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Mathematical Modelling of Fractional-order Covid-19 Pandemic With Memory Effect: A Review

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Abstract

Mathematical models with memory effect play an important role in the field of epidemiology. The fractional-order derivatives are powerful tools to characterize the memory effect in the dynamical systems of infectious diseases. Hence, we attempt to present a systematic survey on the fractional-order compartmental models of Covid-19 pandemic to explain how the fractional-order models have been employed to study and forecast the spread of Covid-19 pandemic. Such non-integer order models can help decision makers in control programs to put strategic plans to control Covid-19 outbreak.

Keywords: Fractional order models-Infectious diseases-COVID-19 pandemic.

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1. Introduction

(COVID-19) pandemic is one of the most serious global challenge during the last few years. It is one of the most deadly infectious diseases caused by Coronavirus [1]. It appeared firstly in Wuhan city, China, by the end of 2019 and has spread in the other countries around the world. Covid-19 pandemic is a great global threat as it caused millions of deaths and it caused enormous economic crisis in several countries during the great lockdown. Mathematical models are effective tools that can help to give a clear understanding of the behavior of COVID-19 pandemic. On the other hand, such models help decision makers to put strategic plans to control the spread of COVID-19 pandemic. Fractional order models can give better understanding of the behavior of COVID-10 pandemic [2]. Such models consider the impact of memory on the dynamics and spread of the pandemic. Motivated by this, in this paper, we present a quick review on different types of fractional-order models such as constant/variable order and discrete fractional order models for COVID-19 pandemic. Fractional order optimal control models and delayed fractional order models are presented as well. The rest of the paper is organized as follows. In section 2, some basic definitions of fractional order derivatives are presented while in section 3 we present some constant fractional order models for COVID-19. Some delayed fractional order models for COVID-19 are presented in section 4. We present some COVID-10 fractional order optimal control models in section 5 while in section 6 we discuss some fractional order discrete COVID-19 models. A Non-integer variable-order model of COVID-19 is presented in section 7. Section 9 is devoted to the conclusion of the paper.

2. Preliminaries

In this section, we give some definitions of variable-order fractional derivative which is an extension of constant-order fractional derivative [3-12]. There exist different approaches for defining the fractional derivatives.

Definition 1. The fractional integral of order $\alpha > 0$ of a function $f: R^+ \to R$ is given by

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \qquad \alpha > 0, x > 0,$$

$$J^0 f(x) = f(x).$$

Hence, we have

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} \mathrm{t}^{\alpha+\gamma}, \qquad \alpha > 0, \gamma > -1, t > 0$$

Definition 2. Riemann–Liouville and Caputo fractional derivatives of order $\alpha$of a continuous function $f: R^+ \to R$ is given respectively by

$$D_*^{\alpha} f(x) = D^m\left(J^{m-\alpha} f(x)\right),$$

$$D^{\alpha} f(x) = J^{m-\alpha}\left(D^m f(x)\right),$$

Where $m-1 < \alpha \le m, m \in N.$

Definition 3 (Riemann–Liouville fractional derivatives of order)
Let $\alpha(t)$ be a continuous and bounded function, then Riemann–Liouville variable-order fractional derivative of $f(t)$: $[a, b] \to \mathbb{R}$ is defined as:

i) Left Riemann–Liouville derivative of order $\alpha(t)$ is defined by

$${}_a^{RL}D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma\left(1-\alpha(t)\right)} \frac{d}{dt} \int_a^t (t-\tau)^{-\alpha(t)} f(\tau) d\tau, \qquad 0 < \alpha(t) \le 1$$

ii) Right Riemann–Liouville derivative of order $\alpha(t)$ is defined by

$${}_t^{RL}D_b^{\alpha(t)} f(t) = \frac{-1}{\Gamma\left(1-\alpha(t)\right)} \frac{d}{dt} \int_t^b (\tau-t)^{-\alpha(t)} f(\tau) d\tau, \qquad 0 < \alpha(t) \le 1$$

Definition 4 (Caputo fractional derivatives of order $\alpha(t)$) [13,14]

