Electronic Journal of Mathematical Analysis and Applications
Vol. 10(2) July 2022, pp. 1-10.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

FIRST-ORDER ITERATIVE DIFFERENTIAL INCLUSION

DORIA AFFANE AND SAMIA GHALIA

Abstract. Through this article, we aim to identify sufficient conditions to study the existence of solutions to a perturbed first-order iterative differential inclusion with maximal monotone operator. We provided examples to demonstrate our results.

1. Introduction

Iterative differential equations are special types of the so-called state-dependent delay-differential equations. This type of equations appears in many fields such as biologic, physics, the engineering technique fields,... They have been extensively and intensively studied in the recent years. One of the first papers which developed the study of iterative differential equations comes back to E. Eder in [21], where the existence of the unique monotone solution for the 2-th iterative differential equation $\dot{u}(t) = u^{[2]}(t);\ u^{[2]}(t) = u(u(t))$ was given. Later, K. Wang [39] obtained a solution of the generally iterative differential equation $\dot{u}(t) = f(u^{[2]}(t)),\ u(T_0) = T_0$, where T_0 is one endpoint of the interval of existence, using Schauder's fixed point theorem. In [23], M. Fečkan showed the existence of local solutions via the contraction mapping principle for the initial value problem for the iterative differential equation $\dot{u}(t) = f(u^{[2]}(t));\ u(0) = 0$. The nonautonomous equation $\dot{u}(t) = f(t, u(t), u^{[2]}(t))$, a.e. $t \in [T_0, T],\ u(T_0) = u_0$, was investigated by A. Pelczar [33] using Picard's successive approximation. In [15] V. Berinde applied the nonexpansive operators to studied the same problem and extended the existence results given in [18]. Also, we mention the paper [42], where P. Zhang and X. Gong established the existence of solutions for general iterative differential equation

$$\begin{cases} \dot{u}(t) = f(t, u(t), u^{[2]}(t), \cdots, u^{[n]}(t)), & \text{a.e.} \quad t \in [T_0, T]; \\ u(T_0) = u_0, \end{cases}$$

where $n \in \mathbb{N}$ and for $1 < i \leq n,\ u^{[i]}(t) = u(u^{[i-1]}(t))$. There were also quite a number of papers and research deal it see for example [22, 24, 26, 27, 29, 30, 31, 34, 35, 36, 37, 40, 41].

2010 *Mathematics Subject Classification.* 34K35, 28B20, 34K35.
Key words and phrases. Iterative differential equations, differential inclusion, maximal monotone operator, perturbation.
Submitted July 21, 2021. Revised Aug. 12, 2021.

On the other hand, differential inclusions have received great interest from researchers who have used them in studying many situations including differential variational inequalities, projected dynamical systems, Moreau's sweeping process, linear and nonlinear complementarity dynamical systems, discontinuous ordinary differential equations. For example, S. Aizicovici and V. Staicu [11] proved the existence of integral solutions to the nonlocal Cauchy problem $\dot{u}(t) \in -Au(t) + F(t, u(t)),\ u(0) = g(u)$ in a Banach space X. Later, the authors in [1, 5] studied the existence of solutions of a boundary second order differential inclusion under conditions that are strictly weaker than the usual assumption of convexity on the values of the right-hand side. For more details, see the papers [9, 25, 32]. Others have also been interested in the study of differential inclusions with operators, see the papers [2, 3, 4, 6, 7, 8, 10, 14] and references therein.
Motivated by the above discussions, the main purpose of this paper is to consider sufficient conditions for studying the existence of solutions to the problem

$$(\mathcal{I}) \begin{cases} \dot{u}(t) = f(t, u(t), u^{[2]}(t), \cdots, u^{[n]}(t), u^{[n]}(\alpha t)), & \text{a.e.} \quad t \in [T_0, T]; \\ u(T_0) = u_0, \end{cases}$$

where $u_0 \in [T_0, T]$ and $\alpha \in]0, 1[$. Moreover, we assume a new problem, which is a perturbed iterative differential inclusion with maximal monotone operators

$$(\mathcal{II}) \begin{cases} -\dot{u}(t) \in A(t)u^{[n]}(t) + f(t, u(t), u^{[2]}(t), \cdots, u^{[n]}(t), u^{[n]}(\alpha t)), & \text{a.e. } t \in [T_0, T]; \\ u(T_0) = u_0, \end{cases}$$

and we prove the existence of solutions.
The present paper is organized as follows: After providing some notation and preliminaries; in Section 3, we provide the existence of solutions for problem $(\mathcal{I})$ using Schauder's fixed point theorem, where f is a bounded Carathéodory mapping. Then, we generalize the first result to the perturbed iterative differential inclusion with maximal monotone operators $(\mathcal{II})$. We provide two examples to demonstrate our results.

