transformation formula

$$\int_0^x \int_0^{x_1} \int_0^{x_2} \cdots \int_0^{x_{n-1}} f(x_n)\, dx_n \cdots dx_1 = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} \odot f(t) dt \tag{6}$$

that transforms any multiple integral to a single integral.

$$\int_0^x \int_0^x \int_0^x f(t)\, dtdtdt = (\frac{1}{2!}) \int_0^x (x-t)^2 \odot f(t) dt \tag{7}$$

And

$$\int_0^x \int_0^x f(t)\, dtdtdt = \int_0^x (x-t) \odot f(t)\, dt \tag{8}$$

are two special cases of the formula given above, and the mostly used formulas that will transform double and triple integrals respectively to a single integral for each.

For simplicity reasons, we prove the first formula (7) that converts double integral to a single integral.

Noting that the right-hand side of (8) is a function of x allows us to set the equation

$$I(x) = \int_0^x (x-t) \odot f(t)\, dt \tag{9}$$

Differentiating both sides of (9),

by Leibniz rule can get

$$I'(x) = \int_0^x f(t)dt \tag{10}$$

Integrating both sides of (10) via 0 to x,

I (0) = 0 from (9), Find that

$$I(x) = \int_0^x \int_0^x f(t)\,dt\ dt \tag{11}$$

*5.3 Converting Volterra Equation to IVP*

Though it is rarely applied, the method of transforming Volterra integral equations to initial value issues will be covered in this section. This could be explained by the fact that initial conditions are incorporated into integral equations, which make them simple to solve. The size of evaluations needed will, however, rise when solving initial value problems, where beginning conditions will be employed, because more steps will be necessary to complete the answer.

To utilize this approach, we simply differentiate both sides of (1), observing that the integral at the right-hand side of (1) should be differentiated using the Leibniz rule (2). Once the integral sign has been eliminated and the integral equation has been transformed into a pure differential, the differentiation process should be repeated sequentially. equivalent to the discussed integral equation. It's noteworthy to observe that beginning conditions must be established at each stage of differentiating by setting x = 0 at u(x) and its obtained derivatives. Following the conventional methods taught in undergraduate courses on ordinary differential equations, the resulting starting value problem is then resolved.

Though not frequently employed, as previously said, the method of transforming Volterra integral equations to initial value issues will be explained by going over the following example.

*Example 3*

Solve the following Volterra integral equation

$$u(x) = 2\mathrm{x}^2 + 1/12\mathrm{x}^4 + \int_0^x (t-x)\,u(t)\,dt \tag{12}$$

via the transformation into an equivalent initial value problem.

Using the Leibniz method and differentiating both sides of (12) with regard to x, we discover

$$u'(x) = 4\mathrm{x} + \frac{1}{3}(\mathrm{x}^3) - \int_0^x u(t)\,dt \tag{13}$$

To cancel the integral symbol at the right-hand side of (13), we differentiate both sides, so we get

$$u''(x) = 4 + \mathrm{x}^2 - u(x) \tag{14}$$

or equivalently the nonhomogeneous second order differential equation

$$u''(x) + u(x) \ = 4 + \mathrm{x}^2 \tag{15}$$

The initial conditions getting by substituting x = 0 into both sides of equations (12) and (13), so

$$u(0) = 0. u'(0) = 0 \tag{16}$$

To solve the resulting initial value problem

$$u''(x) + u(x) = 4 + x^2 \quad . \quad u(0) = 0. u'(0) = 0 \tag{17}$$

We first solve the corresponding homogeneous equation algorithm of solution.

*5.4 A Fuzzy Initial Value Problem (FIVP)*

In this section, can describe a fuzzy initial value problem (FIVP) and concept of solution which we propose. We investigate a fuzzy initial value problem for linear differential equation with triangular fuzzy forcing function and with fuzzy initial values.

Such a FIVP can arise in modelling of a process the dynamics of which is crisp but there are uncertainties in forcing function and in initial values.

*Solution algorithm*

We suggest the following approach to solve the FIVP to fuzzy differential equation transformation in light of the aforementioned justifications:

1. Transform an integral equation into a differential one.

2. Apply a membership shift for the transformed equation.

3. Determine the related crisp problem's solution $u_{cr}(x)$.

4. Determine the solutions $u_{un}(x)$.

5. Find fuzzy IE solution $\tilde{u}(x) = \tilde{u}_{cr}(x) + \tilde{u}_{un}(x)$.

## 6. Examples on method

In this section, can study the algorithm of solution method on the transformed Volterra integral equation into differential equation.