Let $\alpha(t)$ be a continuous and bounded function, then the Caputo variable-order fractional derivative of $f(t)$: $[a, b] \to \mathbb{R}$ is defined as:

i) Left Caputo derivative of order $\alpha(t)$ is defined by

$${}_a^{C}D_t^{\alpha(t)} f(t) = \frac{1}{\Gamma\left(1-\alpha(t)\right)} \int_a^t (t-\tau)^{-\alpha(t)} f'(\tau) d\tau, \qquad 0 < \alpha(t) \le 1$$

ii) Right Caputo fractional order derivative of order $\alpha(t)$ is defined by

$${}_t^{C}D_b^{\alpha(t)} f(t) = \frac{-1}{\Gamma\left(1-\alpha(t)\right)} \int_t^b (\tau-t)^{-\alpha(t)} f'(\tau) d\tau, \qquad 0 < \alpha(t) \le 1$$

Definition 5 The γ-Caputo fractional difference operator is defined as

$${}^{C}\Delta_{\theta}^{\gamma} H(v) = \Delta_{\theta}^{-(m-\gamma)} \Delta^m H(v) = \frac{1}{\Gamma(m-\gamma)} \sum_{\tau=\theta}^{v-(m-\gamma)} (v-1-\tau)^{(m-\gamma-1)} \Delta^m H(\tau)$$

3. Constant Fractional Order Models of COVID-19

Motivated by the importance of fractional order models of infectious disease, several constant fractional order models of COVID-19 have been presented during 2020 to 2023. Some examples of such models are presented in this section.

3.1 A fractional-order model of COVID-19 considering the fear effect of the media and social networks on the community

A compartmental fractional order model of COVID-19 is implemented in [15]. The authors in this paper studied the fear effect of the media and social media platforms on human societies during COVID-19 outbreak through SEIR+D model. The presented model is as follows.

$$D^{\alpha}[S(t)] = \Lambda + S(t)r\left(1 - \frac{S(t)}{K_1}\right)\frac{1}{1+\alpha_1 I(t)} - \beta_1 E(t)S(t) - \gamma_1 I(t)S(t) - \eta S(t)$$
$$D^{\alpha}[E(t)] = E(t)\left(1 - \frac{E(t)}{K_2}\right) + \beta_1(1-\varepsilon_1)E(t)S(t) - \theta E(t) - \eta E(t) - \mu E(t)$$
$$D^{\alpha}[I(t)] = I(t)\left(1 - \frac{I(t)}{K_3}\right) + \beta_1\varepsilon_1 E(t)S(t) + \gamma_1 I(t)S(t) + \theta E(t) - \beta_2 I(t) - \eta I(t) - \mu I(t)$$
$$D^{\alpha}[Q(t)] = Q(t)\left(1 - \frac{Q(t)}{K_4}\right)\frac{1}{1+\alpha_2 D(t)} + \beta_2 I(t) - \eta Q(t) - \mu Q(t) - \gamma_2 Q(t)$$
$$D^{\alpha}[R(t)] = \gamma_2 Q(t) - \eta R(t)$$
$$D^{\alpha}[D(t)] = \mu\big(E(t) + I(t) + Q(t)\big) - \mu_1 D(t)$$

Where $0 < \alpha \leq 1$. On the other hand S, E, I present the susceptible, exposed and infected groups repectively. Q is the quarantined people, while *R* presents the recovered group. The compartment *D* (*t*) presents the death class.

The parameters are given as follows.

α_1 The fear effect of the susceptible class to be infected by COVID-19

α_2 The fear effect of individuals under quarantine to die from COVID-19

β_1 Infection rate from the $S - E$ interaction

γ_1 Infection rate from the $S - I$ interaction

ε_1 Recognition of infection

β_2 The rate of infected people being isolated

γ_2 The rate of recovering from the infection

K_1 Carrying capacity of the susceptible class

K_2 Carrying capacity of the exposed class

K_3 Carrying capacity of the infected class of COVID-19

K_4 Carrying capacity of individuals under quarantine

Λ The rate of offspring per year

μ The death rate of COVID-19 infected

μ_1 The death rate of the infected group died from different symptoms that was activated by the virus COVID-19

η The natural death rate

The numerical simulation in this work proved that, health care management and public awareness are crucial to control the spread of COVID-19 virus.