2. Notations and preliminaries

Throughout all the paper, $[T_0, T]$ $(T_0 \leq 0 \leq T)$ is an interval of $\mathbb{R}$ the set of real numbers. We denote by $L^1_{\mathbb{R}}([T_0, T])$ the space of measurable mappings $u : [T_0, T] \to [T_0, T]$ such that $\int_{T_0}^t |u(t)|dt < +\infty$ with the norm $\|u\|_{L^1_{\mathbb{R}}} = \int_{T_0}^t |u(t)|dt$, by $\mathcal{C}([T_0, T])$ the Banach space of all continuous mappings $u : [T_0, T] \to [T_0, T]$ equipped with the sup-norm and $\mathcal{C}^1([T_0, T])$ the Banach space of all continuous mappings with continuous derivative. For extensive information on these concepts, see the book [16].
Now, we give the definition and some properties of the maximal monotone operator. We refer the reader to [12], [13] and [17] for this concept.
A set-valued mapping $A(t) : \mathbb{R} \rightrightarrows \mathbb{R}$ $(t \in [T_0, T])$ is monotone if and only if

$$\forall x_1, x_2 \in D(A(t)) : \ (A(t)x_1 - A(t)x_2)(x_1 - x_2) \geq 0.$$

If $A(t)$ is monotone and $\mathcal{R}(I + \lambda A(t)) = \mathbb{R}$, we say that $A(t)$ is maximal monotone, here, $D(A(t)) = \{x \in \mathbb{R} : \ A(t)x \neq \emptyset\}$ is the domain of $A(t)$, and $\mathcal{R}(I + \lambda A(t))$ is the range of $(I + \lambda A(t))$.
Let $\lambda > 0$, we denote by $J_\lambda(t) = (I + \lambda A(t))^{-1}$ the resolvent and $A_\lambda(t) = \frac{1}{\lambda}(I - J_\lambda(t))$ the Yosida approximation of $A(t)$.

Proposition 2.1. *Let $A(t) : D(A) \subset \mathbb{R} \rightrightarrows \mathbb{R}$ $(t \in [T_0, T])$ be a maximal monotone operator and $\lambda > 0$. Then*

(1) *$A_\lambda(t)$ is single valued, maximal monotone and Lipschitzean with constant $\frac{2}{\lambda}$ on $\mathcal{R}(I + \lambda A(t))$;*
(2) *$A_\lambda(t)x \in AJ_\lambda(t)x$, $\forall x \in \mathcal{R}(I + \lambda A(t))$;*
(3) *$\dfrac{1}{\lambda}|J_\lambda A(t)x - x| = |A_\lambda(t)x| \le |A(t)x|_0$, $\forall x \in \mathcal{R}(I + \lambda A(t)) \cap D(A(t))$,*

where $|A(t)x|_0 = \inf\{|y|;\ y \in A(t)x\}$, is the element of $A(t)x$ of minimal norm.

The following theorems are very important in proving our results.

Theorem 2.1. [19] *(Scorza Dragoni theorem)*
Let J be a compact metric space, (J, Σ, ν) a Radon measure space. Let X a complete separable metric space, E a finite dimensional space and $h : J \times X \to E$ a Carathéodory function. So, for all real $\varepsilon > 0$, there exists a compact $J_\varepsilon \subset J$ such that $\nu(J \backslash J_\varepsilon) < \varepsilon$ and the restriction from h to $J_\varepsilon \times X$ is continuous.

Theorem 2.2. [28] *(Schauder)*
Let S be a nonempty closed convex subset of a Banach space and let $G : S \to S$ be continuous. If $G(S)$ is relatively compact, then G has a fixed point in S.

3. The main results

3.1. Existence result for a first-order iterative differential equation.

Theorem 3.1. *Let $f : [T_0, T]^{n+2} \to \mathbb{R}$ be a mapping such that:*

i) *for any $x \in [T_0, T]^{n+1}$ fixed, $f(\cdot, x)$ is Lebesgue measurable on $[T_0, T]$;*
ii) *for any $t \in [T_0, T]$ fixed, $f(t, \cdot)$ is continuous on $[T_0, T]^{n+1}$;*
iii) *there is a nonnegative function $m \in L^1_{\mathbb{R}}([T_0, T])$ such that*

$$|f(t, x)| \le m(t),\ \forall (t, x) \in [T_0, T]^{n+2}.$$

Then, the problem $(\mathcal{I})$ has an absolutely continuous solution.

Proof. **Step1**. Suppose that f is continuous on $[T_0, T]^{n+2}$. Let S be a subset defined by

$$S = \{v \in \mathcal{C}([T_0, T]) :\ v \text{ has a continuous derivative and } \|v\|_{\mathcal{C}} \le m_1\}$$

where $m_1 = |u_0| + \|m\|_{L^1_{\mathbb{R}}}$. It is clear that S is a closed convex subset of $\mathcal{C}^1([T_0, T])$. For all $v \in S$, the problem

$$(P_{f,v}) \begin{cases} \dot{u}(t) = f(t, v(t), v^{[2]}(t), \cdots, v^{[n]}(\alpha s)), & \text{a.e. } t \in [T_0, T]; \\ u(T_0) = u_0, \end{cases}$$

admits a solution $u_v \in \mathcal{C}^1([T_0, T])$ defined by

$$u_v(t) = u_0 + \int_{T_0}^{t} f(s, v(s), v^{[2]}(s), \cdots, v^{[n]}(s), v^{[n]}(\alpha s))ds.$$

