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# Solutions of Fractional differential equations with some modifications of Adomian Decomposition method

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**ABSTRACT**
In this paper, we apply the Adomian decomposition method (ADM) for solving Fractional Differential Equations (FDEs) with some modifications to the traditional method. The aim of this paper is to make ADM more efficient, rapid in convergence, and easy to use, so we will discuss two modifications. We use the reliable modification to simplify calculations. For difficulties in symbolic integration, we use a numerical implementation method. All these modifications were applied to the integer-order case, so we would apply it to FDEs. Some numerical results are given from solving these cases and comparing the solution with the ADM method.

***Keywords:*** *Fractional differential equations; Adomian decomposition Method; reliable modification; numerical implementation.*

## 1. Introduction

Fractional Differential equations (FDEs) have many applications in engineering and science such as electrical networks, fluid flow, control theory, fractals theory, electromagnetic theory, viscoelasticity, potential theory, chemistry, biology, optical and neural network systems [1]-[16]. In this paper, the Adomian decomposition method (ADM) [17]-[26] is used to solve some fractional differential equations which have difficulties in solving them. This method has many advantages; it is efficiently works with different types of linear and nonlinear equations in deterministic or stochastic fields and gives an analytic solution for all these types of equations without linearization or discretization.

This paper is organized as follows: In section two, ADM was applied to the problems with the Reliable modification technique [28] under consideration. In section three, some examples are introduced with this modification. In section four, we discussed the nnumerical Implementation of ADM [29,30]. Finally, the numerical results of this technique are obtained using the MATHEMATICA package.

## 2. Description of Reliable Modification Technique

The general form of the fractional differential equation takes the following form:

$$D^{\alpha}y(t) + Ry(t) + Ny(t) = g(t), \tag{1}$$

Subject to the initial conditions,

$$y^{(j)}(0) = c_j, \quad j = 0,1,2,\dots,n. \tag{2}$$

Where $\alpha$ is the highest order derivative, R is a linear differential operator of order less than $\alpha$, $Ny$ represents the nonlinear term, and g is the source term. Applying $I^{\alpha}$ to both sides of equation (1) with the definition of Caputo fractional derivative , and using the given conditions, we obtain

$$y(t) = f - I^{\alpha}(Ry(t)) - I^{\alpha}(Ny(t)), \qquad (3)$$

where the function $f$ represents the terms arising from integrating the source term g and using the given conditions.

By using the standard ADM we obtain,

$$y_0(t) = f$$

$$y_{K+1}(t) = -I^{\alpha}(Ry(t)) - I^{\alpha}(A_k),\ k \geq 0. \quad (4)$$

To achieve our modification, we assume that the function $f$ can be divided into two parts, namely $f_1$ and $f_2$. Under this assumption, we set

$$f = f_1 + f_2. \qquad (5)$$

Based on this, we make a minor change to only on the components $y_0$ and $y_1$. The variation we make is that only the part $f_1$ will be the first term $y_0$, whereas the remaining part $f_2$ be combined with the second term $y_1$. In view of these remarks, we formulate the modified recursive algorithm as follows,

$$y_0(t) = f_1,$$

$$y_1(t) = f_2 - I^{\alpha}(Ry_0(t)) - I^{\alpha}(A_0) \qquad (6)$$

$$y_{k+1(t)} = -I^{\alpha}\left(Ry_{k(t)}\right) - I^{\alpha}(A_k), \qquad k \geq 1.$$

The success of this method depends mainly on the proper choice of the parts $f_1$ and $f_2$ and unfortunately, until now there has been no way of choosing $f_1$ and $f_2$, the trials are the only criteria that can be applied.

## 3. Application of Using Reliable Modification Technique

***Example 1:*** Consider the following linear FDE of Bagley - Torvik

$$\frac{d^2y(t)}{dt^2} + \frac{d^{3/2}y(t)}{dt^{3/2}} + y(t) = t + 1, \qquad (7)$$

Subject to $y(0) = 1, y'(0) = 1$.

Operating with $I^2$ on both sides of equation (7) and then using the formula (8)

$$D^{-\mu}D^{\alpha}f(t) = D^{\alpha-\mu}f(t) - \sum_{k=0}^{l-1} f^{(k)}0^{+}\frac{t^{k+\mu-\alpha}}{\Gamma(\mu-\alpha+k+1)} \qquad (8)$$

we get:

$$y(t) = (1+t) - \left[ I^{\frac{1}{2}} y(t) - \frac{t^{\frac{1}{2}}}{\Gamma\left(\frac{3}{2}\right)} - \frac{t^{\frac{3}{2}}}{\Gamma\left(\frac{5}{2}\right)} \right] - I^2 y(t) + I^2(t+1). \ (9)$$

Applying the Reliable modification we obtain:

$$y_0(t) = 1 + t,$$

$$y_1(t) = \frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + \frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)} + I^2(t+1) - I^2 y_0(t) - I^{1/2} y_0(t)$$

$$= \frac{t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} + \frac{t^{3/2}}{\Gamma\left(\frac{5}{2}\right)} + I^2(t+1) - I^2(t+1) - I^{\frac{1}{2}}(t+1) = 0 \qquad (10)$$

We see that the solution of the this problem is $y(t) = \sum_{n=0}^{\infty} y_n(t) = y_0(t) = 1 + t$, which is the exact solution.