The numerical simulation through the presented fractional order model in [15] explains the importance of the memory effect of the population behaviour during COVID-19 pandemic. In this work proved that, health care management and public awareness are crucial to control the spread of COVID-19 virus.

3.2 A fractional-order SEIHDR model for COVID-19 within inter-city networked coupling effects

The inter-city network coupling effects in different Chines cities have been studied in [16] through a fractional order SEIHDR model proposes a fractional SEIHDR model. The numerical results of the proposed fractional order model in [ ] have a better fitting with the real data more than the results of the corresponding integer order model. The presented fractional order model is presented as follows.

$$ {}_0^C D_t^\alpha S_K = -\sum_{j=1}^{n} \beta_{Kj} \left(\frac{S_K I_j}{N_K} + \frac{S_K E_j}{N_K}\right) $$

$$ {}_0^C D_t^\alpha E_K = \sum_{j=1}^{n} \beta_{Kj} \left(\frac{S_K I_j}{N_K} + \frac{S_K E_j}{N_K}\right) - \mu_{1K} E_K - r_K E_K $$

$$ {}_0^C D_t^\alpha I_K = r_K E_K - \delta_K I_K - \mu_{2K} I_K $$

$$ {}_0^C D_t^\alpha H_K = \delta_K I_K - \lambda_K(t) H_K - \kappa_K(t) H_K $$

$$ {}_0^C D_t^\alpha R_K = \lambda_K(t) H_K $$

$$ {}_0^C D_t^\alpha D_K = \mu_{1K} E_K - \mu_{2K} I_K + \kappa_K(t) H_K $$

S_K, E_K, I_K, H_K, R_K and D_K represent susceptible, exposed, infected individual, hospitalized, recovered and death groups.

The parameters are defined as follows:

Λ_K is the inflow number of susceptible group

$\lambda^\alpha{}_k$ be the recovery rate;

$r_k{}^\alpha$ imply the transit rate of the exposed class E_k

δ_k denote hospitalization rate

β^α , μ^α , δ^α and r^α are positive constants

$\lambda^\alpha{}_k(t)$ and $\kappa_k{}^\alpha(t)$ are bounded function

3.3 A fractional-order mathematical model for COVID-19 outbreak with the effect of symptomatic and asymptomatic transmissions

In [17], the authors studied the transmission of COVID-19 virus through a fractional compartmental fractional order model. The Caputo fractional order derivative is employed to describe the memory effect on the asymptomatic and symptomatic transmissions. The model includes five compartments S, E, I_1, I_2, and R which are defined respectively as susceptible, exposed, asymptomatic infected, symptomatic infected and recovered individuals. The propose model is as follows.

$$^{C}D^{\theta}S = \Lambda - \mu_1 S I_1(1 + \lambda_1 I_1) - \mu_2 S I_2(1 + \lambda_2 I_2) - \alpha S$$

$$^{C}D^{\theta}E = \mu_1 S I_1(1 + \lambda_1 I_1) + \mu_2 S I_2(1 + \lambda_2 I_2) - (\rho_1 + \rho_2)E$$

$$^{C}D^{\theta}I_1 = \rho_1 E - (\alpha_1 + \beta_1)I_1$$

$$^{C}D^{\theta}I_2 = \rho_2 E - (\alpha_2 + \beta_2)I_2$$

$$^{C}D^{\theta}R = \beta_1 I_1 - \beta_2 I_2 - \alpha R$$

The parameters are defined as follows.