Consider the mapping $P : v \mapsto u_v$ defined on S with values in $\mathcal{C}([T_0, T])$ by $P(v) = u_v$. Let us show that $u_v \in S$. We have u is derivative continuous and for all

$t \in [T_0, T]$

$$|u_v(t)| \leq |u_0| + \int_{T_0}^{t} |f(s, v(s), v^{[2]}(s), \cdots, v^{[n]}(\alpha s))| ds$$
$$= |u_0| + \int_{T_0}^{t} m(s) ds = |u_0| + \|m\|_{L^1_{\mathbb{R}}} = m_1.$$

Then,

$$\|u\|_{\mathcal{C}} \leq m_1. \tag{3.1}$$

Let (v_r) be a sequence of elements of S converging to v in S. Then, $(v_r^{[i]})$ converges to $v^{[i]}$ $(i = 2, 3, \cdots, n)$ and we have

$$|u_{v_r}(t) - u_v(t)| \leq \int_{T_0}^{t} |f(s, v_r(s), v_r^{[2]}(s), \cdots, v^{[n]}(\alpha s)) - f(s, v(s), v^{[2]}(s), \cdots, v^{[n]}(\alpha s))| ds.$$

Since f is continuous, so $\big(|f(\cdot, v_r(\cdot), v_r^{[2]}(\cdot), \cdots, v_r^{[n]}(\alpha\cdot)) - f(\cdot, v(\cdot), v^{[2]}(\cdot), \cdots, v^{[n]}(\alpha\cdot))|\big)_r$ converging to 0 when $r \to +\infty$, then

$$\|P(v_r) - P(v)\| = \|u_{v_r} - u_v\|_{\mathcal{C}} \to 0 \text{ when } r \to +\infty.$$

Hence we have the continuity of P.
Now, let us prove that $P(S)$ is relatively compact in $\mathcal{C}([T_0, T])$. For all $t, \tau \in [T_0, T]$, we have

$$|u_v(t) - u_v(\tau)| \leq \int_{\tau}^{t} |f(s, v(s), v^{[2]}(s), \cdots, v^{[n]}(\alpha s))| ds \leq \int_{\tau}^{t} m(s) ds.$$

As $m \in L^1_{\mathbb{R}}([T_0, T])$, we obtain the equicontinuity of the set $\{u_v : v \in S\}$.
On the other hand, for all $v \in S$ and all $t \in [T_0, T]$, $|\dot{u}_v(t)| \leq m(t)$, by the relation (3.1), it is clear that $\{u_v(t) : v \in S\}$ is relatively compact in $[T_0, T]$. The Arzelà-Ascoli theorem gives us its relative compactness in $\mathcal{C}([T_0, T])$. From where $P(S) = \{u_v : v \in S\}$ is relatively compact in $\mathcal{C}([T_0, T])$. The Theorem 2.2 allows us to conclude that P admits a fixed point which is in fact the solution to the problem under consideration.
Step2. Suppose that f satisfies the hypotheses of Theorem 3.1. Let $\varepsilon > 0$, according to the Theorem 2.1, there exists a compact set $J_\varepsilon \subset [T_0, T]$ such that the Lebesgue measure of $([T_0, T] \setminus J_\varepsilon)$ is less than ε and the restriction g_ε of f to $J_\varepsilon \times [T_0, T]^{n+1}$ is continuous. Hence, the existence of an increasing sequence of compact sets (J_r) in $[T_0, T]$ such that the Lebesgue measure of $([T_0, T] \setminus J_r)$ tends to 0 when $r \to \infty$ and the restriction g_r of f to $J_r \times [T_0, T]^{n+1}$ is continuous.
Let $\tilde{f}_r$ be the Dugundji continuous extension of g_r to $[T_0, T]^{n+2}$. We apply the arguments of the demonstration of Step 1 to each $\tilde{f}_r$; we obtain for all $r \in \mathbb{N}$ a solution u_r of the problem

$$\begin{cases} \dot{u}_r(t) = \tilde{f}_r(t, u_r(t), u_r^{[2]}(t), \cdots, u_r^{[n]}(\alpha t)), \ \forall t \in [T_0, T]; \\ u_r(T_0) = u_0. \end{cases}$$

We have for all $r \in \mathbb{N}$ and all $t \in [T_0, T]$, $|\dot{u}_r(t)| \leq m(t)$. So, we can extract from the sequence $(\dot{u}_r(\cdot))$ a subsequence converging weakly* in $L^\infty_{\mathbb{R}}([T_0, T])$ to a function $w(\cdot)$.
On the other hand, we have

$$u_r(t) = u_0 + \int_{T_0}^{t} \dot{u}_r(s) ds,$$

then for all $t, \tau \in [T_0, T]$

$$|u_h(t) - u_h(\tau)| \leq \int_\tau^t m(s)ds$$

therefore the sequence $(u_r(\cdot))$ is equicontinuous and relatively compact. According to Arzelà-Ascoli's theorem (u_r) is relatively compact in $\mathcal{C}([T_0, T])$. By extracting a subsequence we may (u_r) converges uniformly to a function u satisfying $u(T_0) = u_0$. We have to show that

$$\dot{u}(t) = f(t, u(t), u^{[2]}(t), \cdots, u^{[n]}(\alpha t)), \text{ a.e. } t \in [T_0, T].$$

