Electronic Journal of Mathematical Analysis and Applications
Vol. 10(2) July 2022, pp. 11-34.
ISSN: 2090-729X(online)
http://math-frac.org/Journals/EJMAA/

STEPANOV AND WEYL CLASSES OF MULTI-DIMENSIONAL ρ-ALMOST PERIODIC TYPE FUNCTIONS

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Abstract. In this paper, we analyze Stepanov and Weyl classes of multi-dimensional ρ-almost periodic type functions $F : I \times X \to Y$, where $n \in \mathbb{N}$, $\emptyset \neq I \subseteq \mathbb{R}^n$, X and Y are complex Banach spaces and ρ is a binary relation on Y, working in the general setting of Lebesgue spaces with variable exponents. We provide the main structural characterizations for the introduced classes of functions and apply our results to the abstract Volterra integro-differential equations.

1. Introduction and preliminaries

The notion of an almost periodic function was introduced by H. Bohr [6] around 1925 and later generalized by many others. Let I be either $\mathbb{R}$ or $[0, \infty)$, let $c \in \mathbb{C} \setminus \{0\}$ satisfy $|c| = 1$, and let $f : I \to X$ be a given continuous function, where X is a complex Banach space equipped with the norm $\| \cdot \|$. If $\varepsilon > 0$, then a number $\tau > 0$ is called a (ε, c)-period for $f(\cdot)$ if and only if $\|f(t+\tau) - cf(t)\| \leq \varepsilon$, $t \in I$. The set of all (ε, c)-periods for $f(\cdot)$ is denoted by $\vartheta_c(f, \varepsilon)$. The function $f(\cdot)$ is said to be c-almost periodic if and only if for each $\varepsilon > 0$ the set $\vartheta_c(f, \varepsilon)$ is relatively dense in $[0, \infty)$, i.e., there exists $l > 0$ such that any subinterval of $[0, \infty)$ of length l meets $\vartheta_c(f, \varepsilon)$. The usual notion of almost periodicity (almost anti-periodicity) is obtained by plugging $c = 1$ ($c = -1$). For further information concerning almost periodic functions, we refer the reader to the research monographs [5], [7], [13]-[14], [17]-[19], [25], [27] and [29].

The notion of $(w, \mathbb{T})$-periodicity for a continuous function $f : [0, \infty) \to X$, where $\omega > 0$ and $\mathbb{T} : X \to X$ is a linear isomorphism, has recently been introduced by M. Fečkan, K. Liu and J. Wang in [11, Definition 2.2]: a continuous function $f : [0, \infty) \to X$ is called $(w, \mathbb{T})$-periodic if and only if $f(t+\omega) = \mathbb{T}f(t)$ for all $t \geq 0$. The authors have analyzed the existence and uniqueness of $(w, \mathbb{T})$-periodic solutions for various classes of impulsive evolution equations using the strongly continuous

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2010 *Mathematics Subject Classification.* 42A75, 43A60, 47D99.
Key words and phrases. Stepanov multi-dimensional ρ-almost periodic type functions, Weyl multi-dimensional ρ-almost periodic type functions, abstract Volterra integro-differential equations.
Submitted Aug. 25, 2021. Revised Oct. 7,2021.
Marko Kostić is partially supported by grant 451-03-68/2020/14/200156 of Ministry of Science and Technological Development, Republic of Serbia.

semigroups, the Fredholm alternative type theorems and the fixed point theorems. In a joint research article [12] with M. Fečkan, M. T. Khalladi and A. Rahmani, the author of this paper has investigated the basic properties of multi-dimensional ρ-almost periodic type functions with values in complex Banach spaces.

On the other hand, in [2], A. Chávez, K. Khalil, M. Kostić and M. Pinto have analyzed multi-dimensional almost periodic functions of form $F : I \times X \to Y$, where $(Y, \| \cdot \|_Y)$ is a complex Banach spaces and $\emptyset \neq I \subseteq \mathbb{R}^n$; the multi-dimensional c-almost periodic type functions have recently been investigated in [16]. For more details about multi-dimensional Stepanov (Weyl) (c-)almost periodic type functions, multi-dimensional almost automorphic type functions and their applications, we refer the reader to the forthcoming research monograph [19] by M. Kostić; see also the research articles [1], [3]-[4], and [22]-[24].

The organization and main ideas of this paper can be briefly described as follows. After giving some preliminaries about multi-dimensional ρ-almost periodic functions (Subsection 1.1), the basic results and definitions about the Lebesgue spaces with variable exponent (Subsection 1.2), the multi-dimensional Bochner transform, the Stepanov distance and the Stepanov norm (Subsection 1.3), we introduce and analyze the multi-dimensional Stepanov ρ-almost periodic functions in Section 2. The main purpose of Section 3 is to introduce and analyze several various classes of the multi-dimensional Weyl ρ-almost periodic functions. In the final section of paper, we provide certain applications of our theoretical results to the abstract Volterra integro-differential equations. We feel it is our duty to emphasize that we do not present proofs for many structural results clarified below since these proofs can be obtained by insignificant modifications of already known proofs of corresponding results from our former research studies. **Notation and terminology.** Suppose that X, Y, Z and T are given non-empty sets. Let us recall that a binary relation between X and Y is any subset $\rho \subseteq X \times Y$. If $\rho \subseteq X \times Y$ and $\sigma \subseteq Z \times T$ with $Y \cap Z \neq \emptyset$, then we define $\rho^{-1} \subseteq Y \times X$ and $\sigma \cdot \rho = \sigma \circ \rho \subseteq X \times T$ by $\rho^{-1} := \{(y, x) \in Y \times X : (x, y) \in \rho\}$ and $\sigma \circ \rho := \{(x, t) \in X \times T : \exists y \in Y \cap Z \text{ such that } (x, y) \in \rho \text{ and } (y, t) \in \sigma\}$, respectively. As is well known, the domain and range of ρ are defined by $D(\rho) := \{x \in X : \exists y \in Y \text{ such that } (x, y) \in X \times Y\}$ and $R(\rho) := \{y \in Y : \exists x \in X \text{ such that } (x, y) \in X \times Y\}$, respectively; $\rho(x) := \{y \in Y : (x, y) \in \rho\}$ $(x \in X)$, $x\ \rho\ y \Leftrightarrow (x, y) \in \rho$. If ρ is a binary relation on X and $n \in \mathbb{N}$, then we define ρ^n inductively; $\rho^{-n} := (\rho^n)^{-1}$. Set $\rho(X') := \{y : y \in \rho(x) \text{ for some } x \in X'\}$ $(X' \subseteq X)$ and $\mathbb{N}_n := \{1, \cdot \cdot \cdot, n\}$ $(n \in \mathbb{N})$.

We assume henceforth that $(X, \| \cdot \|)$, $(Y, \| \cdot \|_Y)$ and $(Z, \| \cdot \|_Z)$ are three complex Banach spaces, $n \in \mathbb{N}$, $\mathcal{B}$ is a certain collection of subsets of X satisfying that for each $x \in X$ there exists $B \in \mathcal{B}$ such that $x \in B$. By $L(X, Y)$ we denote the Banach space of all linear continuous functions from X into Y; $L(X) \equiv L(X, X)$. We will always use the principal branch of the exponential function to take the powers of complex numbers. If $\mathbf{t_0} \in \mathbb{R}^n$ and $\epsilon > 0$, then we set $B(\mathbf{t}_0, \epsilon) := \{\mathbf{t} \in \mathbb{R}^n : |\mathbf{t} - \mathbf{t_0}| \leq \epsilon\}$, where $| \cdot |$ denotes the Euclidean norm in $\mathbb{R}^n$. Set $I_M := \{\mathbf{t} \in I : |\mathbf{t}| \geq M\}$ $(I \subseteq \mathbb{R}^n;\ M > 0)$. Generally, if $F(\cdot)$ is a function, then we set $\check{F}(\cdot) := F(-\cdot)$.

We will use the following definition from [2]:

Definition 1 Suppose that $\mathbb{D} \subseteq I \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded. By $C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ we denote the vector space consisting of all continuous functions $Q : I \times X \to Y$ such that, for every $B \in \mathcal{B}$, we have $\lim_{t \in \mathbb{D}, |t| \to +\infty} Q(\mathbf{t}; x) = 0$, uniformly for

$x \in B$. If $X = \{0\}$, then we abbreviate $C_{0,\mathbb{D},\mathcal{B}}(I \times X : Y)$ to $C_{0,\mathbb{D},\mathcal{B}}(I : Y)$; furthermore, if $\mathbb{D} = I$, then we omit the term "$\mathbb{D}$" from the notation.

1.1. **Multi-dimensional ρ-almost periodic type functions.** In [12], we have recently introduced and analyzed the following notion:

Definition 2 Suppose that $\emptyset \neq I' \subseteq \mathbb{R}^n$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $F : I \times X \to Y$ is a continuous function, ρ is a binary relation on Y and $I + I' \subseteq I$. Then we say that:

(i) $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I', \rho)$-almost periodic if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exists $l > 0$ such that for each $\mathbf{t}_0 \in I'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap I'$ such that, for every $\mathbf{t} \in I$ and $x \in B$, there exists an element $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ such that

$$\left\| F(\mathbf{t} + \tau; x) - y_{\mathbf{t};x} \right\|_Y \leq \epsilon.$$

(ii) $F(\cdot;\cdot)$ is $(\mathcal{B}, I', \rho)$-uniformly recurrent if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_k) in I' such that $\lim_{k\to+\infty} |\tau_k| = +\infty$ and that, for every $\mathbf{t} \in I$ and $x \in B$, there exists an element $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ such that

$$\lim_{k\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \left\| F(\mathbf{t} + \tau_k; x) - y_{\mathbf{t};x} \right\|_Y = 0.$$

It is clear that the Bohr $(\mathcal{B}, I', \rho)$-almost periodicity of $F(\cdot;\cdot)$ implies the $(\mathcal{B}, I', \rho)$-uniform recurrence of $F(\cdot;\cdot)$; the converse statement is not true in general ([19]). In the case that $\rho = T : Y \to Y$ is a single-valued function (not necessarily linear or continuous), then we obtain the most important case for our further investigations, when the function $F(\cdot;\cdot)$ is $(\mathcal{B}, I', T)$-almost periodic, resp. $(\mathcal{B}, I', T)$-uniformly recurrent. In the case that $X = \{0\}$ $(I' = I)$, we omit the term "$\mathcal{B}$" ("I'") from the notation; furthermore, if $T = c\mathrm{I}$ for some complex number $c \in \mathbb{C} \setminus \{0\}$, then we also say that the function $F(\cdot;\cdot)$ is $(\mathcal{B}, I', c)$-almost periodic, resp. $(\mathcal{B}, I', c)$-uniformly recurrent. Further on, we say that the function $F(\cdot;\cdot)$ is almost periodic (uniformly recurrent) if and only if $F(\cdot;\cdot)$ is $(\mathcal{B}, I', c)$-almost periodic, resp. $(\mathcal{B}, I', c)$-uniformly recurrent with $I' = I$ and $c = 1$; the corresponding notion of almost anti-periodicity (uniform anti-recurrence) is obtained by plugging $I' = I$ and $c = -1$. We will use the following results from [12]:

Lemma 1

(i) Suppose that $\emptyset \neq I' \subseteq \mathbb{R}^n$, $\emptyset \neq I \subseteq \mathbb{R}^n$, $I + I' \subseteq I$, and the function $F : I \times X \to Y$ is Bohr $(\mathcal{B}, I', \rho)$-almost periodic ($(\mathcal{B}, I', \rho)$-uniformly recurrent), where ρ is a binary relation on Y satisfying $R(F) \subseteq D(\rho)$ and $\rho(y)$ is a singleton for any $y \in R(F)$. If for each $\tau \in I'$ we have $\tau + I = I$, then $I + (I' - I') \subseteq I$ and the function $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I' - I', \mathrm{I})$-almost periodic ($(\mathcal{B}, I' - I', \mathrm{I})$-uniformly recurrent).

(ii) Suppose that $\emptyset \neq I' \subseteq \mathbb{R}^n$, and the function $F : \mathbb{R}^n \times X \to Y$ is Bohr $(\mathcal{B}, I', \rho)$-almost periodic ($(\mathcal{B}, I', \rho)$-uniformly recurrent), where ρ is a binary relation on Y satisfying $R(F) \subseteq D(\rho)$ and $\rho(y)$ is a singleton for any $y \in R(F)$. Then the function $F(\cdot;\cdot)$ is Bohr $(\mathcal{B}, I' - I', \mathrm{I})$-almost periodic ($(\mathcal{B}, I' - I', \mathrm{I})$-uniformly recurrent).

(iii) Suppose that $\rho = T \in L(Y)$ is a linear isomorphism.

   (a) Suppose that $\emptyset \neq I \subseteq \mathbb{R}^n$, $I + I \subseteq I$, I is closed, $F : I \times X \to Y$ is Bohr $(\mathcal{B}, T)$-almost periodic and $\mathcal{B}$ is any family of compact subsets of

X. If

$$(\forall l>0)\,(\exists \mathbf{t_0}\in I)\,(\exists k>0)\,(\forall \mathbf{t}\in I)(\exists \mathbf{t_0'}\in I)\\ (\forall \mathbf{t_0''}\in B(\mathbf{t_0'},l)\cap I)\,\mathbf{t}-\mathbf{t_0''}\in B(\mathbf{t_0},kl)\cap I, \tag{1}$$

then for each $B\in\mathcal{B}$ we have that the set $\{F(\mathbf{t};x):\mathbf{t}\in I,\ x\in B\}$ is relatively compact in Y; in particular, $\sup_{\mathbf{t}\in I;x\in B}\|F(\mathbf{t};x)\|_Y<\infty$.