S is susceptible class
E is exposed class,
I_1 asymptomatic infected,
I_2 symptomatic infected,
R is recovered or removed class
μ_1 rate of asymptomatic individuals transmit the infection
μ_2 rate of symptomatic individuals transmit the infection
ρ_1, ρ_2 are infection rates
λ_1 and λ_2 are the positive constants

4. Fractional-Order Models of COVID-19 with time delay.

4.1 A Fractional Order SEIRV Model with time delay

The authors in [18] proposed a SEIRV time-delayed fractional order model of COVID-19 pandemic. The single time delay in this model expresses the time needed to be recovered. The parameters of the model have been computed using real data from India during COVID-19 outbreak. Adams–Bashforth-Moulton method has been used to solve the proposed dynamical system,

The susceptible, exposed, infected, recovered, and vaccinated population (SEIRV) with a single delay incorporated in the

The proposed model is presented as follows

$$^{C}D_t^{v}(S(t)) = \Lambda - \beta S(t) I(t) - \mu_0 S(t) - \delta S(t - \eta_1)$$
$$^{C}D_t^{v}(E(t)) = \beta S(t) I(t) - (\mu_0 + \mu_1) E(t)$$
$$^{C}D_t^{v}(I(t)) = \mu_1 E(t) - (\mu_0 + \mu_2) I(t)$$
$$^{C}D_t^{v}(R(t)) = \mu_2 I(t) - \mu_0 R(t)$$
$$^{C}D_t^{v}(V(t)) = \delta S(t - \eta_1) - \mu_0 V(t)$$

Where:

The compartments S, E, I, R, and V are the susceptible, exposed, infected, recovered, and vaccinated groups respectively. The parameters are defined as follows.

$\delta\, S(t-\eta_1)$ is the susceptible individuals who were vaccinated at time $(t-\eta_1)$ and then entered the vaccinated individuals after time delay η_1.
Λ is the birth rate of susceptible individuals.
β is the infection rate of susceptible individuals
μ_0 is the mortality rate of infected individuals
δ is the rate of vaccination
μ_1 is the rate of progression from exposed to infected individuals
μ_2 is the recovery rate of infected individuals.

4.2 Fractional-Order COVID-19 Delayed Model with immune response

The authors in [19] provided a fractional-order model with time delay to study the dynamics of COVID-19 pandemic under immune system response. The fractional derivative is the index of memory while the time delay parameter τ represents the needed time for the reaction between the infected and effector cells.

The fractional derivative and the time delays are the parameters of memory that naturally represent the memory effects of the immune system. The model give better understand of the dynamics of the virus in the respiratory system. The proposed model is presented as follows.

$$\frac{d^\beta V(t)}{dt^\beta} = \gamma_V I(t) - \gamma_{VA} S(t)A(t)V(t) - \gamma_{VH} H(t)V(t) - \alpha_V V(t) - \frac{a_{V1}V(t)}{1 + a_{V2}V(t)},$$
$$\frac{d^\beta H(t)}{dt^\beta} = b_{HD} D(t)\big(H(t) + R(t)\big) + a_R R(t) - \gamma_{HV} V(t)H(t) - b_{HF} F(t)H(t),$$
$$\frac{d^\beta I(t)}{dt^\beta} = \gamma_{HV} V(t)H(t) - b_{IE} E(t\tau)I(t-\tau) - a_I I(t),$$
$$\frac{d^\beta M(t)}{dt^\beta} = (b_{MD} D(t) + b_{MV} V(t))(1 - M(t)) - a_M M(t),$$
$$\frac{d^\beta R(t)}{dt^\beta} = b_{HF} F(t)H(t) - a_R R(t),$$
$$\frac{d^\beta F(t)}{dt^\beta} = b_F M(t) + c_F I(t) - b_{FH} H(t)F(t) - a_F F(t),$$
$$\frac{d^\beta E(t)}{dt^\beta} = (b_{EM} M(t)E(t) - b_{EI} I(t-\tau)E(t-\tau) + a_E(1 - E(t)),$$
$$\frac{d^\beta P(t)}{dt^\beta} = b_{PM} M(t)P(t) + a_P(1 - P(t)),$$
$$\frac{d^\beta A(t)}{dt^\beta} = b_A P(t) - \gamma_{AV} S(t)A(t)V(t) - a_A A(t),$$
$$\frac{d^\beta S(t)}{dt^\beta} = rP(t)(1 - S(t)),$$
$$D(t) = 1 - I(t) - R(t) - H(t),$$
$where\ 0 < \beta \le 1$

$V(t)$ presents the concentration of free (COVID 19) virus. $H(t)$ is the proportion of healthy cells while $I(t)$ is the proportion of infected cells. $M(t)$ presents the activated anti gen presenting cells per homeostatic level while $F(t)$ presents the interferon per homeostatic level of macrophages. – $R(t)$, proportion of resistant cells;
$E(t)$, effector cells per homeostatic level,

$P(t)$, plasma cells per homeostatic level;
$A(t)$, concentration of antibodies per homeostatic level;
$S(t)$, antigenic distance;
$D(t)$, concentration of damaged cells in 1ml in the compartment of upper respiratory.