By construction, for each $r \in \mathbb{N}$, there is a set $\mathcal{N}_r$ of negligible Lebesgue measure, such as

$$\dot{u}_r(t) = f(t, u_r(t), u_r^{[2]}(t), \cdots, u_r^{[n]}(\alpha t)), \ \forall t \in J_r \setminus \mathcal{N}_r.$$

Let $\mathcal{N}_0 = ([T_0, T] \setminus \cup J_r) \cup (\cup \mathcal{N}_r)$ which is Lebesgue-negligible. If $t \notin \mathcal{N}_0$, there is an integer $p = p(t)$ such that

$$\dot{u}_r(t) = f(t, u_r(t), u_r^{[2]}(t), \cdots, u_r^{[n]}(\alpha t)), \ \forall \ r \geq p,$$

this relation gives us

$$\limsup_{r\to\infty} \langle x', \dot{u}_r(t)\rangle = \limsup_{r\to\infty} \langle x', f(t, u_r(t), u_r^{[2]}(t), \cdots, u_r^{[n]}(\alpha t))\rangle$$
$$\leq \langle x', f(t, u(t), u^{[2]}(t), \cdots, u^{[n]}(\alpha t))\rangle,$$

for all $x' \in \mathbb{R}$ and $r \geq p$. As $(\dot{u}_r)$ converges weakly* to $\dot{u}$ in $L^\infty_{\mathbb{R}}([T_0, T])$, we get for any set $A \subset [T_0, T]$,

$$\int_A \langle x', \dot{u}(t)\rangle dt = \lim_{r\to\infty} \int_A \langle x', \dot{u}_r(t)\rangle dt,$$

using Fatou's lemma, we get

$$\int_A \langle x', \dot{u}(t)\rangle dt = \int_A \limsup_{r\to\infty} \langle x', \dot{u}_r(t)\rangle dt \leq \int_A \langle x', f(t, u(t), u^{[2]}(t), \cdots, u^{[n]}(\alpha t))\rangle dt,$$

so

$$\langle x', \int_A \dot{u}(t)dt\rangle = \langle x', \int_A f(t, u(t), u^{[2]}(t), \cdots, u^{[n]}(\alpha t))dt\rangle,$$

then,

$$\dot{u}(t) = f(t, u(t), u^{[2]}(t), \cdots, u^{[n]}(\alpha t)), \ a.e. \ t \in [T_0, T].$$

3.2. **Existence result for a first-order iterative differential inclusion.** For our proof, we need the following lemma.

Lemma 3.1. [42] *Let*

$$\Phi_{\mathcal{K}} = \{u \in \mathcal{C}([T_0, T]) : |u(t) - u(s)| \leq \mathcal{K}|t - s|, \ \forall t, s \in [T_0, T]\},$$

where $0 < \mathcal{K} < 1$. *If* $\varphi, \psi \in \Phi_{\mathcal{K}}$, *then*

$$\|\varphi^{[j]} - \psi^{[j]}\|_{\mathcal{C}} \leq \frac{1 - \mathcal{K}^j}{1 - \mathcal{K}} \|\varphi - \psi\|_{\mathcal{C}}, \ j = 1, 2, \cdots.$$

Theorem 3.2. *Let* $f : [T_0, T]^{n+2} \to \mathbb{R}$ *be a function satisfies the hypothesis* $i)$ *and* $ii)$ *of Theorem 3.1 and* $A(t) : \mathbb{R} \rightrightarrows \mathbb{R}$ $(t \in [T_0, T])$ *be a maximal monotone operator. Suppose that the following assumptions hold:*

($\mathcal{H}_1$) *for all* $y \in [T_0, T]$ *and all* $\lambda > 0$, $t \mapsto J_\lambda A(t)y$ *is Lebesgue measurable and there exists* $\bar{g} \in L^2_{\mathbb{R}}([T_0, T])$ *such that* $t \mapsto J_\lambda A(t)\bar{g}(t)$ *belongs to* $L^2_{\mathbb{R}}([T_0, T])$;

($\mathcal{H}_2$) *there exists a function* $m \in L^2_{\mathbb{R}}([T_0, T])$ *such that* $\|m\|_{L^2_{\mathbb{R}}} < 1$ *and*

$$|A(t)y|_0 + |f(t,x)| \leq m(t),\ \forall (t,x,y) \in [T_0,T]^{n+2}.$$

Then, the problem ($\mathcal{II}$) *admits a solution.*

Proof. We consider the mapping

$$g_r(t,x) = A_{\lambda_r}(t)y + f(t,x),\ \forall (t,x) \in [T_0,T]^{n+2},$$

where (λ_r) is a decreasing sequence in $]0,1[$ converges to 0 when $r \to \infty$.
According to the property 3) of the Proposition 2.1 and hypothesis ($\mathcal{H}_2$), we have

$$\begin{aligned}|g_r(t,x)| &\leq |A_{\lambda_r}(t)y| + |f(t,x)| \\ &\leq |A(t)y|_0 + |f(t,x)| \leq m(t).\end{aligned}$$