(b) Suppose that $\emptyset\neq I\subseteq\mathbb{R}^n$, $I+I\subseteq I$, I is closed and $F:I\times X\to Y$ is Bohr $(\mathcal{B},T)$-almost periodic, where $\mathcal{B}$ is a family consisting of some compact subsets of X. If the following condition holds

$$(\exists \mathbf{t_0}\in I)\,(\forall\epsilon>0)(\forall l>0)\,(\exists l'>0)\,(\forall \mathbf{t'},\ \mathbf{t''}\in I)\\ B(\mathbf{t_0},l)\cap I\subseteq B(\mathbf{t_0}-\mathbf{t'},l')\cap B(\mathbf{t_0}-\mathbf{t''},l'), \tag{2}$$

then for each $B\in\mathcal{B}$ the function $F(\cdot;\cdot)$ is uniformly continuous on $I\times B$.

(iv) Suppose that $\emptyset\neq I'\subseteq\mathbb{R}^n$, $\emptyset\neq I\subseteq\mathbb{R}^n$, $I+I'\subseteq I$ and $F:I\times X\to Y$ is a $(\mathcal{B},I',\rho)$-uniformly recurrent function, where $\rho=T\in L(Y)$ is a linear isomorphism. Then for each real number $a>0$ we have:

$$\sup_{\mathbf{t}\in I,x\in B}\big\|F(\mathbf{t};x)\big\|_Y\leq\sup_{\mathbf{t}\in I+I',|\mathbf{t}|\geq a,x\in B}\big\|T^{-1}F(\mathbf{t};x)\big\|_Y,$$

and for each $x\in X$ we have

$$\sup_{\mathbf{t}\in I}\big\|F(\mathbf{t};x)\big\|_Y\leq\sup_{\mathbf{t}\in I+I',|\mathbf{t}|\geq a}\big\|T^{-1}F(\mathbf{t};x)\big\|_Y,$$

so that the function $F(\cdot;x)$ is identically equal to zero provided that the function $F(\cdot;\cdot)$ is $(\mathcal{B},I',\rho)$-uniformly recurrent and $\lim_{|\mathbf{t}|\to+\infty,\mathbf{t}\in I+I'}F(\mathbf{t};x)=0$.

We will use the following definitions, as well ([12]):

Definition 3 Suppose that $\mathbb{D}\subseteq I\subseteq\mathbb{R}^n$, the set $\mathbb{D}$ is unbounded, $\emptyset\neq I'\subseteq\mathbb{R}^n$, $\emptyset\neq I\subseteq\mathbb{R}^n$, $F:I\times X\to Y$ is a continuous function, ρ is a binary relation on Y and $I+I'\subseteq I$. Then we say that the function $F(\cdot;\cdot)$ is (strongly) $\mathbb{D}$-asymptotically Bohr $(\mathcal{B},I',\rho)$-almost periodic, resp. (strongly) $\mathbb{D}$-asymptotically $(\mathcal{B},I',\rho)$-uniformly recurrent, if and only if there exists a Bohr $(\mathcal{B},I',\rho)$-almost periodic function, resp. $(\mathcal{B},I',\rho)$-uniformly recurrent function, $(F_0:\mathbb{R}^n\times X\to Y)$ $F_0:I\times X\to Y$ and a function $Q\in C_{0,\mathbb{D},\mathcal{B}}(I\times X:Y)$ such that $F(\mathbf{t};x)=F_0(\mathbf{t};x)+Q(\mathbf{t};x)$, $\mathbf{t}\in I$, $x\in X$.

The functions $F_0(\cdot;\cdot)$ and $Q(\cdot;\cdot)$ are usually called the principal part of $F(\cdot;\cdot)$ and the corrective (ergodic) part of $F(\cdot;\cdot)$, respectively.

Definition 4 Suppose that $\mathbb{D}\subseteq I\subseteq\mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset\neq I'\subseteq\mathbb{R}^n$, $\emptyset\neq I\subseteq\mathbb{R}^n$, $F:I\times X\to Y$ is a continuous function, $I+I'\subseteq I$ and ρ is a binary relation on X. Then we say that:

(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Bohr $(\mathcal{B},I',\rho)$-almost periodic of type 1 if and only if for every $B\in\mathcal{B}$ and $\epsilon>0$ there exist $l>0$ and $M>0$ such that for each $\mathbf{t}_0\in I'$ there exists $\tau\in B(\mathbf{t}_0,l)\cap I'$ such that, for every $\mathbf{t}\in I$ and $x\in B$ with $\mathbf{t},\ \mathbf{t}+\tau\in\mathbb{D}_M$, there exists an element $y_{\mathbf{t},x}\in\rho(F(\mathbf{t};x))$ such that

$$\big\|F(\mathbf{t}+\tau;x)-y_{\mathbf{t},x}\big\|_Y\leq\epsilon.$$

(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically $(\mathcal{B}, I', \rho)$-uniformly recurrent of type 1 if and only if for every $B \in \mathcal{B}$ there exist a sequence (τ_k) in I' and a sequence (M_k) in $(0, \infty)$ such that $\lim_{k\to+\infty} |\tau_k| = \lim_{k\to+\infty} M_k = +\infty$ and that, for every $\mathbf{t} \in I$ and $x \in B$, there exists an element $y_{\mathbf{t};x} \in \rho(F(\mathbf{t}; x))$ such that

$$\lim_{k\to+\infty} \sup_{\mathbf{t},\mathbf{t}+\tau_k\in\mathbb{D}_{M_k};x\in B} \big\|F(\mathbf{t} + \tau_k; x) - y_{\mathbf{t},x}\big\|_Y = 0.$$

In the case that $X = \{0\}$ $(I' = I)$, we omit the term "$\mathcal{B}$" ("I'") from the notation, as before.

1.2. **Lebesgue spaces with variable exponents $L^{p(x)}$.** Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ be a nonempty Lebesgue measurable subset and let $M(\Omega : X)$ denote the collection of all measurable functions $f : \Omega \to X$; $M(\Omega) := M(\Omega : \mathbb{R})$. Furthermore, $\mathcal{P}(\Omega)$ denotes the vector space of all Lebesgue measurable functions $p : \Omega \to [1, \infty]$. For any $p \in \mathcal{P}(\Omega)$ and $f \in M(\Omega : X)$, we define

$$\varphi_{p(x)}(t) := \begin{cases} t^{p(x)}, \quad t \geq 0, \quad 1 \leq p(x) < \infty, \\ \\ 0, \quad 0 \leq t \leq 1, \quad p(x) = \infty, \\ \\ \infty, \quad t > 1, \quad p(x) = \infty \end{cases}$$

and

$$\rho(f) := \int_\Omega \varphi_{p(x)}(\|f(x)\|)\, dx.$$

We define the Lebesgue space $L^{p(x)}(\Omega : X)$ with variable exponent by

$$L^{p(x)}(\Omega : X) := \Big\{f \in M(\Omega : X) : \lim_{\lambda\to 0+} \rho(\lambda f) = 0\Big\}.$$

Equivalently

$$L^{p(x)}(\Omega : X) = \Big\{f \in M(\Omega : X) : \text{ there exists } \lambda > 0 \text{ such that } \rho(\lambda f) < \infty\Big\};$$

see, e.g., [9, p. 73]. For every $u \in L^{p(x)}(\Omega : X)$, we introduce the Luxemburg norm of $u(\cdot)$ by

$$\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega:X)} := \inf\Big\{\lambda > 0 : \rho(u/\lambda) \leq 1\Big\}.$$

Equipped with the above norm, the space $L^{p(x)}(\Omega : X)$ becomes a Banach space (see e.g. [9, Theorem 3.2.7] for the scalar-valued case), coinciding with the usual Lebesgue space $L^p(\Omega : X)$ in the case that $p(x) = p \geq 1$ is a constant function. Further on, for any $p \in M(\Omega)$, we define

$$p^- := \text{essinf}_{x\in\Omega} p(x) \quad \text{and} \quad p^+ := \text{esssup}_{x\in\Omega} p(x).$$

Set

$$D_+(\Omega) := \big\{p \in M(\Omega) : 1 \leq p^- \leq p(x) \leq p^+ < \infty \text{ for a.e. } x \in \Omega\big\}.$$

For $p \in D_+([0, 1])$, the space $L^{p(x)}(\Omega : X)$ behaves nicely, with almost all fundamental properties of the Lebesgue space with constant exponent $L^p(\Omega : X)$ being retained; in this case, we know that

$$L^{p(x)}(\Omega : X) = \Big\{f \in M(\Omega : X) ; \text{ for all } \lambda > 0 \text{ we have } \rho(\lambda f) < \infty\Big\}.$$

We will use the following lemma (cf. [9] for the scalar-valued case):

Lemma 2

(i) (The Hölder inequality) Let $p,\ q,\ r \in \mathcal{P}(\Omega)$ such that

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}, \quad x \in \Omega.$$

Then, for every $u \in L^{p(x)}(\Omega : X)$ and $v \in L^{r(x)}(\Omega)$, we have $uv \in L^{q(x)}(\Omega : X)$ and

$$\|uv\|_{q(x)} \leq 2\|u\|_{p(x)}\|v\|_{r(x)}.$$

(ii) Let Ω be of a finite Lebesgue's measure and let $p,\ q \in \mathcal{P}(\Omega)$ such $q \leq p$ a.e. on Ω. Then $L^{p(x)}(\Omega : X)$ is continuously embedded in $L^{q(x)}(\Omega : X)$, and the constant of embedding is less than or equal to $2(1 + m(\Omega))$.

(iii) Let $f \in L^{p(x)}(\Omega : X)$, $g \in M(\Omega : X)$ and $0 \leq \|g\| \leq \|f\|$ a.e. on Ω. Then $g \in L^{p(x)}(\Omega : X)$ and $\|g\|_{p(x)} \leq \|f\|_{p(x)}$.

We will use the following simple lemma, whose proof can be omitted:

Lemma 3 Suppose that $f \in L^{p(x)}(\Omega : X)$ and $A \in L(X, Y)$. Then $Af \in L^{p(x)}(\Omega : Y)$ and $\|Af\|_{L^{p(x)}(\Omega:Y)} \leq \|A\| \cdot \|f\|_{L^{p(x)}(\Omega:X)}$.

For further information concerning the Lebesgue spaces with variable exponents $L^{p(x)}$, we refer the reader to [9], [10] and [26].

1.3. **Stepanov multi-dimensional Bochner transform, Stepanov distance and Stepanov norm.** In this subsection and the subsequent section, we will always assume that Ω is a fixed compact subset of $\mathbb{R}^n$ with positive Lebesgue measure and $p \in \mathcal{P}(\Omega)$. Further on, Λ denotes a general non-empty subset of $\mathbb{R}^n$ satisfying $\Lambda + \Omega \subseteq \Lambda$ (in [2] and the Subsection 1.1, this region has been denoted by I). Recall that the multi-dimensional Bochner transform $\hat{F}_\Omega : \Lambda \times X \to Y^\Omega$ is defined by

$$\big[\hat{F}_\Omega(\mathbf{t}; x)\big](u) := F(\mathbf{t} + \mathbf{u}; x), \quad \mathbf{t} \in \Lambda,\ \mathbf{u} \in \Omega,\ x \in X.$$

The notion of Stepanov $(\Omega, p(\mathbf{u}))$-boundedness on $\mathcal{B}$ is introduced in [4] as follows:

Definition 5 Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$ and $F : \Lambda \times X \to Y$ satisfies that for each $\mathbf{t} \in \Lambda$ and $x \in X$, the function $F(\mathbf{t} + \mathbf{u}; x)$ belongs to the space $L^{p(\mathbf{u})}(\Omega : Y)$. Then we say that $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded on $\mathcal{B}$ if and only if for each $B \in \mathcal{B}$ we have

$$\sup_{\mathbf{t}\in\Lambda; x\in B} \Big\| \big[\hat{F}_\Omega(\mathbf{t}; x)\big](u) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = \sup_{\mathbf{t}\in\Lambda; x\in B} \Big\| F(\mathbf{t} + \mathbf{u}; x) \Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} < \infty.$$

Denote by $L_{S,\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$ the set consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-bounded functions on $\mathcal{B}$.