*Example 4* Consider the following VFIE:

$$u(x) = \left(\frac{2}{3}x^3 - 3x^2 + 2x - 5\right) \oplus \int_0^x [3 - 2(x - t)]u(t)dt \tag{18}$$

First,

convert (18) into differential equation,

can write the eq (18) as

$$u(x) = \left(\frac{2}{3}x^3 - 3x^2 + 2x - 5\right) + 3\int_0^x u(t)\,dt - 2\int_0^x (x - t)u(t)dt \tag{19}$$

Differentiate (19)

$$u'(x) = (2x^2 - 6x + 2) + 3u(x) - 2\int_0^x u(t)dt \tag{20}$$

Differentiate (20)

$$u''(x) = (4\mathrm{x} - 6) + 3u'(x) - 2u(x) \tag{21}$$

$$u''(x) - 3u'(x) + 2u(x) = 4x - 6 \tag{22}$$

Solve the eq (22) in triangle memberships

$$\begin{Bmatrix} u'' - 3u' + 2u = 4x - 6 \\ u(0) = (1 \cdot 5.2.3) \\ u(1) = (2.3.4) \end{Bmatrix} \tag{23}$$

Apply the algorithm to find the solution

$\tilde{u}(x) = \tilde{u}_{cr}(x) + \tilde{u}_{un}(x)$

1-calculate the crisp solution $u_{cr}$ (x) with operator

$$\begin{Bmatrix} u'' - 3u' + 2u = 4x - 6 \\ u(0) = 2 \\ u(1) = 3 \end{Bmatrix} \tag{24}$$

$$\tilde{u}_{cr}(x) = 2x + \left(\frac{1}{\mathrm{e}^2 - e}\right)[2(\mathrm{e}^{x+2} - \mathrm{e}^{2x+1}) + (\mathrm{e}^{2x} - \mathrm{e}^{x})] \tag{25}$$

2-the fuzzy solution and its α=0.5-cut different time, the fuzzy homogenous problem

$$\begin{Bmatrix} u'' - 3u' + 2u = 0 \\ u(0) = (-0 \cdot 5.0.1) \\ u(1) = (-1.0.1) \end{Bmatrix} \tag{26}$$

$$(\mathrm{D}^2 - 3D + 2)u = 0 \tag{27}$$

$$\mathrm{u}_1(x) = \mathrm{e}^x \& \mathrm{u}_2(x) = \mathrm{e}^{2x} \tag{28}$$

$$s(x) = [\mathrm{u}_1(x) \quad \mathrm{u}_2(x)] = [\mathrm{e}^x \ \mathrm{e}^{2x}]$$

$$M = \begin{bmatrix} \mathrm{u}_1(0) & \mathrm{u}_2(0) \\ \mathrm{u}_1(1) & \mathrm{u}_2(1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e & \mathrm{e}^2 \end{bmatrix} \tag{29}$$

$$w = s(x)\mathrm{M}^{-1} = \left(\frac{1}{\mathrm{e}^2 - e}\right)[\mathrm{e}^x \;\; \mathrm{e}^{2x}]\begin{bmatrix}\mathrm{e}^2 & -1\\ -e & 1\end{bmatrix} \quad (30)$$

$$w = \left(\frac{1}{\mathrm{e}^2 - e}\right)[\mathrm{e}^{x+2} - \mathrm{e}^{2x+1} - \mathrm{e}^x + \mathrm{e}^{2x}] = \left(\frac{1}{\mathrm{e}^2 - e}\right)[w1 \;\; w2] \quad (31)$$

$$\mathrm{u}\tilde{}_{\mathrm{un}}(x) = \mathrm{w}_1(x)a\tilde{} + \mathrm{w}_2(x)b\tilde{}$$

$$\mathrm{u}\tilde{}_{\mathrm{un}}(x) = \mathrm{w}_1(x)u(0) + \mathrm{w}_2(x)u(1)$$

$$\mathrm{u}\tilde{}_{\mathrm{un}}(x) = \left(\frac{1}{\mathrm{e}^2 - e}\right)[(\mathrm{e}^{x+2} - \mathrm{e}^{2x+1})(-0\cdot 5.0.1) + (\mathrm{e}^{2x} - \mathrm{e}^x)(-1.0.1)] \quad (32)$$

$$\mathrm{u}\tilde{}(x) = \mathrm{u}\tilde{}_{\mathrm{cr}}(x) + \mathrm{u}\tilde{}_{\mathrm{un}}(x)$$

$$= 2x + \left(\frac{1}{\mathrm{e}^2 - e}\right)[2(\mathrm{e}^{2+x} - \mathrm{e}^{1+2x}) + (\mathrm{e}^{2x} - \mathrm{e}^x)]$$
$$+ \left(\frac{1}{\mathrm{e}^2 - e}\right)[(\mathrm{e}^{x+2} - \mathrm{e}^{2x+1})(-0\cdot 5.0.1) + (\mathrm{e}^{2x} - \mathrm{e}^x)(-1.0.1)] \quad (33)$$

$$= 2x + \left(\frac{1}{\mathrm{e}^2 - e}\right)[(\mathrm{e}^{x+2} - \mathrm{e}^{2x+1})(1\cdot 5.2.3) + (\mathrm{e}^{2x} - \mathrm{e}^x)(0.1.2)] \quad (34)$$