Note that hypothesis ($\mathcal{H}_1$) and property 1) in Proposition 2.1 implies that $(t,y) \mapsto A_{\lambda_r}(t)y$ is a Carathéodory mapping. By applying Theorem 3.1, we obtain for all $r \in \mathbb{N}$, the existence of a solution u_r for the differential equation

$$(P_{g_r}) \begin{cases} -\dot{u}_r(t) = g_r(t, u_r(t), u_r^{[2]}(t), \cdots, u_r^{[n]}(\alpha t)) \text{ a.e. } t \in [T_0,T]; \\ u_r(T_0) = u_0, \end{cases}$$

with

$$u_r(t) = u_0 + \int_{T_0}^t g_r(s, u_r(s), u_r^{[2]}(s), \cdots, u_r^{[n]}(\alpha s))ds.$$

By applying the arguments of the proof of Theorem 3.1, we conclude that $(u_r(\cdot))$ is relatively compact. By extracting a subsequence, we may $(u_r(\cdot)), (u_r^{[i]}(\cdot))$ and $(u_r^{[i]}(\alpha\cdot))$ uniformly converge to $u(\cdot)), u^{[i]}(\cdot)$ and $u^{[i]}(\alpha\cdot)$ with $u(T_0) = u_0$ and that $(\dot{u}_r(\cdot))$ converges $\sigma(L^2_{\mathbb{R}}([T_0,T]), L^2_{\mathbb{R}}([T_0,T]))$ to $\dot{u}(\cdot)$.
On the other hand, by the hypotheses on f we have $(f(\cdot, u_r(\cdot), u_r^{[2]}(\cdot), \cdots, u_r^{[n]}(\alpha\cdot)))_r$ converges to the function $f(\cdot, u(\cdot), u^{[2]}(\cdot), \cdots, u^{[n]}(\alpha\cdot))$ a.e. and also

$$|f(t, u_r(t), u_r^{[2]}(t), \cdots, u_r^{[n]}(\alpha t))| \leq m(t),\ \forall t \in [T_0,T].$$

by Lebesgue's theorem, we conclude that

$$|f(t, u(t), u^{[2]}(t), \cdots, u^{[n]}(\alpha t))| \leq m(t),$$

$(f(\cdot, u_r(\cdot), u_r^{[2]}(\cdot), \cdots, u_r^{[2]}(\alpha\cdot)))_r$ converges to the function $f(\cdot, u(\cdot), u^{[2]}(\cdot), \cdots, u^{[n]}(\alpha\cdot))$ in $L^2_{\mathbb{R}}([T_0,T])$ and therefore this convergence is true for the weak topology.
According to property 2) of Proposition 2.1, we have for a.e. $t \in [T_0,T]$,

$$-\dot{u}_r(t) - f(t, u_r(t), u_r^{[2]}(t), \cdots, u_r^{[n]}(\alpha t)) = A_{\lambda_r}(t)u_r^{[n]}(t) \in A(t)J_{\lambda_r}A(t)u_r^{[n]}(t). \quad (3.2)$$

On the other hand, we have

$$|J_{\lambda_r}A(t)u_r^{[n]}(t) - u^{[n]}(t)| \leq |J_{\lambda_r}A(t)u_r^{[n]}(t) - u_r^{[n]}(t)| + |u_r^{[n]}(t) - u^{[n]}(t)|. \quad (3.3)$$

Using property 3) of Proposition 2.1 and hypothesis (H_2), we obtain

$$|J_{\lambda_r}A(t)u_r^{[n]}(t) - u_r^{[n]}(t)| = \lambda_r|A_{\lambda_r}(t)u_r^{[n]}(t)| \leq \lambda_r|A(t)u_r^{[n]}(t)|_0 \leq \lambda_r m(t). \quad (3.4)$$

We have $\lambda_r m(t) \to 0$ when $r \to \infty$. By the relation (3.4), we can see that

$$|J_{\lambda_r}A(t)u_r^{[n]}(t) - u_r^{[n]}(t)| \to 0 \text{ when } r \to \infty,$$

and so

$$|J_{\lambda_r}A(t)u_r^{[n]}(t) - u^{[n]}(t)| \to 0 \text{ when } r \to \infty.$$

By the relations (3.2), (3.3) and (3.4) we have

$$|J_{\lambda_r}A(t)u_r^{[n]}(t) - u^{[n]}(t)| \leq |J_{\lambda_r}A(t)u_r^{[n]}(t) - u_r^{[n]}(t)| + |u_r^{[n]}(t) - u^{[n]}(t)|.$$