If $n = 1$, $X = \{0\}$, $\Omega = [0, 1]$ and $\Lambda = [0, \infty)$ or $\Lambda = \mathbb{R}$, then the notion introduced above reduces to the notion introduced recently in [8, Definition 4.1]. If $X = \{0\}$, then we abbreviate $L_{S,\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$ to $L_S^{\Omega,p(\mathbf{u})}(\Lambda : Y)$; in this case, we say that the function $F(\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded and define $\|F\|_{S^{\Omega,p(\mathbf{u})}} := \sup_{\mathbf{t}\in\Lambda} \|F(\mathbf{t} + \mathbf{u})\|_{L^{p(\mathbf{u})}(\Omega:Y)}$; in the usually considered case $\Omega = [0, 1]^n$, then we also say that the function $F(\cdot; \cdot)$ is Stepanov $p(\mathbf{u})$-bounded. Let $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfy $\Lambda + \Omega \subseteq \Lambda$. Suppose first that $p(\mathbf{u}) \equiv p \in [1, \infty)$ and $F : \Lambda \to Y$ and $G : \Lambda \to Y$ are two functions for which $\|F(\mathbf{t} + \mathbf{u}) - G(\mathbf{t} + \mathbf{u})\|_Y \in L^p(\Omega : \mathbb{C})$ for all $\mathbf{t} \in \Lambda$. We define the Stepanov distance $D_{S_\Omega}^p(F, G)$ of functions $F(\cdot)$ and $G(\cdot)$ by

$$D_{S_\Omega}^p(F, G) := \sup_{\mathbf{t}\in\Lambda} \Bigg[\Big(\frac{1}{m(\Omega)} \Big)^{1/p} \big\| F(\mathbf{t} + \mathbf{u}) - G(\mathbf{t} + \mathbf{u}) \big\|_{L^p(\Omega:Y)} \Bigg].$$

Suppose now that $p,\ q \in \mathcal{P}(\Omega)$, $1/p(\mathbf{u})+1/q(\mathbf{u})=1$ for a.e. $\mathbf{u}\in\Omega$ and $q(\mathbf{u})<+\infty$ for a.e. $\mathbf{u}\in\Omega$. In this case (the definition is consistent with the above given provided that $p(\mathbf{u})\equiv p\in(1,\infty)$), we define the Stepanov distance $D_{S_\Omega}^{p(\cdot)}(F,G)$ of functions $F(\cdot)$ and $G(\cdot)$ by

$$D_{S_\Omega}^{p(\cdot)}(F,G) := \sup_{\mathbf{t}\in\Lambda}\Biggl[m(\Omega)^{-1}\|1\|_{L^{q(\mathbf{u})}(\Omega)}\bigl\|F(\mathbf{t}+\mathbf{u})-G(\mathbf{t}+\mathbf{u})\bigr\|_{L^{p(\mathbf{u})}(\Omega:Y)}\Biggr].$$

Clearly, if $1\leq p_1(\mathbf{u})\equiv p_1\leq p_2\equiv p_2(\mathbf{u})$ for a.e. $\mathbf{u}\in\Omega$, then we have $D_{S_\Omega}^{p_1}(F,G)\leq D_{S_\Omega}^{p_2}(F,G)$. If $\Omega\equiv[0,l]^n$ for some $l>0$, then we also write $D_{S_l}^{p}(F,G)\equiv D_{S_\Omega}^{p}(F,G)$ and $D_{S_l}^{p(\cdot)}(F,G)\equiv D_{S_\Omega}^{p(\cdot)}(F,G)$. By $S_\Omega^p(\Lambda:Y)$ we denote the vector space of all functions $F:\Lambda\rightarrow Y$ for which $\|F(\mathbf{t}+\mathbf{u})\|_Y\in L^p(\Omega:Y)$ for all $\mathbf{t}\in\Lambda$ and the Stepanov norm

$$\|F\|_{S_\Omega^p} := \sup_{\mathbf{t}\in\Lambda}\Biggl[\Bigl(\frac{1}{m(\Omega)}\Bigr)^{1/p}\bigl\|F(\mathbf{t}+\mathbf{u})\bigr\|_{L^p(\Omega:Y)}\Biggr]$$

is finite. If $p,\ q\in\mathcal{P}(\Omega)$, $1/p(\mathbf{u})+1/q(\mathbf{u})=1$ for a.e. $\mathbf{u}\in\Omega$ and $q(\mathbf{u})<+\infty$ for a.e. $\mathbf{u}\in\Omega$, then (the definition is consistent with the above given provided that $p(\mathbf{u})\equiv p\in(1,\infty)$), we define the Stepanov norm $\|F\|_{S_\Omega^{p(\mathbf{u})}}$ by

$$\|F\|_{S_\Omega^{p(\mathbf{u})}} := \sup_{\mathbf{t}\in\Lambda}\Biggl[m(\Omega)^{-1}\|1\|_{L^{q(\mathbf{u})}(\Omega)}\bigl\|F(\mathbf{t}+\mathbf{u})\bigr\|_{L^{p(\mathbf{u})}(\Omega:Y)}\Biggr];$$

again, $S_\Omega^{p(\mathbf{u})}(\Lambda:Y)$ denotes the vector space consisting of all functions $F:\Lambda\rightarrow Y$ satisfying that $\|F(\mathbf{t}+\mathbf{u})\|_Y\in L^{p(\mathbf{u})}(\Omega:Y)$ for all $\mathbf{t}\in\Lambda$ and $\|F\|_{S_\Omega^{p(\mathbf{u})}}<\infty$. We know that $S_\Omega^{p(\mathbf{u})}(\Lambda:Y)$ is a Banach space equipped with the norm $\|\cdot\|_{S_\Omega^{p(\mathbf{u})}}$. For simplicity and better exposition, we will always assume that the following two conditions hold henceforth:

(A1) The binary relation ρ on Y is a function, i.e., $\rho(y)$ is a singleton for each $y\in D(\rho)$.

(A2) We have $p\in D_+(\Omega)$. Then the continuity of mapping $F:\Lambda\times X\rightarrow Y$ implies the continuity of mapping $\hat{F}_\Omega:\Lambda\times X\rightarrow L^{p(\mathbf{u})}(\Omega:Y)$; see [4].

2. Stepanov multi-dimensional ρ-almost periodic functions in Lebesgue spaces with variable exponents

In this section, we will tacitly assume that, for any considered function $F:\Lambda\times X\rightarrow Y$, the function $\hat{F}_\Omega:\Lambda\times X\rightarrow L^{p(\mathbf{u})}(\Omega:Y)$ is continuous. Unless stated otherwise, our standing assumptions will be that $\emptyset\neq\Lambda'\subseteq\mathbb{R}^n$, $\emptyset\neq\Lambda\subseteq\mathbb{R}^n$, $\Lambda+\Lambda'\subseteq\Lambda$, $\Lambda+\Omega\subseteq\Lambda$, and $F:\Lambda\times X\rightarrow Y$. We introduce the notion of a Stepanov multi-dimensional ρ-almost periodic function in the following way (in any concept proposed, Stepanov or Weyl, we omit the term "ρ" if $\rho=\mathrm{I}$, the identity operator on Y):

Definition 6 Suppose that $\emptyset\neq\Lambda'\subseteq\mathbb{R}^n$, $\emptyset\neq\Lambda\subseteq\mathbb{R}^n$, $\Lambda+\Lambda'\subseteq\Lambda$, $\Lambda+\Omega\subseteq\Lambda$, and $F:\Lambda\times X\rightarrow Y$.

(i) Then we say that $F(\cdot;\cdot)$ is Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathcal{B},\Lambda',\rho)$-almost periodic (Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathcal{B},\rho)$-almost periodic, if $\Lambda'=\Lambda$) if and only if for every $B\in\mathcal{B}$ and $\epsilon>0$ there exists $l>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that, for every $\mathbf{t} \in I$ and $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{t}+\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, belongs to the space $L^{p(\mathbf{u})}(\Omega : Y)$ and

$$\left\|F(\mathbf{t}+\tau+\mathbf{u}; x) - \rho(F(\mathbf{t}+\mathbf{u}; x))\right\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \epsilon, \quad \mathbf{t} \in \Lambda,\ x \in B.$$

By $APS^{\Omega,p(\mathbf{u}),\rho}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y)$ and $APS^{\Omega,p(\mathbf{u}),\rho}_{\mathcal{B}}(\Lambda \times X : Y)$ we denote the spaces consisting of all Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic functions and Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \rho)$-almost periodic functions, respectively.

(ii) Then we say that $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}), \rho)$-$(\mathcal{B}, \Lambda')$-uniformly recurrent (Stepanov $(\Omega, p(\mathbf{u}), \rho)$-$\mathcal{B}$-uniformly recurrent, if $\Lambda' = \Lambda$) if and only if for every $B \in \mathcal{B}$ there exists a sequence (τ_n) in Λ' such that $\lim_{n\to+\infty} |\tau_n| = +\infty$ and that, for every $\mathbf{t} \in I$ and $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{t}+\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, belongs to the space $L^{p(\mathbf{u})}(\Omega : Y)$ and

$$\lim_{n\to+\infty} \sup_{\mathbf{t}\in I; x\in B} \left\|F(\mathbf{t}+\tau_n+\mathbf{u}; x) - \rho(F(\mathbf{t}+\mathbf{u}; x))\right\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0.$$

By $URS^{\Omega,p(\mathbf{u}),\rho}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y)$ and $URS^{\Omega,p(\mathbf{u}),\rho}_{\mathcal{B}}(\Lambda \times X : Y)$ we denote the spaces consisting of all Stepanov $(\Omega, p(\mathbf{u}), \rho)$-$(\mathcal{B}, \Lambda')$-uniformly recurrent functions and Stepanov $(\Omega, p(\mathbf{u}), \rho)$-$\mathcal{B}$-uniformly recurrent functions, respectively.

If $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', ρ)-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', ρ)-uniformly recurrent) [Stepanov $(\Omega, p(\mathbf{u}))$-ρ-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-ρ-uniformly recurrent), if $\Lambda = \Lambda'$].
Employing Lemma 2, we immediately get the following (the same conclusions hold for the corresponding spaces of Stepanov uniformly recurrent functions):
Proposition 1 Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda + \Omega \subseteq \Lambda$, and $F : \Lambda \times X \to Y$.

(i) For every $p \in \mathcal{P}(\Omega)$, we have that $APS^{\Omega,p(\mathbf{u}),\rho}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y)$ is a subset of $APS^{\Omega,1,\rho}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y)$
(ii) For every $p,\ q \in \mathcal{P}(\Omega)$, we have that the assumption $q(\mathbf{u}) \leq p(\mathbf{u})$ for a.e. $\mathbf{u} \in \Omega$ implies that $APS^{\Omega,p(\mathbf{u}),\rho}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y)$ is a subset of $APS^{\Omega,q(\mathbf{u}),\rho}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y)$.
(iii) If $1 \leq p^- \leq p(\mathbf{u}) \leq p^+ < +\infty$ for a.e. $\mathbf{u} \in \Omega$, then

$$APS^{\Omega,p_+,\rho}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y) \subseteq APS^{\Omega,p(\mathbf{u}),\rho}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y) \subseteq APS^{\Omega,p_-,\rho}_{\mathcal{B},\Lambda'}(\Lambda \times X : Y).$$

In the next proposition, we will reconsider the statements of [15, Proposition 2.9] and [21, Proposition 2.13] (see also [4, Example 2.8]) for Stepanov multi-dimensional ρ-almost periodic type functions:
Proposition 2 Suppose that $\rho = T \in L(Y)$, $l \in \mathbb{N}$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, and $F : \Lambda \times X \to Y$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', T)$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', T)$-uniformly recurrent). Then $\Lambda + l\Lambda' \subseteq \Lambda$ and the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, l\Lambda', T^l)$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, l\Lambda', T^l)$-uniformly recurrent).

Proof. We will consider only Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', T)$-almost periodic functions. Inductively, we easily get $\Lambda + l\Lambda' \subseteq \Lambda$. Let $\epsilon > 0$ and $B \in \mathcal{B}$ be given. Further on, if $\mathbf{t}_0,\ \tau \in l\Lambda'$, then $\mathbf{t}_0/l,\ \tau/l \in \Lambda'$ and the result follows from the corresponding definition and the computation ($\mathbf{t} \in \Lambda,\ \mathbf{u} \in \Omega,\ x \in B$):

$$
\begin{aligned}
&\Big\| F\big(\mathbf{t}+\tau+\mathbf{u};x\big) - T^l F(\mathbf{t}+\mathbf{u};x)\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&= \Bigg\| \sum_{j=0}^{l-1} T^j \Big[F\big(\mathbf{t}+((l-j)\tau/l)+\mathbf{u};x\big) - TF\big(\mathbf{t}+((l-j-1)\tau/l)+\mathbf{u};x\big)\Big]\Bigg\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\leq \sum_{j=0}^{l-1} \|T\|^j \Big\| F\big(\mathbf{t}+((l-j)\tau/l)+\mathbf{u};x\big) - TF\big(\mathbf{t}+((l-j-1)\tau/l)+\mathbf{u};x\big)\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \\
&\leq \epsilon \sum_{j=0}^{l-1} \|T\|^j.
\end{aligned}
$$

□

The next simple result follows almost immediately from Definition 5 and Definition 6; this result enables us to deduce many structural properties of Stepanov multi-dimensional ρ-almost periodic functions using the corresponding properties of multi-dimensional ρ-almost periodic functions:

Proposition 3 Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, and $F : \Lambda \times X \to Y$.

(i) Define a binary relation ρ_1 on $L^{p(\mathbf{u})}(\Omega : Y)$ by $\rho_1([\hat{F}_\Omega(\mathbf{t};x)](\cdot)) := \rho(F(\mathbf{t} + \cdot;x))$, $\mathbf{t} \in \Lambda$, $x \in X$. If the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent), then the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathcal{B}, \Lambda', \rho_1)$-almost periodic ($(\mathcal{B}, \Lambda', \rho_1)$-uniformly recurrent).