4.3 Time delay SEIR fractional COVID-19 model

In [20], a time delay SEIR fractional order model of COVID-19 via Caputo fractional derivatives is presented. Caputo fractional order derivative is implemented in the proposed model to study the memory effects on the numerical simulations while the time delay τ shows the impact of healthcare process. The numerical solutions have been obtained via predictor–corrector method. The presented system is as follows.

$$^{C}D_t^{\zeta} S(t) = b - \beta S(t) I(t) - dS(t),$$

$$^{C}D_t^{\zeta} E(t) = \beta S(t) I(t) - \gamma \beta S(t-\tau) I(t-\tau) e^{-d\tau} - dE(t) - \delta E(t),$$

$$^{C}D_t^{\zeta} I(t) = \gamma \beta S(t-\tau) I(t-\tau) e^{-d\tau} - [\nu_1 + \nu_2(1 - c(t))] I(t) - [\theta_1 + \theta_2 c(t)] I(t) - dI(t),$$

$$^{C}D_t^{\zeta} R(t) = [\theta_1 + \theta_2 c(t)] I(t) + \delta E(t) - dR(t),$$

$$^{C}D_t^{\zeta} D(t) = [\nu_1 + \nu_2(1 - c(t))] I(t),$$

Where $S(t)$, $E(t)$, $I(t)$, $R(t)$ and *D(t)* are the susceptible, exposed, infectious, recovered and deaths groups respectively.
The parameters are given as:
b is the Birth rate
β is the Contact rate
d Rate of Natural death
γ Exposed to infected Rate
δ Exposed to removed rate
c Health care systems opportunity level
θ_1 Natural recovery rate
θ_1 Recovery rate
ν_1 Minimum disease-induced death rate
ν_2 Maximum disease-induced death rate

5. Fractional order optimal control model for the COVID-19 Pandemic

5.1 Fractional order optimal control model for COVID-19 with two time dependents controls measures

A fractional order optimal control model for COVID-19 with two time dependents controls measures μ_1 and μ_2 is presented in [21]. μ_1 represents the measures that reduce the rate of contacts while μ_2 represents quarantine and treatment controlling processes. Numerical solutions are obtained using RK4 method. The model is presented as follows.

$${}_{0}^{C}D_t^{\alpha} S_b\,(t) = \lambda_b^{\alpha} - \mu_1^{\alpha} S_b - \beta_1^{\alpha} S_b I_b$$

$${}_{0}^{C}D_t^{\alpha} I_b(t) = \beta_1^{\alpha} S_b I_b - (\mu_2^{\alpha} + \delta_1^{\alpha})\, I_b - \beta_2^{\alpha} S_h I_b$$

$${}_{0}^{C}D_t^{\alpha} S_h(t) = \lambda_h^{\alpha} - \mu_3^{\alpha} S_h - \beta_2^{\alpha}(1 - u1) S_h I_b$$

$${}_{0}^{C}D_t^{\alpha} I_h(t) = \beta_2^{\alpha}(1 - u1) S_h I_b - (\mu_4^{\alpha} + \delta_2^{\alpha}) I_h - \beta_3^{\alpha}(1 - u2) I_h H_h$$

$${}_{0}^{C}D_t^{\alpha} H_h(t) = \beta_3^{\alpha}(1 - u2) I_h H_h - (\mu_5^{\alpha} + \delta_3^{\alpha}) H_h - \beta_4^{\alpha} H_h F_m$$

$${}_{0}^{C}D_t^{\alpha} F_m(t) = \beta_4^{\alpha} H_h F_m - (\mu_6^{\alpha} + \delta_4^{\alpha}) F_m - \beta_5^{\alpha} P_C F_m$$

$${}_{0}^{C}D_t^{\alpha} P_C(t) = \beta_5^{\alpha} P_C F_m - (\mu_7^{\alpha} + \delta_5^{\alpha}) P_C - \beta_6^{\alpha} P_C C_C$$

$${}_{0}^{C}D_t^{\alpha} C_C(t) = \beta_6^{\alpha} P_C C_C - (\mu_8^{\alpha} + \delta_6^{\alpha}) C_C$$