Using Lemma 3.1, we get

$$\begin{aligned}|J_{\lambda_r}A(t)u_r^{[n]}(t) - u^{[n]}(t)| &\leq \lambda_r m(t) + \frac{1-\|m\|^n_{L^1_{\mathbb{R}}}}{1-\|m\|_{L^1_{\mathbb{R}}}}\|u_r - u\|_{\mathcal{C}} \\ &\leq \lambda_r m(t) + \frac{1-\|m\|^n_{L^1_{\mathbb{R}}}}{1-\|m\|_{L^1_{\mathbb{R}}}}(\|u_r\|_{\mathcal{C}} + \|u\|_{\mathcal{C}}) \\ &\leq \lambda_r m(t) + 2\frac{1-\|m\|^n_{L^1_{\mathbb{R}}}}{1-\|m\|_{L^1_{\mathbb{R}}}}(|u_0| + \|m\|_{L^1_{\mathbb{R}}}).\end{aligned}$$

Since $\lambda_r < 1$, for all $r \in \mathbb{N}$ we obtain for a.e. $t \in [T_0, T]$,

$$|J_{\lambda_r}A(t)u_r^{[n]}(t) - u^{[n]}(t)| < m(t) + \frac{2}{1-\|m\|_{L^1_{\mathbb{R}}}}(|u_0| + \|m\|_{L^1_{\mathbb{R}}}).$$

As $m \in L^2_{\mathbb{R}}([T_0, T])$, we conclude by using Lebesgue's theorem that $J_{\lambda_r}A(t)u_r^{[n]}(\cdot)$ converges to $u^{[n]}(\cdot)$ in $L^2_{\mathbb{R}}([T_0, T])$.
Let $\mathcal{A} : L^2_{\mathbb{R}}([T_0, T]) \rightrightarrows L^2_{\mathbb{R}}([T_0, T])$ be an operator defined by

$$v \in \mathcal{A}u^{[n]} \Leftrightarrow v(t) \in A(t)u^{[n]}(t) \ \text{ a.e. } t \in [T_0, T].$$

Using the proof of Lemma 3.1 in [20] and thanks to hypothesis $(\mathcal{H}_1)$ we conclude that $\mathcal{A}$ is a maximal monotone operator in $L^2_{\mathbb{R}}([T_0, T])$ by ([38], Theorem 1.5.2) its graph is sequentially strongly-weakly closed.
As $(\dot{u}_r(\cdot) + f(\cdot, u_r(\cdot), u_r^{[2]}(\cdot), \cdots, u_r^{[n]}(\alpha\cdot)))_r$ converges $\sigma(L^2_{\mathbb{R}}, L^2_{\mathbb{R}})$ to $\dot{u}(\cdot) + f(\cdot, u(\cdot), u^{[2]}(\cdot), \cdots, u^{[n]}(\alpha\cdot))$, we conclude, by relation(3.2) that the problem $(\mathcal{II})$ admits a solution.

4. Applications

Example 4.1. *Consider the following problem*

$$(P_1)\begin{cases}-\dot{u}(t) \in \partial|u^{[2]}(t)| + \frac{1}{8}t\left(\cos(u(t)) + \sin(u^{[2]}(\frac{t}{2}))\right), \ \text{a.e. } t \in [-\frac{\pi}{2}, \frac{\pi}{2}]; \\ u(0) = 0.\end{cases}$$

where the set-valued mapping

$$\partial|x_2| = \begin{cases}-\frac{1}{2} & \text{if } \ x_2 < 0, \\ \frac{1}{2} & \text{if } \ x_2 > 0, \\ [-\frac{1}{2}, \frac{1}{2}] & \text{if } \ x_2 = 0,\end{cases}$$

is a maximal monotone operator. The function $f(t, x_1, x_2, x_3) = \frac{1}{8}t(\cos x_1 + \sin x_3)$, for $(t, x_1, x_2, x_3) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^4$ satisfies the hypotheses i) and ii) of Theorem 3.1. Let us show that the hypotheses $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$ of Theorem 3.2 are satisfied.

$(\mathcal{H}_1)$ *for all $\lambda > 0$,*

$$J_\lambda \partial|x_2| = \left(I + \lambda\partial|x_2|\right)^{-1} = \begin{cases}0 & \text{if } \ x_2 \in [-\frac{\lambda}{2}, \frac{\lambda}{2}], \\ \frac{1-\lambda}{2} & \text{if } \ x_2 \geq \frac{\lambda}{2}, \\ \frac{1+\lambda}{2} & \text{if } \ x_2 \leq -\frac{\lambda}{2}.\end{cases}$$

Therefore, $t \mapsto J_\lambda \partial |x_2|$ *is Lebesgue measurable and there exists* $\bar{g} \in L^2_{\mathbb{R}}([-\frac{\pi}{2}, \frac{\pi}{2}])$ *such that* $t \mapsto J_\lambda \partial |\bar{g}(t)|$ *belongs to* $L^2_{\mathbb{R}}([-\frac{\pi}{2}, \frac{\pi}{2}])$.