(ii) Let ρ_1 be a binary relation on $L^{p(\mathbf{u})}(\Omega : Y)$ such that $\rho_1(G)$ is a singleton for all functions $G \in R(\hat{F}_\Omega)$. Define a binary relation ρ on Y by $\rho(F(\mathbf{t} + \mathbf{u};x)) := \rho_1([\hat{F}_\Omega(\mathbf{t};x)](\mathbf{u}))$, $\mathbf{t} \in \Lambda$, $x \in X$, $\mathbf{u} \in \Omega$. If the function $\hat{F}_\Omega : \Lambda \times X \to L^{p(\mathbf{u})}(\Omega : Y)$ is $(\mathcal{B}, \Lambda', \rho_1)$-almost periodic ($(\mathcal{B}, \Lambda', \rho_1)$-uniformly recurrent), then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent).

For example, using Lemma 1 and Proposition 3 (see also Lemma 3 for the issue (iv)), we may deduce the following:

Theorem 1

(i) Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, and $F : \Lambda \times X \to Y$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent). If for each $\tau \in \Lambda'$ we have $\tau + \Lambda = \Lambda$, then $\Lambda + (\Lambda' - \Lambda') \subseteq \Lambda$ and the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda' - \Lambda', \mathrm{I})$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda' - \Lambda', \mathrm{I})$-uniformly recurrent).

(ii) Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, and the function $F : \mathbb{R}^n \times X \to Y$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent). Then the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda' - \Lambda, \mathrm{I})$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda' - \Lambda, \mathrm{I})$-uniformly recurrent).

(iii) Suppose that $\rho = T \in L(Y)$ is a linear isomorphism.

(a) Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda \subseteq \Lambda$, Λ is closed, $F : \Lambda \times X \to Y$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, T)$-almost periodic and $\mathcal{B}$ is any family of compact subsets of X. If (1) holds with the region I replaced with the region Λ therein, then for each $B \in \mathcal{B}$ we have that the set $\{\hat{F}_\Omega(\mathbf{t};x) : \mathbf{t} \in I,\ x \in B\}$ is relatively compact in $L^{p(\mathbf{u})}(\Omega : Y)$; in particular, $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-bounded on $\mathcal{B}$.

(b) Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda \subseteq \Lambda$, Λ is closed and $F : \Lambda \times X \to Y$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, T)$-almost periodic, where $\mathcal{B}$ is a family consisting of some compact subsets of X. If (2) holds with the region I replaced with the region Λ therein, then for each $B \in \mathcal{B}$ the function $\hat{F}_\Omega(\cdot;\cdot)$ is uniformly continuous on $\Lambda \times B$.

(iv) Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$ and $F : \Lambda \times X \to Y$ is a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent function, where $\rho = T \in L(Y)$ is a linear isomorphism. Define the function $\tilde{T} : L^{p(\mathbf{u})}(\Omega : Y) \to L^{p(\mathbf{u})}(\Omega : Y)$ by

$$\big[\tilde{T}F\big](\mathbf{u}) := TF(\mathbf{u}), \quad \mathbf{u} \in \Omega,\ F \in L^{p(\mathbf{u})}(\Omega : Y). \tag{3}$$

Then $\tilde{T} \in L(L^{p(\mathbf{u})}(\Omega : Y))$ is a linear isomorphism, for each real number $a > 0$ we have:

$$\sup_{\mathbf{t} \in \Lambda, x \in B} \big\|\hat{F}_\Omega(\mathbf{t};x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \sup_{\mathbf{t} \in \Lambda+\Lambda', |\mathbf{t}| \geq a, x \in B} \big\|\tilde{T}^{-1}\hat{F}_\Omega(\mathbf{t};x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)},$$

and for each $x \in X$ we have

$$\sup_{\mathbf{t} \in \Lambda} \big\|\hat{F}_\Omega(\mathbf{t};x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \sup_{\mathbf{t} \in \Lambda+\Lambda', |\mathbf{t}| \geq a} \big\|\tilde{T}^{-1}\hat{F}_\Omega(\mathbf{t};x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)},$$

so that the function $F(\cdot;x)$ is almost everywhere equal to zero on the set $\Lambda + \Omega$, provided that the function $F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent and $\lim_{|\mathbf{t}| \to +\infty, \mathbf{t} \in \Lambda+\Lambda'} \hat{F}_\Omega(\mathbf{t};x) = 0$.

Further on, the following analogue of [12, Theorem 2.11] holds true:

Theorem 2 Suppose that $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, and $F : \Lambda \times X \to Y$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent). Then the following holds:

(i) Set $\sigma := \{(\|y_1\|_Y, \|y_2\|_Y) \mid \exists \mathbf{t} \in \Lambda + \Omega\ \exists x \in X\ :\ y_1 = F(\mathbf{t};x) \text{ and } y_2 \in \rho(y_1)\}$. Then the function $\|F(\cdot;\cdot)\|_Y$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \sigma)$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \sigma)$-uniformly recurrent).

(ii) Suppose that $\lambda \in \mathbb{C} \setminus \{0\}$. Set $\rho_\lambda := \{\lambda(y_1, y_2) \mid \exists \mathbf{t} \in \Lambda + \Omega\ \exists x \in X\ :\ y_1 = F(\mathbf{t};x) \text{ and } y_2 \in \rho(y_1)\}$. Then the function $\lambda F(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho_\lambda)$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho_\lambda)$-uniformly recurrent).

(iii) Suppose $a \in \mathbb{C}$ and $x_0 \in X$. Define $G : (\Lambda - a) \times X \to Y$ by $G(\mathbf{t};x) := F(\mathbf{t} + a; x + x_0)$, $\mathbf{t} \in \Lambda - a$, $x \in X$, as well as $\mathcal{B}_{x_0} := \{-x_0 + B : B \in \mathcal{B}\}$, $\Lambda'_a := \Lambda'$ and $\rho_{a,x_0} := \{(y_1, y_2) \mid \exists \mathbf{t} \in \Lambda + \Omega - a\ \exists x \in X\ :\ y_1 = F(\mathbf{t} + a; x + x_0) \text{ and } y_2 \in \rho(y_1)\}$. Then the function $G(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}_{x_0}, \Lambda'_a, \rho_{a,x_0})$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}_{x_0}, \Lambda'_a, \rho_{a,x_0})$-uniformly recurrent).

(iv) Suppose that $a,\ b \in \mathbb{C} \setminus \{0\}$. Define the function $G : (\Lambda/a) \times X \to Y$ by $G(\mathbf{t};x) := F(a\mathbf{t}; bx)$, $\mathbf{t} \in \Lambda/a$, $x \in X$, as well as $\mathcal{B}_b := \{b^{-1}B : B \in \mathcal{B}\}$, $\Lambda'_a := \Lambda'/a$ and $\rho_{a,b} := \{(y_1, y_2) \mid \exists \mathbf{t} \in (\Lambda + \Omega)/a\ \exists x \in X\ :\ y_1 =$

$F(a\mathbf{t}; bx)$ and $y_2 \in \rho(y_1)\}$. Then the function $G(\cdot;\cdot)$ is Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}_b, \Lambda'_a, \rho_{a,b})$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}_b, \Lambda'_a, \rho_{a,b})$-uniformly recurrent).

(v) Assume that for each $B \in \mathcal{B}$ there exists $\epsilon_B > 0$ such that for each element $x \in B^\circ \cup \bigcup_{y \in \partial B} B(y, \epsilon_B)$, the sequence $(F_k(\cdot; x))$ of Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic functions (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent functions) converges to a function $F(\cdot; x)$ in the norm $\|\cdot\|_{S_\Omega^{p(\mathbf{u})}}$, uniformly with respect to $x \in B^\circ$. Then the function $F(\cdot;\cdot)$ is likewise Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent), provided that $\{F(\mathbf{t}+\mathbf{u}; x) : \mathbf{t} \in \Lambda,\ \mathbf{u} \in \Omega,\ x \in X\} \subseteq D(\rho)$ and

(C_ρ) For each $\epsilon > 0$ there exists $\delta > 0$ such that, for every $y_1,\ y_2 \in \{F(\mathbf{t} + \cdot; x) : \mathbf{t} \in \Lambda,\ x \in X\} \cup_{k \in \mathbb{N}} \{F_k(\mathbf{t} + \cdot; x) : \mathbf{t} \in \Lambda,\ x \in X\}$ with $\|y_1 - y_2\|_{L^{p(\mathbf{u})}(\Omega:Y)} < \delta$, we have $\|z_1 - z_2\|_Y < \epsilon/3$ with $z_1 = \rho_1(y_1)$, $z_2 = \rho_1(y_2)$ and ρ_1 being defined in Proposition 3(i).

We continue by providing the following illustrative example:

Example 1 It is worth noting that Theorem 1(i) does not hold if there exists a point $\tau \in \Lambda'$ such that $\tau + \Lambda \neq \Lambda$. For example, in [12, Example 2.21], we have considered the situation in which $\Lambda = [0, \infty)$, $\Lambda' = (0, \infty)$, and $Y = \mathbb{C}^2$. Let it be the case, let $a \in \mathbb{C}$ satisfy $|a| > 1$, and let the function $u : [0, \infty) \to \mathbb{C}$ be almost periodic; further on, let

$$T = \begin{bmatrix} a & 1-a \\ a & 1-a \end{bmatrix}$$

and $\Omega = [0, 1]$. Then $N(A) = \{(\alpha, \beta) \in \mathbb{C}^2 : \alpha a + \beta(1-a) = 0\}$; suppose that $q = (q_1, q_2) : [0, \infty) \to N(A)$ is any continuous function satisfying $\lim_{t \to +\infty} q(t) = 0$. We have shown that the function $t \mapsto \vec{u}(t) := (u(t) + q_1(t), u(t) + q_2(t))$, $t \geq 0$ is (Λ', T)-almost periodic but not almost periodic. Since any uniformly continuous, Stepanov almost periodic function $f : [0, \infty) \to Y$ is almost periodic (see e.g., [19]), the function $t \mapsto \vec{u}(t)$, $t \geq 0$ cannot be Stepanov almost periodic ($p(\mathbf{u}) \equiv 1$), and therefore, the function $t \mapsto \vec{u}(t)$, $t \geq 0$ cannot belong to the space $APS_{\Lambda'}^{\Omega, p(\mathbf{u}), \mathrm{I}}(\Lambda : Y)$ due to Proposition 1(i).

Concerning Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', T)$-almost periodic functions with values in the finite-dimensional space $Y = \mathbb{C}^k$, where $T \in \mathbb{C}^{k,k}$ is a complex matrix of format $k \times k$, we will clarify only one result closely connected with our conclusions established in Example 1. This is an analogue of [4, Proposition 2.20] for Stepanov classes of ρ-almost periodic type functions; the proof follows from Proposition 3 and the argumentation used in the proof of the afore-mentioned result:

Proposition 4 Suppose that $k \in \mathbb{N}$, $T = [a_{ij}]$ is a complex matrix of format $k \times k$, $\Omega = [0, 1]$, $\Lambda = \mathbb{R}$ or $\Lambda = [0, \infty)$, $\Lambda' \subseteq \mathbb{R}$, $\Lambda + \Lambda' \subseteq \Lambda$, and the function $F : \Lambda \to \mathbb{C}^k$ is Stepanov $(\Omega, p(u))$-(Λ', T)-almost periodic (Stepanov $(\Omega, p(u))$-(Λ', T)-uniformly recurrent and the function $\hat{F}_\Omega(\cdot)$ is Stepanov $p(u)$-bounded). If $F = (F_1, \cdots, F_k)$, then there exists a non-trivial linear combination F of functions $F_1, \cdots, F_k$ which is Stepanov $(\Omega, p(u))$-(Λ', I)-almost periodic (Stepanov $(\Omega, p(u))$-(Λ', I)-uniformly recurrent and the function $\hat{\mathrm{F}}_\Omega$ is Stepanov $p(u)$-bounded).

Further on, in [4, Proposition 2.22], we have clarified a sufficient condition for a function $F : \Lambda \times X \to Y$ to be Stepanov $(\Omega, p(\mathbf{u}))$-$\mathcal{B}$-almost periodic. This result can be extended in the following way:

Proposition 5 Let $T \in L(Y)$, $\Lambda + \Lambda \subseteq \Lambda$, $\Lambda + \Omega \subseteq \Lambda$, $\mathcal{B}$ is any family of compact subsets of X and $F : \Lambda \times X \to Y$ satisfy the following conditions:

(i) For each $x \in X$, $F(\cdot; x) \in APS^{\Omega,p(\mathbf{u}),T}(\Lambda : Y)$.

(ii) $F(\cdot; \cdot)$ is $S^{p(\mathbf{u})}$-uniformly continuous with respect to the second argument on each compact subset B in $\mathcal{B}$, i.e., for each $\varepsilon > 0$ there exists $\delta_{B,\varepsilon} > 0$ such that for all $x_1, x_2 \in B$ we have

$$\|x_1 - x_2\| \leq \delta_{B,\varepsilon} \Longrightarrow \Big\|F(\mathbf{t} + \cdot; x_1) - F(\mathbf{t} + \cdot; x_2)\Big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \varepsilon \quad \text{for all } \ \mathbf{t} \in \Lambda. \tag{4}$$

Then $F(\cdot; \cdot)$ is Stepanov $(\Omega, p(\mathbf{u}), T)$-$\mathcal{B}$-multi-almost periodic.