With the objective function given as:

$$J(u_1, u_2) = \int_0^{t_f} (a\, S_h + b I_h + c u_1^2 + d\, u_2^2)\, dt$$

Where S_h is the susceptible human population and I_h is the infected human population, t_f is the final time and the coefficients a, b, c, d are positive weights. Our aim is to minimize the susceptible and infected human populations while minimizing the cost of control u_1 , u_2 . Thus, we search for an optimal control u_1^*, u_2^* such that minimize

$$J(u_1^*, u_2^*) = {}_{u1}\, min_{u2} \{J(u_1, u_2) \mid u_1, u_2 \in \Omega\}$$

5.2 SEIR Fractional order optimal control model for COVID-19 with two controls [22].

The authors in [22] presents a SEIR Fractional order optimal control model for COVID-19 with two controls. The first control is the media awareness campaigns while the second is the quarantine. The proposed model consists of four compartments S(t) (susceptible), E(t) exposed, I(t) (infected), R(t) (removed) as follows.

$${}_{0}^{C}D_t^{\alpha}\, S(t) = \Lambda - \beta_1 S(t)E(t) - \beta_2 S(t)I(t) - \mu S(t) + \tau R(t) - u_1(t)S(t)$$
$${}_{0}^{C}D_t^{\alpha}\, E(t) = \beta_1 S(t)E(t) + \beta_2 S(t)I(t) - (\mu + \rho)E(t) - u_2(t)S(t)$$
$${}_{0}^{C}D_t^{\alpha} I(t) = \rho E(t) - (\gamma + d + \mu)I(t) + (1 - p)u_2(t)E(t)$$
$${}_{0}^{C}D_t^{\alpha} R(t) = \gamma I(t) - (\mu + \tau)R(t) + u_1(t)S(t) + p u_2(t)E(t)$$

The authors in [22] minimize the number of susceptible and infected groups, while maximizing the number of recovered group. $0 < \alpha \leq 1$ represents the memory and the learning behaviour of the population.

6. COVID-19 Fractional- Order Discrete Model

6.1 Fractional-order discrete COVID-19 pandemic model with new daily cases, additional severe cases and deaths

In [23], a new discrete Fractional-order discrete COVID-19 pandemic model is presented. It consists of three classes, the new daily cases C, new additional severe cases S and deaths D as follows.

$$^{C}\Delta_{\theta}^{\gamma_1}C(v)=\beta_1(D(v-1+\gamma_1))^2+\beta_2(C(v-1+\gamma_1))^2+\beta_3(S(v-1+\gamma_1)(D(v-1+\gamma_1)+\beta_4C(v-1+\gamma_1))$$

$$^{C}\Delta_{\theta}^{\gamma_2}S(v)=\beta_5C(v-1+\gamma_2)+\beta_6S(v-1+\gamma_2)+\beta_7(D(v-1+\gamma_2))^2$$

$$^{C}\Delta_{\theta}^{\gamma_3}D(v)=\beta_8C(v-1+\gamma_3)D(v-1+\gamma_3)+\beta_9C(v-1+\gamma_3)S(v-1+\gamma_3)+\beta_{10}D(v-1+\gamma_3)+\beta_{11}(C(v-1+\gamma_3))^2$$

6.2 COVID-19 Fractional order discrete model with vaccination

A Fractional order discrete model with vaccination is proposed in [24] to study the spread of COVID-10 pandemic. The fractional order derivatives $\boldsymbol{\gamma_1}$ and $\boldsymbol{\gamma_1}$ are used to characterize the memory in the proposed dynamical system. The authors in this paper deduced reasonable ranges for the fractional order derivatives $\boldsymbol{\gamma_1}$ and $\boldsymbol{\gamma_1}$. The system is presented as follows.