To find $\mathrm{u}^{-}$, $\mathrm{u}\_$

$$B\tilde{} = (a\tilde{}.b\tilde{}).\, a\tilde{} = (a\_.0.a^{-}).\, b\tilde{} = (b\_.0.b^{-}) \quad (35)$$

$$\mathrm{u}\tilde{}(x) = w(x)v.\, v = \begin{bmatrix}a\tilde{}\\ b\tilde{}\end{bmatrix} \quad (36)$$

$$\mathrm{u}\tilde{}(x) = \mathrm{w}_1(x)a\tilde{} + \mathrm{w}_2(x)b\tilde{} \quad (37)$$

$$a = [\mathrm{a}_{-\alpha}.\mathrm{a}^{-}{}_{\alpha}];\, b = [\mathrm{b}_{-\alpha}.\mathrm{b}^{-}{}_{\alpha}] \quad (38)$$

$$B = [\mathrm{a}_{-\alpha}.\mathrm{a}^{-}{}_{\alpha}] \odot [\mathrm{b}_{-\alpha}.\mathrm{b}^{-}{}_{\alpha}]$$

Then

$$\mathrm{u}_{\alpha}(t) = [\mathrm{u}_{-\alpha}(x).\mathrm{u}^{-}{}_{\alpha}(x)]$$

$$\mathrm{u}^{-}{}_{\alpha}(x) = \max\{\mathrm{a}_{-\alpha}\,\mathrm{w}_1(x).\mathrm{a}^{-}{}_{\alpha}\mathrm{w}_1(x)\} + \max\{\mathrm{b}_{-\alpha}\,\mathrm{w}_2(x).\mathrm{b}^{-}{}_{\alpha}\mathrm{w}_2(x)\} \quad (39)$$

$$w_1(x) = \left(\frac{1}{e^2 - e}\right)(e^{x+2} - e^{2x+1}) \tag{40}$$

$$w_2(x) = \left(\frac{1}{e^2 - e}\right)(e^{2x} - e^{x}) \tag{41}$$

$$a\tilde{} = (1 \cdot 5.2.3).\, b\tilde{} = (0.1.2) \tag{42}$$

$$a\bar{} = 3;\, a\_ = 1 \cdot 5;\, b\bar{} = 2;\, b\_ = 0$$

So,

$$u\bar{}_{\alpha}(x) = 2x + \left(\frac{1}{e^2 - e}\right)[(e^{x+2} - e^{2x+1}) \odot 3 + (e^{2x} - e^{x}) \odot 2] \tag{43}$$

if

$$u\_{\alpha}(x) = \max\{a\_{\alpha}\, w_1(x).\, a\bar{}_{\alpha} w_1(x)\} + \max\{b\_{\alpha}\, w_2(x).\, b\bar{}_{\alpha} w_2(x)\} \tag{44}$$

So,

$$u\_{\alpha}(x) = 2x + \left(\frac{1}{e^2 - e}\right)[(e^{x+2} - e^{2x+1}) \odot\ 1 \cdot 5 + (e^{2x} - e^{x}) \odot 0] \tag{45}$$

$$u_{\alpha}(x) = [u\_{\alpha}(x).\, u\bar{}_{\alpha}(x)]$$

$$u\tilde{}_{un}(x) = (u\_{un}(x).\, 0.\, u\bar{}_{un}(x))$$

$$u\tilde{}_{un.\alpha}(x) = (1 - \alpha)[u\_{un}(x).\, u\bar{}_{un}(x)]$$

Finally, solution of fuzzy IE:

$$[u\_{un}(x).\, u\bar{}_{un}(x)] = 2x + (1 - \alpha) \odot \left(\frac{1}{e^2 - e}\right)\left[(e^{x+2} - e^{2x+1})[1 \cdot 5.3] + (e^{2x} - e^{x})[0.2]\right] \tag{46}$$

## 7- Conclusion

The utilization of RI, HI, and the connections between them in the fuzzy integral equation have all been proven. The algorithm was developed to find the Volterra fuzzy integral equation's solution. we consider a differential equation with fuzzy forcing function and with fuzzy initial values, assume the forcing function be in a special form, which we call triangular fuzzy function. We present solution as a fuzzy set of functions by some membership degree. We propose a method to find the fuzzy solution. The earlier work can be restated by taking into account several elements, such as picture fuzzy. Applications in engineering can use it, including Desalination of drinking water, which is the solubility of salt in water, is another application for preventing the spread of Corona by understanding the rate of infection spread, recovery rate, and rate of infection with the disease.

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