Proof. The proof is almost the same as the corresponding proof of the aforementioned proposition, and we will provide the main details in the case that $p(\mathbf{u}) \equiv p \in [1, \infty)$. Suppose that $\varepsilon > 0$ and $B \subseteq X$ is a compact set. It follows that there exists a finite subset $\{x_1, ..., x_n\} \subseteq B$ ($n \in \mathbb{N}$) such that $B \subseteq \bigcup_{i=1}^{n} B(x_i, \delta_{B,\varepsilon})$. Therefore, for every $x \in B$, there exists $i \in \mathbb{N}_n$ satisfying $\|x - x_i\| \leq \delta_{B,\varepsilon}$. Let $\tau \in \Lambda$. Then we have

$$\begin{aligned}
&\left(\int_\Omega \|F(\mathbf{t}+\mathbf{s}+\tau; x) - TF(\mathbf{t}+\mathbf{s}; x)\|_Y^p \, d\mathbf{s}\right)^{\frac{1}{p}} \\
\leq &\left(\int_\Omega \|F(\mathbf{t}+\mathbf{s}+\tau; x) - F(\mathbf{t}+\mathbf{s}+\tau; x_i)\|_Y^p \, d\mathbf{s}\right)^{\frac{1}{p}} \\
&+\left(\int_\Omega \|F(\mathbf{t}+\mathbf{s}+\tau; x_i) - TF(\mathbf{t}+\mathbf{s}; x_i)\|_Y^p \, d\mathbf{s}\right)^{\frac{1}{p}} \\
&+\left(\int_\Omega \|TF(\mathbf{t}+\mathbf{s}; x_i) - TF(\mathbf{t}+\mathbf{s}; x)\|_Y^p \, d\mathbf{s}\right)^{\frac{1}{p}}, \ \mathbf{t} \in \Lambda.
\end{aligned} \tag{5}$$

Using (i) to conclude that for each $i = 1, \ldots, n$ there exists $l_{B,\varepsilon} > 0$ such that for all $\mathbf{t}_0 \in \Lambda$ there exists $\tau \in B(\mathbf{t}_0, l_{B,\varepsilon})$ satisfying

$$\begin{aligned}
&\left(\int_\Omega \|F(\mathbf{t}+\mathbf{s}+\tau; x_i) - TF(\mathbf{t}+\mathbf{s}; x_i)\|_Y^p \, d\mathbf{s}\right)^{\frac{1}{p}} \\
\leq &\left(\int_\Omega \|F(\mathbf{t}+\mathbf{s}+\tau; x_i) - F(\mathbf{t}+\mathbf{s}; x_i)\|_Y^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda.
\end{aligned} \tag{6}$$

Since $\|x - x_i\| \leq \delta_{K,\delta}$ and $T \in L(Y)$, it follows that

$$\left(\int_\Omega \|F(\mathbf{t}+\mathbf{s}+\tau; x) - F(\mathbf{t}+\mathbf{s}+\tau; x_i)\|_Y^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda, \tag{7}$$

and

$$\left(\int_\Omega \|TF(\mathbf{t}+\mathbf{s}; x) - TF(\mathbf{t}+\mathbf{s}; x_i)\|_Y^p \, d\mathbf{s}\right)^{\frac{1}{p}} \tag{8}$$

$$\leq \|T\| \left(\int_\Omega \|F(\mathbf{t}+\mathbf{s}; x) - F(\mathbf{t}+\mathbf{s}; x_i)\|_Y^p \, d\mathbf{s}\right)^{\frac{1}{p}} \leq \frac{\varepsilon}{3} \quad \text{for all } \mathbf{t} \in \Lambda. \tag{9}$$

Inserting (6), (7) and (9) in (5), we obtain

$$\sup_{x\in B}\left(\int_\Omega \|F(\mathbf{t}+\mathbf{s}+\tau;x)-TF(\mathbf{t}+\mathbf{s};x)\|_Y^p\,d\mathbf{s}\right)^{\frac{1}{p}}\leq\varepsilon\quad\text{for all }\mathbf{t}\in\Lambda.$$

Hence, $F(\cdot;\cdot)$ is Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathcal{B},T)$-almost periodic. □

Concerning the convolution invariance of Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathcal{B},\Lambda',\rho)$-almost periodic type functions, we will clarify the following result:

Theorem 3 Suppose that $h\in L^1(\mathbb{R}^n)$ and $F:\mathbb{R}^n\times X\to Y$ satisfies that for each $x\in X$ the function $\mathbf{t}\mapsto F(\mathbf{t};x)$, $\mathbf{t}\in\mathbb{R}^n$ is measurable as well as that for each set $B\in\mathcal{B}$ there exists a real number $\epsilon_B>0$ such that $\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B^\cdot}\|F(\mathbf{t},x)\|_Y<+\infty$, where $B^\cdot\equiv B^\circ\cup\bigcup_{x\in\partial B}B(x,\epsilon_B)$. Suppose, further, that $\rho=A$ is a closed linear operator on Y satisfying that:

(B) For each $\mathbf{t}\in\mathbb{R}^n$ and $x\in B$, the function $\mathbf{s}\mapsto AF(\mathbf{t}-\mathbf{s}+\cdot;x)\in L^{p(\mathbf{u})}(\Omega:Y)$, $\mathbf{s}\in\mathbb{R}^n$ is well defined and bounded.

Then the function

$$(h*F)(\mathbf{t};x):=\int_{\mathbb{R}^n}h(\sigma)F(\mathbf{t}-\sigma;x)\,d\sigma,\quad \mathbf{t}\in\mathbb{R}^n,\ x\in X \tag{10}$$

is well defined and for each $B\in\mathcal{B}$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B^\cdot}\|(h*F)(\mathbf{t};x)\|_Y<+\infty$; furthermore, if $F(\cdot;\cdot)$ is Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathcal{B},\Lambda',A)$-almost periodic (Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathcal{B},\Lambda',A)$-uniformly recurrent), then the function $(h*F)(\cdot;\cdot)$ is likewise Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathcal{B},\Lambda',A)$-almost periodic (Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathcal{B},\Lambda',A)$-uniformly recurrent).

Proof. We will consider only Stepanov $(\Omega,p(\mathbf{u}))$-$(\mathcal{B},\Lambda',A)$-almost periodic functions. It is clear that the function $(h*F)(\cdot;\cdot)$ is well defined and $\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B^\cdot}\|(h*F)(\mathbf{t};x)\|_Y<+\infty$ for all $B\in\mathcal{B}$; furthermore, $\sup_{\mathbf{t}\in\mathbb{R}^n,x\in B^\cdot}\|\hat{F}_\Omega(\mathbf{t};x)\|_{L^{p(\mathbf{u})}(\Omega:Y)}<\infty$ for all $B\in\mathcal{B}$. Define

$$\hat{A}\equiv\bigl\{(F,G)\in L^{p(\mathbf{u})}(\Omega:Y)\times L^{p(\mathbf{u})}(\Omega:Y):AF(\mathbf{u})=G(\mathbf{u})\text{ for a.e. }u\in\Omega\bigr\}.$$

Using the fact that the space $L^{p(\mathbf{u})}(\Omega:Y)$ is continuously embedded in $L^1(\Omega:Y)$ as well as the fact that any sequence of functions converging in $L^1(\Omega:Y)$ converges pointwisely for a.e. $\mathbf{u}\in\Omega$, we can easily show that $\hat{A}$ is a closed linear operator in $L^{p(\mathbf{u})}(\Omega:Y)$. Using condition (B) and [12, Theorem 2.14], it follows that the function $(h*\hat{F}_\Omega)(\cdot;\cdot)\in L^{p(\mathbf{u})}(\Omega:Y)$ is Bohr $(\mathcal{B},\Lambda',\hat{A})$-almost periodic. Then the final conclusion follows from Proposition 3 and the obvious equality

$$h*\hat{F}_\Omega=(h\hat{*}F)_\Omega.$$

□

Keeping in mind the proofs of Theorem 3, [4, Theorem 5.1] and [17, Proposition 2.6.11] (see also [8, Proposition 6.1]), it is straightforward to deduce the following result about the inheritance of Stepanov $(\Omega,p(\mathbf{u}))$-(Λ',ρ)-almost periodicity (-uniform recurrence) under the actions of the infinite convolution products:

Theorem 4 Let $\rho=A$ be a closed linear operator on Y, $\Omega=[0,1]^n$, $q\in\mathcal{P}(\Omega)$, $1/p(x)+1/q(x)=1$ for all $x\in\Omega$, and $(R(\mathbf{t}))_{\mathbf{t}\in(0,\infty)^n}\subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $\sum_{\mathbf{k}\in\mathbb{N}_0^n}\|R(\cdot+\mathbf{k})\|_{L^{q(\mathbf{u})}(\Omega)}<\infty$ and

$R(\mathbf{t})A \subseteq AR(\mathbf{t})$ for all $\mathbf{t} \in (0,\infty)^n$. If $\check{f} : \mathbb{R}^n \to X$ is Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', A)-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', A)-uniformly recurrent), and the following conditions hold:

(i) The functions $\check{f}(\cdot)$ and $A\check{f}(\cdot)$ are Stepanov $(\Omega, q(\mathbf{u}))$-bounded on $\mathcal{B}$;
(ii) The Bochner transform of function $\check{f}(\cdot)$ is uniformly continuous on $\mathbb{R}^n$ (with values in $L^{p(\mathbf{u})}(\Omega : Y)$).

Then the function $F : \mathbb{R}^n \to Y$, given by

$$F(\mathbf{t}) := \int_{-\infty}^{t_1} \int_{-\infty}^{t_2} \cdots \int_{-\infty}^{t_n} R(\mathbf{t} - \mathbf{s}) f(\mathbf{s})\, d\mathbf{s}, \quad \mathbf{t} \in \mathbb{R}^n,$$

is well defined and almost periodic (bounded uniformly recurrent).
Remark 1

(i) It is worth noticing that conditions (i)-(ii) from the formulation of Theorem 4 hold if $\Lambda' = \mathbb{R}^n$ and $A \in L(Y)$ is a linear isomorphism; see Theorem 1(iii).
(ii) The most important applications of Theorem 4, and Proposition 8 below, can be given in the one-dimensional setting, for various classes of abstract (degenerate) Volterra integro-differential equations; basically, we will not consider here such applications; see [17] for more details.

The following illustrative example can be formulated in the multi-dimensional setting (see e.g., [19, Example 6.2.9], where we have considered case $c = 1$, only):
Example 2 Let $f : \mathbb{R} \to \mathbb{R}$ be a Bohr almost anti-periodic function; $\Omega \equiv [0,1]$. Define $\mathrm{sign}(0) := 0$ and $F : \mathbb{R} \to \mathbb{R}$ by $F(t) := \mathrm{sign}(f(t))$, $t \in \mathbb{R}$. Then the function $F(\cdot)$ is Stepanov $p(u)$-almost anti-periodic. This can be shown by using the argumentation given in the proof of [25, Theorem 5.3.1, p. 210] and the computation carried out in [17, Example 2.2.2(i)].
The following important result about extensions of Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', ρ)-almost periodic type functions follows from Proposition 3 and the argumentation contained in the proof of [4, Theorem 2.15]:
Theorem 5 Suppose that $\rho = T \in L(Y)$ is a linear isomorphism, the linear isomorphism $\tilde{T}$ of space $L^{p(\mathbf{u})}(\Omega : Y)$ is given through (3), the set Λ' is unbounded, $m(\partial\Lambda) = 0$, $\Omega^\circ \neq \emptyset$, $F : \Lambda \to Y$ satisfies that $\hat{F}_\Omega : \Lambda \to L^{p(\mathbf{u})}(\Omega : Y)$ is a uniformly continuous, Bohr $(\Lambda', \tilde{T})$-almost periodic function, resp. a uniformly continuous, $(\Lambda', \tilde{T})$-uniformly recurrent function, $S \subseteq \mathbb{R}^n$ is bounded and, for every $\mathbf{t}' \in \mathbb{R}^n$, there exists a finite real number $M > 0$ such that $\mathbf{t}' + \Lambda'_M \subseteq \Lambda$. Define $\Lambda_S := \Lambda' \cup S$. Then there exists a Stepanov $(\Omega, p(\mathbf{u}))$-(Λ_S, T)-almost periodic, resp. a Stepanov $(\Omega, p(\mathbf{u}))$-(Λ_S, T)-uniformly recurrent, function $\tilde{F} : \mathbb{R}^n \to Y$ such that $\tilde{F}(\mathbf{t}) = F(\mathbf{t})$ for a.e. $\mathbf{t} \in \Lambda$; furthermore, in Stepanov almost periodic case, if $\mathbb{R}^n \setminus \Lambda_S$ is a bounded set and any $\tilde{T}$-almost periodic function defined on $\mathbb{R}^n$ is almost automorphic, then any such extension of function $F(\cdot)$ is unique.
Remark 2 As in all previous investigations of the multi-dimenaional almost periodicity and its generalizations, it is worth noting that Theorem 5 is applicable provided that $(\mathbf{v}_1, \cdot \cdot \cdot, \mathbf{v}_n)$ is a basis of $\mathbb{R}^n$,

$$\Lambda' = \Lambda = \left\{ \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n : \alpha_i \geq 0 \text{ for all } i \in \mathbb{N}_n \right\}$$

is a convex polyhedral in $\mathbb{R}^n$ and Ω is any compact subset of Λ with non-empty interior.