$$^{C}\Delta_{a}^{\gamma_1}S(s)=-pS(s-1+\gamma_1)+\frac{\alpha}{N}I(s-1+\gamma_1)S(s-1+\gamma_1)+\beta(N-S(s-1+\gamma_2)),$$

$$^{C}\Delta_{a}^{\gamma_2}I(s)=\frac{\alpha}{N}I(s-1+\gamma_2)S(s-1+\gamma_2)-(\beta+\sigma)I(s-1+\gamma_2),$$

Where S and I represent the susceptible and infected classes.

6.3 A Fractional-Order Discrete SIR Model for COVID-19

In [25], a fractional-order discrete SIR model is presented to study the dynamics of the COVID-19 pandemic in Germany. The proposed model can adapt to the periodic change in the number of infections. The model consists of three classes, S(t) (susceptible), I(t) (infected), R(t) (removed) as follows.

$$^{C}\Delta_{0}^{\alpha}S(t+1-\alpha)=\theta+\eta R(t)-bS(t)I(t)-\delta S(t),$$

$$^{C}\Delta_{0}^{\alpha}I(t+1-\alpha)=bS(t)I(t)-(\mu+\delta+e)I(t),$$

$$^{C}\Delta_{0}^{\alpha}R(t+1-\alpha)=eI(t)-(\delta+\eta)R(t),$$

The parameters are given as follows

μ Corona death rate
δ Natural death rate
θ The number of new births
b Infection rate
e Recovery rate
η The rate at which a recovering person is at risk of infection

Picard Lindelöf method has been used in this paper to investigate the existence and uniqueness of the presented model solution.

7. Fractional Variable Order Model of COVID-19 Pandemic

The fractional variable-order derivatives are considered are considered as the extension of the constant fractional-order derivatives. The fractional variable-order derivative is used to describe the memory that changes as a function of time. In other words, the variable order derivative is more generalization of the constant fractional order derivative. In [26], a Fractional Variable Order Model of COVID-19 Pandemic is proposed to describe dynamics of bats, hosts, people and seafood markets during the pandemic as follows.

$$\boldsymbol{D^{\alpha(t)}(S_P) = \Pi_P - \mu_p S_P - \frac{\eta_P S_P(I_P + \psi A_P)}{N_P} - \eta_w S_P M,}$$

$$\boldsymbol{D^{\alpha(t)}(E_P) = \frac{\eta_P S_P(I_P + \psi A_P)}{N_P} + \eta_w S_P M - (1 - \theta_P)\omega_P E_P - \theta_P \rho_P E_P - \mu_p E_P,}$$

$$\boldsymbol{D^{\alpha(t)}(I_P) = (1 - \theta_P)\omega_P E_P - (\tau_P + \mu_P) I_P,}$$

$$\boldsymbol{D^{\alpha(t)}(A_P) = \theta_P \rho_P E_P - (\tau_{aP} + \mu_P) A_P,}$$

$$\boldsymbol{D^{\alpha(t)}(R_P) = \tau_P I_P + \tau_{aP} A_P - \mu_P R_P}$$

$$\boldsymbol{D^{\alpha(t)}(M) = \varrho_P I_P + \bar{\omega}_P A_P - \pi M}$$

Where $0 < \alpha(t) \leq 1$.
N_p is the total population, while: S_p, E_p, I_p, A_p, R_p, and M represent the susceptible, exposed, symptomatically infected, asymptomatically infected, recovered/removed people, and the seafood market respectively. Numerical simulations indicate that using fractional variable-order derivative $\alpha(t)$ can give a clear description of the memory that changes over time.

8. Conclusion

In this work, several fractional order dynamical systems models of COVID-19 are presented in order to study the impact of the fractional order derivatives on the dynamical systems solutions of COVID-19. Such non-integer order derivatives are considered as the parameters of memory. Constant/variable fractional order models and constant fractional order models with time delay of COVID-19 are presented in this paper. In addition, discrete fractional order models and fractional order optimal control model for the COVID –19 pandemic are presented as well.

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