$(\mathcal{H}_2)$ *For all* $(t, x_1, x_2, x_3) \in [-\frac{\pi}{2}, \frac{\pi}{2}]^4$, *we have*

$$\big|\partial |x_2|\big|_0 + |f(t, x_1, x_2, x_3)| \le m(t) = \frac{1}{4}t + \frac{1}{2}, \text{ with } \|m\|_{L^2_{\mathbb{R}}} < 1.$$

The hypotheses of Theorem 3.2 are satisfied, then (P_1) *has a solution.*

Example 4.2. *Let* $C : [-1, 1] \rightrightarrows [-1, 1]$ *be a set-valued mapping and consider the problem*

$$(P_2) \begin{cases} -\dot{u}(t) \in \partial I_{C(t)}(u^{[3]}(t)) + \frac{1}{4}t(u(t) + u^{[2]}(t) + u^{[3]}(t)) + \frac{1}{5}u^{[3]}(\frac{t}{3}), & a.e. \quad t \in [-1, 1]; \\ u(0) = 0, \end{cases}$$

where

$$I_{C(t)}(x_3) = \begin{cases} 0 & \text{if } x_3 \in C(t), \\ +\infty & \text{if } x_3 \notin C(t). \end{cases}$$

For all $\lambda > 0$, *we have*

$$\partial I_{C(t)}(x_3) = \begin{cases} \mathbb{R}_- & \text{if } x_3 = -1; \\ \mathbb{R}_+ & \text{if } x_3 = 1; \\ 0 & \text{if } x_3 \in]-1, 1[. \end{cases}$$

Hence

$$J_\lambda \partial I_{C(t)}(x_3) = \begin{cases} x_3 & \text{if } x_3 \in [-1, 1]; \\ 1 & \text{if } x_3 \ge 1; \\ -1 & \text{if } x_3 \le -1. \end{cases}$$

Therefore, $t \mapsto J_\lambda \partial I_{C(t)}(x_3)$ *is Lebesgue measurable and there exists* $\bar{g} \in L^2_{\mathbb{R}}([-1, 1])$ *such that* $t \mapsto J_\lambda \partial I_{C(t)}(\bar{g}(t))$ *belongs to* $L^2_{\mathbb{R}}([-1, 1])$.
For all $(t, x_1, x_2, x_3) \in [-1, 1]^4$, *we put*

$$f(t, x_1, x_2, x_3, x_4) = \frac{1}{4}t(x_1 + x_2 + x_3) + \frac{1}{5}x_4$$

which is a Carathéodory mapping, since $|\partial I_{C(t)}(x_3)|_0 = \{0\}$, *we get*

$$|\partial I_{C(t)}(x_3)|_0 + |f(t, x_1, x_2, x_3, x_3)| \le m(t) = \frac{3}{4}t + \frac{1}{5}, \text{ with } \|m\|_{L^2_{\mathbb{R}}} < 1.$$

The hypotheses of Theorem 3.2 are satisfied, then (P_2) *has a solution.*

Acknowledgment

Research supported by the General direction of scientific research and technological development (DGRSDT) under project PRFU No. C00L03UN180120180001.

References

[1] D. Affane, Quelques Problèmes de Contrôle Optimal pour des Inclusions Différentielles, Ph.D. thesis, MSBY University of Jijel, Algeria, 2012.

[2] D. Affane, M. Aissous and M. F. Yarou, Existence results for sweeping process with almost convex perturbation. Bull. Math. Soc. Sci. Math. Roumanie, Vol. 2, 119–134, 2018.

[3] D. Affane, M. Aissous and M. F. Yarou, Almost mixed semi-continuous perturbation of Moreau's sweeping process. Evol. Equ. Control Theory, Vol.1, 27–38, 2020.

[4] D. Affane and D. Azzam-Laouir, A control problem governed by a second-order differential inclusion. Applicable Analysis, Vol. 88, 1677-1690, 2009.

[5] D. Affane and D. Azzam-Laouir, Second-order differential inclusions with almost convex right-hand sides. Electronic Journal of Qualitative Theory of Differential Equations, 1-14, 2011.

[6] D. Affane, S. Boudada and M. F. Yarou, Unbounded perturbation to time-dependent subdifferential operators with delay. EJMAA, Vol. 8(2), 209-219, 2020.

[7] D. Affane and L. Boulkemh, Topological properties for a perturbed first order sweeping process. Acta Univ. Sapientiae, Mathematica, Vol. 13, 1-22, 2021.

[8] D. Affane and M. F. Yarou, Well-posed control problems related to second-order differential inclusions. Evol. Equ. Control Theory, 21 pages, doi:10.3934/eect.2021042.

[9] D. Affane and M. F. Yarou, Second-order perturbed state-dependent sweeping process with subsmooth sets. Computational Mathematics and Applications. Springer, Singapore, 147-169, 2020.

[10] D. Affane and M. F. Yarou, Unbounded perturbation for a class of variational inequalities, Discuss. Math. Diff. inclu. control optim., Vol. 37, 83-99, 2017.

[11] S. Aizicovici and V. Staicu, Multivalued evolution equations with nonlocal initial conditions in Banach spaces. Nonlinear Differential Equations and Applications NoDEA, Vol. 14, 361-376, 2007.

[12] J. P. Aubin and A. Cellina, Differential inclusions: set-valued maps and viability theory. Springer-Verlag, Berlin, 1984.