2.1. **$\mathbb{D}$-Asymptotically Stepanov ρ-almost periodic type functions.** We start this subsection by recalling the following notion from [4]:

Definition 7 Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda+\Omega \subseteq \Lambda$ and the set $\mathbb{D}$ is unbounded. By $S_{0,\mathbb{D},\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda\times X : Y)$ we denote the vector space consisting of all functions $Q : \Lambda\times X \rightarrow Y$ such that, for every $\mathbf{t} \in \Lambda$ and $x \in X$, we have $[\hat{Q}_\Omega(\mathbf{t}; x)](\mathbf{u}) \in L^{p(\mathbf{u})}(\Omega : Y)$ as well as that, for every $B \in \mathcal{B}$, we have $\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\rightarrow+\infty}[\hat{Q}_\Omega(\mathbf{t}; x)](\mathbf{u}) = 0$ in $L^{p(\mathbf{u})}(\Omega : Y)$, uniformly for $x \in B$. In the case that $X = \{0\}$ and $\mathcal{B} = \{X\}$, then we abbreviate $S_{0,\mathbb{D},\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$ to $S_{0,\mathbb{D}}^{\Omega,p(\mathbf{u})}(\Lambda : Y)$.

For the sake of completeness, we will provide all details of the proof of the following simple result (which remains true for all $p \in \mathcal{P}(\Omega)$):

Lemma 4 Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Omega \subseteq \Lambda$ and the set $\mathbb{D}$ is unbounded. If $\mathbb{D} + \Omega \subseteq \mathbb{D}$, then $C_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y) \subseteq S_{0,\mathbb{D},\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$.

Proof. Let $Q \in C_{0,\mathbb{D},\mathcal{B}}(\Lambda \times X : Y)$. Then the mapping $\mathbf{u} \mapsto Q(\mathbf{t} + \mathbf{u}; x)$, $\mathbf{u} \in \Omega$ is continuous and bounded, so that it belongs to the space $L^\infty(\Omega : Y)$. Applying Lemma 2(ii), we get that this mapping belongs to the space $L^{p(\mathbf{u})}(\Omega : Y)$. We need to prove that $\lim_{\mathbf{t}\in\mathbb{D},|\mathbf{t}|\rightarrow+\infty} Q(\mathbf{t} + \mathbf{u}; x) = 0$ in $L^{p(\mathbf{u})}(\Omega : Y)$, uniformly for $x \in B$. Let $\epsilon > 0$ be given. The required limit equality follows from the existence of a finite real number $M_\epsilon > 0$ such that $\|Q(\mathbf{t} + \mathbf{u}; x)\|_Y \leq \epsilon$ for any $x \in B$ and $\mathbf{t} \in \mathbb{D}$ with $|\mathbf{t}| \geq M_\epsilon$, where we have employed our assumption $\mathbb{D} + \Omega \subseteq \mathbb{D}$, and an application of Lemma 2(ii), which shows that

$$\big\|Q(\mathbf{t} + \mathbf{u}; x)\big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq 2(1 + m(\Omega))\big\|Q(\mathbf{t} + \mathbf{u}; x)\big\|_{L^\infty(\Omega:Y)} \leq 2(1 + m(\Omega))\epsilon,$$

provided $\mathbf{t} \in \mathbb{D}$ and $|\mathbf{t}| \geq M_\epsilon$. □

Now we are ready to introduce the following notion:

Definition 8 Suppose that $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$ satisfies $\Lambda+\Omega \subseteq \Lambda$, $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$, the set $\mathbb{D}$ is unbounded, and $F : \Lambda\times X \rightarrow Y$. Then we say that the function $F(\cdot; \cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic, resp. (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent, if and only if there exist a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic function $(H : \mathbb{R}^n \times X \rightarrow Y)$ $H : \Lambda \times X \rightarrow Y$, resp. a Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent function $(H : \mathbb{R}^n \times X \rightarrow Y)$ $H : \Lambda \times X \rightarrow Y$, and a function $Q \in S_{0,\mathbb{D},\mathcal{B}}^{\Omega,p(\mathbf{u})}(\Lambda \times X : Y)$ such that $F(\mathbf{t}; x) = H(\mathbf{t}; x) + Q(\mathbf{t}; x)$ for a.e. $\mathbf{t} \in \Lambda$ and all $x \in X$.

If $X \in \mathcal{B}$, then we also say that the function $F(\cdot; \cdot)$ is (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', ρ)-almost periodic ((strongly) Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', ρ)-uniformly recurrent). If $\mathbb{D} = \Lambda$, then we omit the "prefix $\mathbb{D}$-" and say that the function $F(\cdot; \cdot)$ is asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', ρ)-almost periodic ((strongly) Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', ρ)-uniformly recurrent).

Arguing similarly as in the proof of Lemma 4, we can prove that any Bohr $(\mathcal{B}, \Lambda', \rho)$-almost periodic function ($(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent function) is automatically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-almost periodic (Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda')$-unifromly recurrent); see also [8, Proposition 4.5]. Keeping in mind the afore-mentioned lemma and the corresponding definitions, we may deduce the following:

Proposition 6 Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \rightarrow Y$, $\Lambda + \Lambda' \subseteq \Lambda$ and $\mathbb{D} + \Omega \subseteq \mathbb{D}$. Then a (strongly) $\mathbb{D}$-asymptotically Bohr $(\mathcal{B}, \Lambda', \rho)$-almost periodic function, resp. (strongly) $\mathbb{D}$-asymptotically $(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent function, is (strongly)

$\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic, resp. (strongly) $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent.

The following slightly weaker notion of $\mathbb{D}$-asymptotical Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodicity is important, as well (see also [12, Proposition 2.26], which can be formulated for the Stepanov classes):

Definition 9 Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $F : \Lambda \times X \to Y$ and $\Lambda + \Lambda' \subseteq \Lambda$. Then we say that:

(i) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic of type 1 if and only if for every $B \in \mathcal{B}$ and $\epsilon > 0$ there exist $l > 0$ and $M > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, l) \cap \Lambda'$ such that

$$\big\| F(\mathbf{t}+\tau+\mathbf{u};x) - \rho(F(\mathbf{t}+\mathbf{u};x)) \big\|_{L^{p(\mathbf{u})}(\Omega:Y)} \leq \epsilon, \text{ provided } \mathbf{t},\ \mathbf{t}+\tau \in \mathbb{D}_M,\ x \in B.$$

(ii) $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent of type 1 if and only if for every $B \in \mathcal{B}$ there exist a sequence (τ_n) in Λ' and a sequence (M_n) in $(0,\infty)$ such that $\lim_{n\to+\infty} |\tau_n| = \lim_{n\to+\infty} M_n = +\infty$ and

$$\lim_{n\to+\infty} \sup_{\mathbf{t},\mathbf{t}+\tau_n \in \mathbb{D}_{M_n}; x \in B} \big\| F(\mathbf{t}+\tau_n+\mathbf{u};x) - \rho(F(\mathbf{t}+\mathbf{u};x)) \big\|_{L^{p(\mathbf{u})}(\Omega:Y)} = 0.$$

If $\Lambda' = \Lambda$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \rho)$-almost periodic of type 1 ($\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \rho)$-uniformly recurrent of type 1); furthermore, if $X \in \mathcal{B}$, then it is also said that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', ρ)-almost periodic of type 1 ($\mathbb{D}$-asymptotically Stepanov (Λ', ρ)-uniformly recurrent of type 1). If $\Lambda' = \Lambda$ and $X \in \mathcal{B}$, then we also say that $F(\cdot;\cdot)$ is $\mathbb{D}$-asymptotically Stepanov ρ-almost periodic of type 1 ($\mathbb{D}$-asymptotically Stepanov ρ-uniformly recurrent of type 1). As before, we remove the prefix "$\mathbb{D}$-" in the case that $\mathbb{D} = \Lambda$ and remove the prefix "$(\mathcal{B},)$" in the case that $X \in \mathcal{B}$.

The question when a given uniformly continuous, bounded function $F : \Lambda \to Y$ which is both Λ-asymptotically Bohr T-almost periodic function of type 1 and Λ-asymptotically Bohr I-almost periodic function of type 1 is Λ-asymptotically Bohr T-almost periodic, where $T \in L(Y)$, has been examined in [4, Theorem 2.27]. This statement can be formulated for the corresponding Stepanov classes using the multi-dimensional Bochner transform and Proposition 3; details can be left to the interested reader.

The following analogue of Proposition 6 holds true:

Proposition 7 Suppose that $\mathbb{D} \subseteq \Lambda \subseteq \mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded, as well as $\emptyset \neq \Lambda' \subseteq \Lambda \subseteq \mathbb{R}^n$, $\Lambda + \Lambda' \subseteq \Lambda$ and $\mathbb{D} + \Omega \subseteq \mathbb{D}$. Then any $\mathbb{D}$-asymptotically $(\mathcal{B}, \Lambda', \rho)$-almost periodic function $F : \Lambda \times X \to Y$ of type 1 ($\mathbb{D}$-asymptotically $(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent function $F : \Lambda \times X \to Y$ of type 1) is $\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-almost periodic of type 1 ($\mathbb{D}$-asymptotically Stepanov $(\Omega, p(\mathbf{u}))$-$(\mathcal{B}, \Lambda', \rho)$-uniformly recurrent of type 1).

Further on, define $I_{\mathbf{t}} := (-\infty, t_1] \times (-\infty, t_2] \times \cdots \times (-\infty, t_n]$ and $\mathbb{D}_{\mathbf{t}} := I_{\mathbf{t}} \cap \mathbb{D}$ for any $\mathbf{t} = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n$. Keeping in mind Theorem 4 and the proof of [4, Proposition 5.3], we may deduce the following result about the invariance of strong $\mathbb{D}$-asymptotical Stepanov $(\Omega, p(\mathbf{u}))$-(Λ', ρ)-almost periodicity (-uniform recurrence) under the actions of the finite convolution product:

Proposition 8 Suppose that $\rho = A$ is a closed linear operator on Y, $\Omega = [0,1]^n$, $q \in \mathcal{P}(\Omega)$, $1/p(x) + 1/q(x) = 1$ for all $x \in \Omega$, and $(R(\mathbf{t}))_{\mathbf{t}>\mathbf{0}} \subseteq L(X,Y)$ is a strongly continuous operator family satisfying that $\sum_{\mathbf{k}\in\mathbb{N}_0^n}\|R(\cdot+\mathbf{k})\|_{L^{q(\mathbf{u})}(\Omega)}<\infty$ and $R(\mathbf{t})A\subseteq AR(\mathbf{t})$ for all $\mathbf{t}\in(0,\infty)^n$. Suppose that $\check{g}:\mathbb{R}^n\rightarrow X$ is Stepanov $(\Omega,p(\mathbf{u}))$-(Λ',A)-almost periodic (Stepanov $(\Omega,p(\mathbf{u}))$-(Λ',A)-uniformly recurrent), and the following conditions hold:

(i) The functions $\check{g}(\cdot)$ and $A\check{g}(\cdot)$ are Stepanov $(\Omega,q(\mathbf{u}))$-bounded on $\mathcal{B}$;

(ii) The Bochner transform of function $\check{g}(\cdot)$ is uniformly continuous and bounded on $\mathbb{R}^n$ (with values in $L^{p(\mathbf{u})}(\Omega:Y)$).

Suppose, further, that $\emptyset\neq\Lambda\subseteq\mathbb{R}^n$ satisfies $\Lambda+\Omega\subseteq\Lambda$, $\mathbb{D}\subseteq\Lambda\subseteq\mathbb{R}^n$ and the set $\mathbb{D}$ is unbounded. Let $q:\Lambda\rightarrow X$, and let $f(\mathbf{t}):=g(\mathbf{t})+q(\mathbf{t})$ for all $\mathbf{t}\in\Lambda$. Then the function $F:\Lambda\rightarrow Y$, defined by

$$F(\mathbf{t}):=\int_{\mathbb{D}_{\mathbf{t}}}R(\mathbf{t}-\mathbf{s})f(\mathbf{s})\,ds,\quad \mathbf{t}\in\Lambda,$$

is strongly $\mathbb{D}$-asymptotically Stepanov $(\Omega,p(\mathbf{u}))$-(Λ',A)-almost periodic (strongly $\mathbb{D}$-asymptotically Stepanov $(\Omega,p(\mathbf{u}))$-(Λ',A)-uniform recurrent), provided that

$$\lim_{|\mathbf{t}|\rightarrow\infty,\mathbf{t}\in\mathbb{D}}\sum_{\mathbf{k}\in\mathbb{N}_0^n}\big\|R(\mathbf{s}+\mathbf{k})\big\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[I_{\mathbf{t}}\cap\mathbb{D}^c])\cap\Omega)}=0,$$

and for each $\epsilon>0$ there exists $r>0$ such that for each $\mathbf{t}\in\mathbb{D}$ with $|\mathbf{t}|\geq r$ there exists a finite real number $r_{\mathbf{t}}>0$ such that

$$\sum_{\mathbf{k}\in\mathbb{N}_0^n}\Biggl\{\big\|R(\mathbf{s}+\mathbf{k})\big\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[I_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega)}\times\big\|\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\big\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})])\cap\Omega)}\Biggr\}<\epsilon/2$$

and

$$\sum_{\mathbf{k}\in\mathbb{N}_0^n}\Biggl\{\big\|R(\mathbf{s}+\mathbf{k})\big\|_{L^{q(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega)}\times\big\|\check{q}(\mathbf{s}+\mathbf{k}-\mathbf{t})\big\|_{L^{p(\mathbf{s})}((\mathbf{t}-\mathbf{k}-[\mathbb{D}_{\mathbf{t}}\cap B(0,r_{\mathbf{t}})^c])\cap\Omega)}\Biggr\}<\epsilon/2.$$

We close this subsection with the observation that the statement of [4, Theorem 2.23] can be formulated for Stepanov classes, as well.