***Example 2:*** Consider the following linear FDE of Basset problem

$$Du(t) + 2D^{1/2}u(t) + \frac{1}{2}u(t) = 0, \quad u(0) = 1 \ (11)$$

Applying the reliable modification we obtain:

$$u_0(t) = 1,$$

$$u_1(t) = \frac{2\, t^{1/2}}{\Gamma\left(\frac{3}{2}\right)} - 2I^{1/2}[u_0(t)] - \frac{1}{2} I^1[u_0(t)]$$

$$u_{k+1}(t) = -2I^{\frac{1}{2}}[u_k(t)] - \frac{1}{2} I^1[u_k(t)], \qquad n \geq 2. \qquad (12)$$

From these relations, $(8)$ $and$ $(11)$, the first terms of the series solution by using the reliable modification and the classical method will be:

The modification terms:

$$u_0(t) = 1$$

$$u_1(t) = -\frac{t}{2}$$

$$u_2(t) = \frac{4\, t^{3/2}}{3\sqrt{\pi}} + \frac{t^2}{8}$$

$$u_3(t) = -t^2 - \frac{8t^{5/2}}{15\sqrt{\pi}} - \frac{t^3}{48}$$

… …

The classical ADM terms:

$$u_0(t) = 1 + \frac{4\sqrt{t}}{\sqrt{\pi}}$$

$$u_1(t) = -\sqrt{t}(27\sqrt{\pi}\sqrt{t} + 8(3+t))/6\sqrt{\pi}$$

$$u_2(t) = \frac{1}{120\sqrt{\pi}}(t\left(32\sqrt{t}(50+t)\right) + 15\sqrt{\pi}(32+17t))$$

$$u_3(t) = -\frac{(t^{\frac{3}{2}}(35\sqrt{\pi}\sqrt{t}(528+25t) + 64(280+189t+t^2)))}{1680\sqrt{\pi}}$$

$$\dots\dots$$

From the above, we see that the terms resulting from our modification are simpler than the others obtained by the classical method. Moreover, we reach the exact solution more rapidly than in the classical manner; that is clear in figures [1,2,3,4]

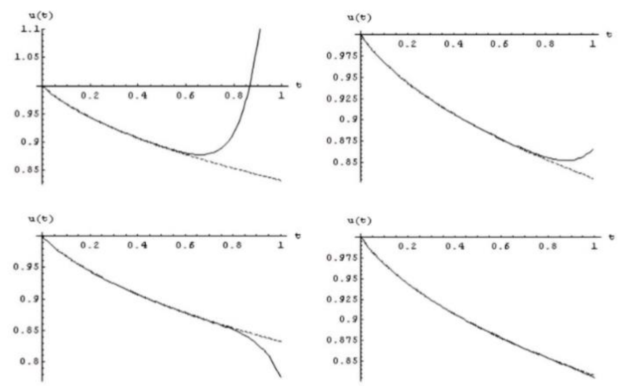

*Fig. 1,2,3,4.*

**4. Description of Numerical Implementation of ADM**

Consider the nonlinear FDE,

$$ {}_0D_t^{\alpha} y(t) + f(y) = g(t), \quad m-1 < \alpha \le m, \ \alpha \ge 1 \quad (13)$$

$$y^{(k)}(0) = c_k, \quad k = 0,1,2,\dots,m-1.$$

Operating with $I^{\alpha}$ on both sides of the equation (10) , hence we obtain

$$y(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + I^{\alpha}[g(t)] - I^{\alpha}[f(y)] \qquad (14)$$

Applying the classical ADM to the equation we get,

$$y_0(t) = h(t) \qquad (15)$$

$$y_{n+1}(t) = -I^{\alpha}[f(y)], \qquad (16)$$

Where $h(t) = \sum_{k=0}^{m-1} c_k \frac{t^k}{k!} + I^{\alpha}[g(t)]$.

Now, we will use the numerical method; which is given in [31-33], to approximate the integral term in the equation. We choose a regular mesh in t, thus setting $t = t_i = ih$ where $h = \frac{1}{n}$ is the fixed step length. Therefore, the integral can be approximated as

$$\int_0^{t_i} (t_i - x)^{\alpha-1} f(y_n(x))dx \simeq h \sum_{j=0}^{i} w_{ij} (t_i - x_j)^{\alpha-1} f\left(y_n(x_j)\right) \quad (17)$$

where $t_i = x_i,\ i = 0,1,2, \ldots n$. This leads to the following set of nonlinear equations,

$$y_{n+1}(t_i) \simeq \frac{h}{\Gamma(\alpha)} \sum_{j=0}^{i} w_{ij} (t_i - t_j)^{\alpha-1} f\left(y_n(t_j)\right),\ \ i, n = 0,1,2, \ldots \quad (18)$$