[13] V. Barbu, Nonlinear semigroups and differential equations in Banach spaces. Noordho Int. Publ. Leyden, 1976.

[14] V. Barbu, The Cauchy Problem in Banach Spaces. Nonlinear Differential Equations of Monotone Types in Banach Spaces. Springer, New York, NY 127-192, 2011.

[15] V. Berinde, Existence and approximation of solutions of some first order iterative differential equations. Miskolc Math. Notes, Vol. 1, 13-26, 2010.

[16] V.I. Bogachev, Measure theory. Springer Science and Business Media, 2007.

[17] H. Brézis, Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North Holland, 1973.

[18] A. Buică, Existence and continuous dependence of solutions of some functional-differential equations. Seminar on Fixed Point Theory, 3, 1-14, 1995.

[19] C. Castaing, Une nouvelle extension du théorème de Dragoni-Scorza. C.R. Acad. Sc. Série A, Paris 271, 396-398, 1970.

[20] C. Castaing, A.G. Ibrahim and M. F. Yarou, Existence problems in second order evolution inclusions: discretization and variational approach. Taiwanese J. Math., Vol. 12, 1433-1475, 2008.

[21] E. Eder, The functional differential equation $\dot{x}(t) = x(x(t))$. J. Diff. Equ., Vol. 54, 390-400, 1984.

[22] A. M. A. El-Sayed and R. G. Ahmed, Solvability of the functional integro-differential equation with self-reference and state-dependence. J. Nonlinear Sci. Appl, Vol. 13, 1-8, 2020.

[23] M. Fečkan, On a certain type of functional differential equations. Math. Slovaca, Vol. 43, 39-43, 1993.

[24] W. Ge W and Y. Mo, Existence of solutions to differential iterative equation. J. Beijing Inst. Tech., Vol. 3, 192-200, 1997.

[25] A. Granas and M. Frigon, Topological methods in differential equations and inclusions. Springer Science and Business Media, 2012.

[26] E.R. Kaufmann, Existence and uniqueness of solutions for a second-order iterative boundary-value problem. Electron. J. Differential Equations, Vol. 150, 1-6, 2015.

[27] S. D. Kendre, V. V . Kharat and R. Narute, On existence of solution for iterative integrodifferential equations. Nonlinear Anal. Differ. Equ., Vol. 3, 123-131, 2015.
[28] M. Kisielewicz, Differential inclusions and optimal control. PWN-Polish Scientific Publishers. Kluwer Academic Publishers. Dordrecht/Boston/London.
[29] M. Kostić, Almost periodic and almost automorphic solutions to integro-differential equations. W. de Gruyter, Berlin, 2019.
[30] M. Lauran, Existence results for some differential equations with deviating argument. Filomat Vol. 25, 2131, 2011.
[31] B. Liu and C. Tun, Pseudo almost periodic solutions for a class of first order differential iterative equations. Applied Mathematics Letters, Vol. 40, 29-34, 2015.
[32] N. S. Papageorgiou and V. Staicu, The method of upperlower solutions for nonlinear second order differential inclusions. Nonlinear Analysis: Theory, Methods and Applications, Vol. 67, 708-726, 2007.
[33] A. Pelczar, On some iterative differential equations I. Zeszyty Naukowe Uniwersytetu Jagiellonskiego, Prace Matematyczne, Vol. 12, 53-56, 1968.
[34] G. T. Stamov, Almost periodic solutions of impulsive differential equations. Springer-Verlag, Berlin, 2012.
[35] I. Stamova and G. Stamov, Applied impulsive mathematical models. Springer International Publishing, Cham, 2016.
[36] S. Unhaley and S. Kendre, On existence and uniqueness results for iterative fractional integro-differential equation with deviating arguments. Appl. Math. E-Notes, Vol. 19, 116-127, 2019.
[37] S. I. Unhale and S. D. Kendre, On existence and uniqueness results for iterative mixed integrodifferential equation of fractional order. Journal of Applied Analysis, Vol. 2, 263-272, 2020.
[38] I. L. Vrabie, Compactness methods for nonlinear evolutions. Pitman Monographs and Surveys in Pure and Applied mathematics. Longman Scientific and Technical. John Wiley and Sons, Inc. New York, 1987.
[39] K. Wang, On the equation $\dot{x}(t) = f(x(x(t)))$. Funk. Ekva, Vol. 33, 405-425, 1990.
[40] D. Yang D and W. Zhang, Solution of equivariance for iterative differential equations. Appl. Math. Lett., Vol. 17, 759-765, 2004.
[41] P. Zhang, Analytic solutions for iterative functional differential equations. Electron. J. Differential Equations, 180, 1-7, 2012.
[42] P. Zhang and X. Gong, Existence of solutions for iterative differential equations. Electron. J. Differential Equations, Vol. 7, 1-10, 2014.

Doria Affane
LMPA Laboratory, Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
E-mail address: affanedoria@yahoo.fr

Samia Ghalia
LMPA Laboratory, Department of Mathematics, Mohamed Seddik Ben Yahia University, Jijel, Algeria
E-mail address: ghalia.samia02@gmail.com