3. Multi-dimensional Weyl ρ-almost periodic type functions

In our recent joint research study with V. E. Fedorov [23], we have analyzed various classes of multi-dimensional Weyl almost periodic type functions. Further on, in [22, Section 5], we have expanded this study by exploring various classes of multi-dimensional Weyl c-almost periodic type functions, where $c\in\mathbb{C}\setminus\{0\}$. The main aim of this section is to briefly describe how the structural results obtained in [22] can be slightly generalized for the general class of multi-dimensional Weyl ρ-almost periodic type functions.

In the first concept, we assume that the following condition holds:

(WM1): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega) > 0$, $p \in \mathcal{P}(\Lambda)$, $\Lambda' + \Lambda + l\Omega \subseteq \Lambda$, $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi : [0, \infty) \to [0, \infty)$ and $\mathbb{F} : (0, \infty) \times \Lambda \to (0, \infty)$.

We introduce the following classes of multi-dimensional Weyl ρ-almost periodic functions:

Definition 10

(i) By $e-W^{(p(\mathbf{u}),\phi,\mathbb{F},\rho)}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, and

$$\sup_{x \in B} \sup_{\mathbf{t} \in \Lambda} \mathbb{F}(l, \mathbf{t}) \phi\Bigl(\bigl\| F(\tau + \mathbf{u}; x) - \rho(F(\mathbf{u}; x)) \bigr\|_Y \Bigr)_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega)} < \epsilon. \tag{11}$$

(ii) By $W^{(p(\mathbf{u}),\phi,\mathbb{F}),\rho}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, and

$$\limsup_{l \to +\infty} \sup_{x \in B} \sup_{\mathbf{t} \in \Lambda} \mathbb{F}(l, \mathbf{t}) \phi\Bigl(\bigl\| F(\tau + \mathbf{u}; x) - \rho(F(\mathbf{u}; x)) \bigr\|_Y \Bigr)_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega)} < \epsilon.$$

Definition 11

(i) By $e-W^{(p(\mathbf{u}),\phi,\mathbb{F},\rho)_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ belongs to the space $L^{p(\mathbf{u})}(\mathbf{t} + l\Omega : Y)$ for each $\mathbf{t} \in \Lambda$, and

$$\sup_{x \in B} \sup_{\mathbf{t} \in \Lambda} \mathbb{F}(l, \mathbf{t}) \phi\Bigl(\bigl\| F(\tau + \mathbf{u}; x) - \rho(F(\mathbf{u}; x)) \bigr\|_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega:Y)} \Bigr) < \epsilon.$$

(ii) By $W^{(p(\mathbf{u}),\phi,\mathbb{F},\rho)_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ belongs to the space $L^{p(\mathbf{u})}(\mathbf{t} + l\Omega : Y)$ for each $\mathbf{t} \in \Lambda$, and

$$\limsup_{l \to +\infty} \sup_{x \in B} \sup_{\mathbf{t} \in \Lambda} \mathbb{F}(l, \mathbf{t}) \phi\Bigl(\bigl\| F(\tau + \mathbf{u}; x) - \rho(F(\mathbf{u}; x)) \bigr\|_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega:Y)} \Bigr) < \epsilon.$$

Definition 12

(i) By $e-W^{(p(\mathbf{u}),\phi,\mathbb{F},\rho)_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ belongs to the space $L^{p(\mathbf{u})}(\mathbf{t} + l\Omega : Y)$ for each $\mathbf{t} \in \Lambda$, and

$$\sup_{x \in B} \sup_{\mathbf{t} \in \Lambda} \phi\Bigl(\mathbb{F}(l, \mathbf{t}) \bigl\| F(\tau + \mathbf{u}; x) - \rho(F(\mathbf{u}; x)) \bigr\|_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega:Y)} \Bigr) < \epsilon.$$

(ii) By $W^{(p(\mathbf{u}),\phi,\mathbb{F},\rho)_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ belongs to the space $L^{p(\mathbf{u})}(\mathbf{t} + l\Omega : Y)$ for each $\mathbf{t} \in \Lambda$, and

$$\limsup_{l\to+\infty} \sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} \phi\Big(\mathbb{F}(l,\mathbf{t})\big\|F(\tau + \mathbf{u}; x) - \rho(F(\mathbf{u}; x))\big\|_{L^{p(\mathbf{u})}(\mathbf{t}+l\Omega:Y)}\Big) < \epsilon.$$

In the second concept, we intend to ensure the translation invariance of multi-dimensional Weyl ρ-almost periodic functions. Assume now that the following condition holds:

(WM2): $\emptyset \neq \Lambda \subseteq \mathbb{R}^n$, $\emptyset \neq \Lambda' \subseteq \mathbb{R}^n$, $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ is a Lebesgue measurable set such that $m(\Omega) > 0$, $p \in \mathcal{P}(\Omega)$, $\Lambda' + \Lambda + l\Omega \subseteq \Lambda$, $\Lambda + l\Omega \subseteq \Lambda$ for all $l > 0$, $\phi : [0,\infty) \to [0,\infty)$ and $\mathbb{F} : (0,\infty) \times \Lambda \to (0,\infty)$.

We introduce the following classes of functions:

Definition 13

(i) By $e - W^{[p(\mathbf{u}),\phi,\mathbb{F},\rho]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $\mathbf{t} \in \Lambda$ and $x \in B$, we have $F(\mathbf{t} + l\mathbf{u}; x) \in D(\rho)$ and

$$\sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} l^n \mathbb{F}(l,\mathbf{t})\phi\Big(\big\|F(\mathbf{t} + \tau + l\mathbf{u}; x) - \rho(F(\mathbf{t} + l\mathbf{u}; x))\big\|_Y\Big)_{L^{p(\mathbf{u})}(\Omega)} < \epsilon.$$

(ii) By $W^{[p(\mathbf{u}),\phi,\mathbb{F},\rho]}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $l > 0$, $\mathbf{t} \in \Lambda$ and $x \in B$, we have $F(\mathbf{t} + l\mathbf{u}; x) \in D(\rho)$ and

$$\limsup_{l\to+\infty} \sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} l^n \mathbb{F}(l,\mathbf{t})\phi\Big(\big\|F(\mathbf{t} + \tau + l\mathbf{u}; x) - \rho(F(\mathbf{t} + l\mathbf{u}; x))\big\|_Y\Big)_{L^{p(\mathbf{u})}(\Omega:Y)} < \epsilon.$$

Definition 14

(i) By $e - W^{[p(\mathbf{u}),\phi,\mathbb{F},\rho]_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exist two finite real numbers $l > 0$ and $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $\mathbf{t} \in \Lambda$ and $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{t}+l\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, belongs to the space $L^{p(\mathbf{u})}(\Omega : Y)$ and

$$\sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} l^n \mathbb{F}(l,\mathbf{t})\phi\Big(\big\|F(\mathbf{t} + \tau + l\mathbf{u}; x) - \rho(F(\mathbf{t} + l\mathbf{u}; x))\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\Big) < \epsilon.$$

(ii) By $W^{[p(\mathbf{u}),\phi,\mathbb{F},\rho]_1}_{\Omega,\Lambda',\mathcal{B}}(\Lambda \times X : Y)$ we denote the set consisting of all functions $F : \Lambda \times X \to Y$ such that, for every $\epsilon > 0$ and $B \in \mathcal{B}$, there exists a finite real number $L > 0$ such that for each $\mathbf{t}_0 \in \Lambda'$ there exists $\tau \in B(\mathbf{t}_0, L) \cap \Lambda'$ such that, for every $l > 0$, $\mathbf{t} \in \Lambda$ and $x \in B$, the mapping $\mathbf{u} \mapsto \rho(F(\mathbf{t} + l\mathbf{u}; x))$, $\mathbf{u} \in \Omega$ is well defined, belongs to the space $L^{p(\mathbf{u})}(\Omega : Y)$ and

$$\limsup_{l\to+\infty} \sup_{x\in B} \sup_{\mathbf{t}\in\Lambda} l^n \mathbb{F}(l,\mathbf{t})\phi\Big(\big\|F(\mathbf{t} + \tau + l\mathbf{u}; x) - \rho(F(\mathbf{t} + l\mathbf{u}; x))\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\Big) < \epsilon.$$

Definition 15

(i) By $e-W^{[p(\mathbf{u}),\phi,\mathbb{F},\rho]_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X:Y)$ we denote the set consisting of all functions $F:\Lambda\times X\to Y$ such that, for every $\epsilon>0$ and $B\in\mathcal{B}$, there exist two finite real numbers $l>0$ and $L>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{t}_0,L)\cap\Lambda'$ such that, for every $\mathbf{t}\in\Lambda$ and $x\in B$, the mapping $\mathbf{u}\mapsto\rho(F(\mathbf{t}+l\mathbf{u};x))$, $\mathbf{u}\in\Omega$ is well defined, belongs to the space $L^{p(\mathbf{u})}(\Omega:Y)$ and

$$\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda}\phi\Big(l^n\mathbb{F}(l,\mathbf{t})\big\|F(\mathbf{t}+\tau+l\mathbf{u};x)-\rho(F(\mathbf{t}+l\mathbf{u};x))\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\Big)<\epsilon.$$

(ii) By $W^{[p(\mathbf{u}),\phi,\mathbb{F},\rho]_2}_{\Omega,\Lambda',\mathcal{B}}(\Lambda\times X:Y)$ we denote the set consisting of all functions $F:\Lambda\times X\to Y$ such that, for every $\epsilon>0$ and $B\in\mathcal{B}$, there exists a finite real number $L>0$ such that for each $\mathbf{t}_0\in\Lambda'$ there exists $\tau\in B(\mathbf{t}_0,L)\cap\Lambda'$ such that, for every $l>0$, $\mathbf{t}\in\Lambda$ and $x\in B$, the mapping $\mathbf{u}\mapsto\rho(F(\mathbf{t}+l\mathbf{u};x))$, $\mathbf{u}\in\Omega$ belongs to the space $L^{p(\mathbf{u})}(\Omega:Y)$ and

$$\limsup_{l\to+\infty}\sup_{x\in B}\sup_{\mathbf{t}\in\Lambda}\phi\Big(l^n\mathbb{F}(l,\mathbf{t})\big\|F(\mathbf{t}+\tau+l\mathbf{u};x)-\rho(F(\mathbf{t}+l\mathbf{u};x))\big\|_{L^{p(\mathbf{u})}(\Omega:Y)}\Big)<\epsilon.$$

It can be simply shown that the both concepts are equivalent in the constant coefficient case; see also [24] for the study of Weyl one-dimensional almost periodic functions.

We will only mention in passing that [22, Example 5.4] can be reformulated in our new framework (cf. also [22, Example 2.10]), as well as the statement of [12, Proposition 5.5]. The situation is a little bit complicated with the statement of [22, Theorem 5.6, Proposition 5.9]; this problem can be left to the interested readers. Concerning [22, Example 5.7], we would like to stress the following:

Example 3 Suppose that $\emptyset\neq K\subseteq\mathbb{R}^n$ is a compact set, $F(\mathbf{t}):=\chi_K(\mathbf{t})$, $\mathbf{t}\in\mathbb{R}^n$, $\Lambda:=\Lambda':=\mathbb{R}^n$, $\phi(x)\equiv x$, $\Omega:=[0,1]^n$ and $\rho(z):=z^k$ for some non-negative integer $k\in\mathbb{N}_0$. Then for each $p\in D_+(\Omega)$ we have $F\in e-W^{[p(\mathbf{u}),x,l^{-\sigma},\rho]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n:\mathbb{C})$. Keeping in mind Lemma 2(ii), we get that ($\tau\in\mathbb{R}^n$; $l>0$):

$$\begin{aligned}
&\sup_{\mathbf{t}\in\mathbb{R}^n}l^{n-\sigma}\Big\|\chi_K(\mathbf{t}+\tau+l\mathbf{u})-\chi_K^k(\mathbf{t}+l\mathbf{u})\Big\|_{L^{p(\mathbf{u})}(\Omega)}\\
&\quad\le 4\sup_{\mathbf{t}\in\mathbb{R}^n}l^{n-\sigma}\Big\|\chi_K(\mathbf{t}+\tau+l\mathbf{u})-\chi_K^k(\mathbf{t}+l\mathbf{u})\Big\|_{L^{p^+}(\Omega)}\\
&\quad=4\sup_{\mathbf{t}\in\mathbb{R}^n}l^{-\sigma}\Big\|\chi_K(\mathbf{t}+\tau+\mathbf{u})-\chi_K^k(\mathbf{t}+\mathbf{u})\Big\|_{L^{p^+}(l\Omega)}\\
&\quad\le 4\sup_{\mathbf{t}\in\mathbb{R}^n}l^{-\sigma}\Bigg[\big\|\chi_K(\cdot)\big\|_{L^{p^+}(l\Omega\cap[K-\mathbf{t}-\tau])}+\big\|\chi_K^k(\cdot)\big\|_{L^{p^+}(l\Omega\cap[K-\mathbf{t}])}\Bigg]\\
&\quad\le 4l^{-\sigma}m(K).
\end{aligned}$$

This simply implies that $F\in e-W^{[p(\mathbf{u}),x,l^{-\sigma},\rho]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n:\mathbb{C})$, as claimed.