For choosing suitable weights $w_{ij}$, we note that for each $i$ , the set $w_{ij}$, j = 0, 1,..i represents the weights for an *(i + 1)*-points quadrature rule of Newton Cotes type for the interval *[0, ih]* . We implement the above idea in the following examples with n = 20 or h = $\frac{1}{20}$

## 5. Application Using Numerical Implementation of ADM

**Example 3**: Consider the following nonlinear FDE

$$D^{3/2} y(t) = t + \tan^{-1} y\,,\quad 0 < t < 1, \quad (19)$$

$$y(0) = 0, \qquad y'(0) = 0.$$

In this example, the integration of the nonlinear term ($\tan^{-1} y$) is difficult, so we will solve it by using the Numerical implementation technique. From the standard ADM, we have the following recursive relations,

$$y_0(t) = \frac{t^{5/2}}{\Gamma\left(\frac{7}{2}\right)},$$

$$y_{n+1} = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \int_0^t (t - x)^{1/2} \tan^{-1} y_n(x) dx. \quad (20)$$

Using our modification we get,

$$y_0(t_i) = \frac{{t_i}^{5/2}}{\Gamma\left(\frac{7}{2}\right)},$$

$$y_{n+1}(t_i) = \frac{h}{\Gamma\left(\frac{3}{2}\right)} \sum_{j=0}^{i} w_{ij} (t_i - t_j)^{\frac{1}{2}} \tan^{-1} y_n(t_j),\ \ i, n = 0,1,2, \ldots, 20. \quad (21)$$

The solution by using this modification is given in Figure 5

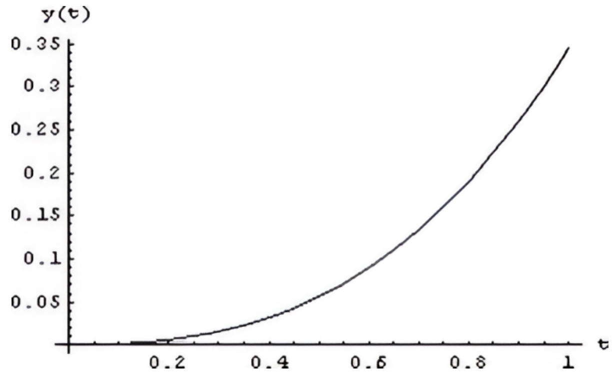


***Fig. 5***

**Example 4**: Consider the following nonlinear FDE

$$D^{7/2}y(t) = t^2 + e^{y^2}, \quad 0 < t < 1, \quad (22)$$

$$y(0) = 0, y'(0) = 0, \; y''(0) = 0, \; y'''(0) = 0.$$

In this example, the integration of the nonlinear term ($e^{y^2}$) is difficult, so we will solve it by using the Nnumerical implementation technique. From the standard ADM, we have the following recursive relations,

$$y_0(t) = \frac{2t^{11/2}}{\Gamma\left(\frac{13}{2}\right)},$$

$$y_{n+1} = \frac{1}{\Gamma\left(\frac{7}{2}\right)}\int_0^t (t-x)^{\frac{5}{2}} \exp\left(y_n{}^2(x)\right)dx. \qquad (23)$$

Using our modification we get,

$$y_0(t_i) = \frac{2t_i{}^{11/2}}{\Gamma\left(\frac{13}{2}\right)},$$

$$y_{n+1}(t_i) = \frac{h}{\Gamma\left(\frac{7}{2}\right)}\sum_{j=0}^{i} w_{ij}\left(t_i - t_j\right)^{\frac{5}{2}} \exp\left(y_n{}^2(t_j)\right), \; i, n = 0,1,2,\ldots,20. \qquad (24)$$

The solution by using this modification is given in Figure 6

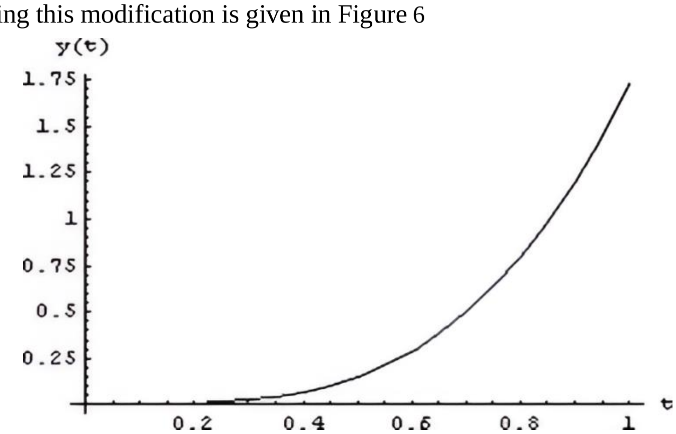


***Fig. 6***

**Conclusion**

In this paper, we use the ADM method to solve several problems of equations with fractional orders. Two modifications to ADM have been introduced. The first one is the Rreliable modification, which find that the series solution compared with the classical ADM was accelerated. Although this technique needs only a slight variation from the classical ADM, the size of the calculations is minimized. The second technique is the Nnumerical iimplementation of ADM, used due to the difficult calculations of the integration, also difficult or impossible to analyze.

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The author reports no conflicts of interest in this work.

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