Concerning the function $F(\mathbf{t}):=\chi_{[0,\infty)^n}(\mathbf{t})$, $\mathbf{t}\in\mathbb{R}^n$, we want only to recall that $F\in W^{[p,x,l^{-\sigma},1]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n:\mathbb{C})$ if and only if $\sigma>(n-1)/p$, as well as that there is no $\sigma>0$ such that $F\in e-W^{[p,x,l^{-\sigma},1]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n:\mathbb{C})$. We also know that there is no $\sigma>0$ and $c\in\mathbb{C}\setminus\{0\}$ such that $F\in e-W^{[p,x,l^{-\sigma},c\mathrm{I}]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n:\mathbb{C})$ as well as that there is no $c\in\mathbb{C}\setminus\{0,1\}$ such that $F\in W^{[p,x,l^{-\sigma},c\mathrm{I}]}_{\Omega,\mathbb{R}^n}(\mathbb{R}^n:\mathbb{C})$ for $n\ge\sigma>(n-1)/p$, which is

the optimal result.
Concerning the convolution invariance of spaces $(e-)W^{(p(\mathbf{u}),\phi,\mathbb{F},\rho)}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$ and $(e-)W^{[p(\mathbf{u}),\phi,\mathbb{F},\rho]}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$, we will only state the following result; the corresponding proof is very similar to the proof of [23, Theorem 2.9] and therefore omitted; see also [22, Theorem 5.8].
Theorem 6 Suppose that $\varphi : [0,\infty) \to [0,\infty)$, $\phi : [0,\infty) \to [0,\infty)$ is a convex monotonically increasing function satisfying $\phi(xy) \leq \varphi(x)\phi(y)$ for all $x,\ y \geq 0$, $\rho = T \in L(Y)$, $h \in L^1(\mathbb{R}^n)$, $\Omega = [0,1]^n$, $F \in (e-)W^{(p(\mathbf{u}),\phi,\mathbb{F},T)}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$, $1/p(\mathbf{u}) + 1/q(\mathbf{u}) = 1$, and for each $x \in X$ we have $\sup_{\mathbf{t}\in\mathbb{R}^n} \|F(\mathbf{t}; x)\|_Y < \infty$. If $\mathbb{F}_1 : (0,\infty) \times \mathbb{R}^n \to (0,\infty)$, $p_1 \in \mathcal{P}(\mathbb{R}^n)$ and if, for every $\mathbf{t} \in \mathbb{R}^n$ and $l > 0$, there exists a sequence $(a_k)_{k\in l\mathbb{Z}^d}$ of positive real numbers such that $\sum_{k\in l\mathbb{Z}^n} a_k = 1$ and

$$\int_{\mathbf{t}+l\Omega} \varphi_{p_1(\mathbf{u})}\Biggl(2\sum_{k\in l\mathbb{Z}^n} a_k l^{-n}\Bigl[\varphi\bigl(a_k^{-1}l^n h(\mathbf{u}-\mathbf{v})\bigr)\Bigr]_{L^{q(\mathbf{v})}(\mathbf{u}-k+l\Omega)}\mathbb{F}_1(l,\mathbf{t})\bigl[\mathbb{F}(l,\mathbf{u}-k)\bigr]^{-1}\Biggr)\,d\mathbf{u} \leq 1,$$

then $h * F \in (e-)W^{(p_1(\mathbf{u}),\phi,\mathbb{F}_1,T)}_{\Omega,\Lambda',\mathcal{B}}(\mathbb{R}^n \times X : Y)$.

4. Applications to the abstract Volterra integro-differential equations

In this section, we apply our results in the analysis of existence and uniqueness of generalized multi-dimensional ρ-almost periodic type solutions for various classes of abstract Volterra integro-differential equations.

4.1. Newtonian potential and logarithmic potential.

Concerning the notion introduced in Definition 10-Definition 12, we would like to note that there exist some important cases in which it is extremely important that the function $\mathbb{F}(l,\mathbf{t})$ depends not only on $l > 0$ but also on $\mathbf{t} \in \Lambda$. We will illustrate this fact by considering the second-order partial differential equation $\Delta u = -f$, where $f \in C^2(\mathbb{R}^3)$ has a compact support. It is well known that the Newtonian potential of $f(\cdot)$, defined by

$$u(x) := \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{f(x-y)}{|y|}\,dy, \quad x \in \mathbb{R}^3,$$

is a unique function belonging to the class $C^2(\mathbb{R}^3)$, vanishing at infinity and satisfying $\Delta u = -f$; see e.g. [28, Theorem 3.9, pp. 126-127]. For simplicity, suppose that $p = p_1 = 1$, $\Omega = [0,1]^n$, $\Lambda' \subseteq \Lambda = \mathbb{R}^3$ and

$$\sup_{l>0;\mathbf{t}\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\mathbb{F}_1(l,\mathbf{t})}{|y|\cdot\mathbb{F}(l,\mathbf{t}-y)}\,dy < +\infty. \tag{12}$$

Then we have the following (we consider here case $\rho = \mathrm{I}$ but the same conclusions hold if $\rho(z) = z^k$, $z \in \mathbb{C}$ for some $k \in \mathbb{N}$; see [12] for more details):
Theorem 7 Suppose that $f \in (e-)W^{1,x,\mathbb{F},\mathrm{I}}_{[0,1]^n,\Lambda'}(\mathbb{R}^3 : \mathbb{C})$ and (12) holds. Then $u \in (e-)W^{1,x,\mathbb{F}_1,\mathrm{I}}_{[0,1]^n,\Lambda'}(\mathbb{R}^3 : \mathbb{C})$.

Proof. Suppose that $l > 0$ and $\mathbf{t} \in \mathbb{R}^3$ are arbitrary; consider the class $e-W^{1,x,\mathbb{F},\mathrm{I}}_{[0,1]^n,\Lambda'}(\mathbb{R}^3 : \mathbb{C})$ for brevity. Let a point $\tau \in \mathbb{R}^3$ satisfy (11). Using the Fubini theorem and (12), we have

$$\begin{aligned}
&\big\|u(\cdot+\tau)-u(\cdot)\big\|_{L^1(\mathbf{t}+l\Omega)} \\
&\quad\leq \frac{1}{4\pi}\int_{\mathbf{t}+l\Omega}\int_{\mathbb{R}^3}\frac{|f(x+\tau-y)-f(x-y)|}{|y|}\,dy\,dx \\
&\quad\leq \frac{1}{4\pi}\int_{\mathbb{R}^3}\Biggl[\int_{\mathbf{t}+l\Omega}|f(x+\tau-y)-f(x-y)|\,dx\Biggr]\frac{dy}{|y|} \\
&=\frac{1}{4\pi}\int_{\mathbb{R}^3}\Biggl[\int_{\mathbf{t}-y+l\Omega}|f(x+\tau)-f(x)|\,dx\Biggr]\frac{dy}{|y|} \\
&\quad\leq \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\epsilon\cdot dy}{|y|\cdot\mathbb{F}(l,\mathbf{t}-y)}\leq\frac{\epsilon}{\mathbb{F}_1(l,\mathbf{t})}.
\end{aligned}$$

This simply implies the required. □

Concerning Theorem 7, we would like to emphasize that the function $y\mapsto|y|^{-1}$, $y\in\mathbb{R}^3$ does not belong to the class $L^1(\mathbb{R}^3)$ so that the results on convolution invariance of multi-dimensional Weyl ρ-almost periodicity cannot be applied here.

We can similarly analyze the two-dimensional analogue of this example by considering the logarithmic potential of $f(\cdot)$, given by

$$u(x):=\frac{(-1)}{2\pi}\int_{\mathbb{R}^2}\ln(|y|)\cdot f(x-y)\,dy,\quad x\in\mathbb{R}^2.$$

In this case, we only need to replace condition (12) by

$$\sup_{l>0;\mathbf{t}\in\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{\ln(|y|)\cdot\mathbb{F}_1(l,\mathbf{t})}{\mathbb{F}(l,\mathbf{t}-y)}\,dy<+\infty;$$

see also [28, Remark 3.7, p. 128].

It will be very complicated to reconsider here many other formulas from the classical theory of partial differential equations which can be employed for our purposes.

4.2. **Applications to the Gaussian semigroup in $\mathbb{R}^n$.** In a great number of recent research papers concerning multi-dimensional almost periodic type functions, we have presented certain applications to the Gaussian semigroup in $\mathbb{R}^n$ and the Poisson semigroup in $\mathbb{R}^n$, with obvious applications to the inhomogeneous heat equation in $\mathbb{R}^n$.

Let Y be one of the spaces $L^p(\mathbb{R}^n)$, $C_0(\mathbb{R}^n)$ or $BUC(\mathbb{R}^n)$, where $1\leq p<\infty$. Then we know that the Gaussian semigroup

$$(G(t)F)(x):=\bigl(4\pi t\bigr)^{-(n/2)}\int_{\mathbb{R}^n}F(x-y)e^{-\frac{|y|^2}{4t}}\,dy,\quad t>0,\ f\in Y,\ x\in\mathbb{R}^n,$$

can be extended to a bounded analytic C_0-semigroup of angle $\pi/2$, generated by the Laplacian Δ_Y acting with its maximal distributional domain in Y. It is clear that our results about the convolution invariance of Stepanov multi-dimensional ρ-almost periodic functions and the convolution invariance of Weyl multi-dimensional ρ-almost periodic functions (see Theorem 3 and Theorem 6) can be applied to the function $x\mapsto(G(t_0)F)(x)$, $x\in\mathbb{R}^n$, where $t_0>0$ is a fixed real number. It is also worth noting that Theorem 3 and Theorem 6 can be applied in the qualitative analysis of solutions of the abstract ill-posed Cauchy problems of first order whose solutions are governed by integrated semigroups or C-regularized semigroups (see [19] for more details); the applications can be given also to the abstract ill-posed Cauchy problems of second (fractional) order.

5. Conclusions and final remarks

The present paper is devoted to the study of various classes of Stepanov multi-dimensional ρ-almost periodic type functions with values in complex Banach spaces. In the investigation of all examined classes of generalized multi-dimensional ρ-almost periodic type functions, we use definitions and results from the theory of Lebesgue spaces with variable exponents. We also provide some relevant applications to the abstract Volterra integro-differential equations.

Finally, let us mention some intriguing topics not analyzed here. In [12], we have recently introduced and analyzed the following notion (cf. also [20]):

Definition 16 Let $\omega \in \mathbb{R}^n \setminus \{0\}$, ρ be a binary relation on X and $\omega + I \subseteq I$. A continuous function $F : I \to X$ is said to be (ω, ρ)-periodic if and only if $F(\mathbf{t}+\omega) \in \rho(F(\mathbf{t}))$, $\mathbf{t} \in I$.

Definition 17 Let $\omega_j \in \mathbb{R} \setminus \{0\}$, $\rho_j \in \mathbb{C} \setminus \{0\}$ is a binary relation on X and $\omega_j e_j + I \subseteq I$ $(1 \leq j \leq n)$. A continuous function $F : I \to X$ is said to be $(\omega_j, \rho_j)_{j\in\mathbb{N}_n}$-periodic if and only if $F(\mathbf{t} + \omega_j e_j) \in \rho_j(F(\mathbf{t}))$, $\mathbf{t} \in I$, $j \in \mathbb{N}_n$.

In the case that $\rho_j = c_j\mathrm{I}$ for some non-zero complex numbers c_j $(1 \leq j \leq n)$, then we also say that the function $F(\cdot)$ is $(\omega_j, c_j)_{j\in\mathbb{N}_n}$-periodic; furthermore, if $c_j = 1$ for all $j \in \mathbb{N}_n$, then we say that $F(\cdot)$ is $(\omega_j)_{j\in\mathbb{N}_n}$-periodic. In this paper, we have not analyzed Stepanov and Weyl classes of multi-dimensional (ω, ρ)-periodic functions ($(\omega_j, \rho_j)_{j\in\mathbb{N}_n}$-periodic functions).

Concerning composition principles for Stepanov one-dimensional ρ-almost periodic functions, we would like to note that the statements of [19, Theorem 4.2.38, Theorem 4.2.39] can be straightforwardly reformulated for Stepanov one-dimensional T-almost periodic functions, where $T \in L(Y)$ is not necessarily linear isomorphism. A possible application can be given to the abstract semilinear Cauchy inclusions analyzed in [19, Subsection 4.2.2], provided that the operator T commutes with the closed multivalued linear operator $\mathcal{A}$ generating the fractional resolvent family $(R_\gamma(t))_{t>0}$ appearing therein; the pivot Banach space should be

$$BUR_{(\alpha_k);T,uc} := \Bigg\{ u : \mathbb{R} \to Y : u(\cdot) \text{ is bounded, uniformly continuous,}$$
$$T - \text{uniformly recurrent and } \lim_{k\to+\infty} \sup_{t\in\mathbb{R}} \|f(t + \alpha_k) - Tf(t)\| = 0 \Bigg\}.$$

See [19, Theorem 4.2.40] for more details. We have not analyzed, among many other topics, composition principles for Stepanov (Weyl) multi-dimensional ρ-almost periodic functions here.

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