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A GEOMETRIC INTERPRETATION TO FIXED-POINT THEORY ON S_b-METRIC SPACES

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ABSTRACT. In this paper we present some fixed-figure theorems as a geometric approach to the fixed-point theory when the number of fixed points of a self-mapping is more than one. To do this, we modify the Jleli-Samet type contraction and define new contractions on S_b-metric spaces. Also, we give some necessary examples to show the validity of our theoretical results.

1. Introduction and background

Classical fixed-point theory started with the Banach fixed-point theorem [3]. This theory is one of the useful tool of mathematical studies and is an applicable area to topology, analysis, geometry, applied mathematics, engineering etc. Metric fixed-point theory has been studied and generalized with various aspects. One of these aspects is to generalize the used contractive condition (for example, see [5]). Another aspect is to generalize the used metric space such as, a b-metric space, an S-metric space and an S_b-metric space as follows:

Definition 1.1. [2] *Let X be a nonempty set, $b \geq 1$ a given real number and $d : X \times X \to [0, \infty)$ a function satisfying the following conditions for all $x, y, z \in X$:*
($b1$) $d(x, y) = 0$ *if and only if* $x = y$,
($b2$) $d(x, y) = d(y, x)$,
($b3$) $d(x, z) \leq b[d(x, y) + d(y, z)]$.
Then the function d is called a b-metric on X and the pair (X, d) is called a b-metric space.

Definition 1.2. [20] *Let X be a nonempty set and $S : X \times X \times X \to [0, \infty)$ be a function satisfying the following conditions for all $x, y, z, a \in X$:*
($S1$) $S(x, y, z) = 0$ *if and only if* $x = y = z$,
($S2$) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.
Then S is called an S-metric on X and the pair (X, S) is called an S-metric space.

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Definition 1.3. [21] *Let X be a nonempty set and $b \geq 1$ be a given real number. A function $S_b : X \times X \times X \to [0, \infty)$ is said to be S_b-metric if and only if for all $x, y, z, a \in X$ the following conditions are satisfied:*
(S_b1) $S_b(x, y, z) = 0$ if and only if $x = y = z$,
(S_b2) $S_b(x, y, z) \leq b[S_b(x, x, a) + S_b(y, y, a) + S_b(z, z, a)]$.
The pair (X, S_b) is called an S_b-metric space.

An S_b-metric space is also a generalization of an S-metric space because every S-metric is an S_b-metric with $b = 1$. But the converse of this statement is not always true as seen in the following example.

Example 1.1. [22] *Let $X = \mathbb{R}$ and the function S_b be defined by*

$$S_b(x, y, z) = S(x, y, z)^2 = \frac{1}{16}(|x - y| + |y - z| + |x - z|)^2,$$

for all $x, y, z \in \mathbb{R}$. Then the function S_b is an S_b-metric with $b = 4$, but it is not an S-metric.

We see that the relationships between a b-metric and an S_b-metric as follows:

Lemma 1.1. [22] *Let (X, S_b) be an S_b-metric space, S_b be a symmetric S_b-metric with $b \geq 1$ and the function $d : X \times X \to [0, \infty)$ be defined by*

$$d(x, y) = S_b(x, x, y),$$

for all $x, y \in X$. Then d is a b-metric on X.

Lemma 1.2. [22] *Let (X, d) be a b-metric space with $b \geq 1$ and the function $S_b : X \times X \times X \to [0, \infty)$ be defined by*

$$S_b(x, y, z) = d(x, z) + d(y, z),$$

for all $x, y, z \in X$. Then S_b is an S_b-metric on X.

Many authors have been studied various fixed-point results on different generalized metric spaces (for example, see [1], [7], [9] and the references therein).

Recently, as a geometric generalization of a fixed-point theory, fixed-circle problem has been studied. This problem was occurred in [12] and investigated some solutions to the this problem using different approaches (for example, see [8], [11], [12], [13], [14], [16], [17], [18], [23] and the references therein). Especially, this problem was studied on S_b-metric space in [13] and obtained some fixed-circle results using the following basic definitions.

Definition 1.4. [13] *Let (X, S_b) be an S_b-metric space with $b \geq 1$ and $x_0 \in X$, $r \in (0, \infty)$. The circle centered at x_0 with radius r is defined by*

$$C^{S_b}_{x_0,r} = \{x \in X : S_b(x, x, x_0) = r\}.$$

Definition 1.5. [13] *Let (X, S_b) be an S_b-metric space with $b \geq 1$, $C^{S_b}_{x_0,r}$ be a circle on X and $T : X \to X$ be a self-mapping. If $Tx = x$ for all $x \in C^{S_b}_{x_0,r}$ then the circle $C^{S_b}_{x_0,r}$ is called as the fixed circle of T.*

The notion of a fixed figure was defined as a generalization of the notions of a fixed circle and a fixed disc as follows:

A geometric figure $\mathcal{F}$ (a circle, an ellipse, a hyperbola, a Cassini curve etc.) contained in the fixed point set $Fix\,(T) = \{x \in X : x = Tx\}$ is called a *fixed figure* (a fixed circle, a fixed ellipse, a fixed hyperbola, a fixed Cassini curve, etc.) of the self-mapping T (see [15]). For this purpose, some fixed-figure theorems were obtained using different aspects (see, [4], [6], [15] and [25] for more details).

By the above motivation, the main of this paper is to obtain some fixed-figure results on an S_b-metric space. To do this, we define new Jleli-Samet type contractions. Using these new contractions, we prove fixed-disc results, fixed-ellipse results, fixed-hyperbola results, fixed-Cassini curve results and fixed-Apollonius circle results on an S_b-metric space. Also, we give an example to show the validity of our obtained theorems.

2. Main results

In this section, we present some fixed-figure results on an S_b-metric space. Before these results, we give the following definitions:

Definition 2.1. *Let (X, S_b) be an S_b-metric space with $b \geq 1$ and $x_0, x_1, x_2 \in X$, $r \in [0, \infty)$.*

(1) The disc centered at x_0 with radius r is defined by
$$D_{x_0,r}^{S_b} = \{x \in X : S_b(x, x, x_0) \leq r\}.$$
(2) The ellipse $E_r^{S_b}(x_1, x_2)$ is defined by
$$E_r^{S_b}(x_1, x_2) = \{x \in X : S_b(x, x, x_1) + S_b(x, x, x_2) = r\}.$$
(3) The hyperbola $H_r^{S_b}(x_1, x_2)$ is defined by
$$H_r^{S_b}(x_1, x_2) = \{x \in X : |S_b(x, x, x_1) - S_b(x, x, x_2)| = r\}.$$
(4) The Cassini curve $C_r^{S_b}(x_1, x_2)$ is defined by
$$C_r^{S_b}(x_1, x_2) = \{x \in X : S_b(x, x, x_1) S_b(x, x, x_2) = r\}.$$
(5) The Apollonius circle $A_r^{S_b}(x_1, x_2)$ is defined by
$$A_r^{S_b}(x_1, x_2) = \left\{x \in X - \{x_2\} : \frac{S_b(x, x, x_1)}{S_b(x, x, x_2)} = r\right\}.$$

Now, we give the following example.

Example 2.1. *Let (X, d) be a metric space and let us consider the S_b-metric space (X, S_b) with the S_b-metric $S_b : X \times X \times X \to [0, \infty)$ defined as*
$$S_b(x, y, z) = [d(x, y) + d(y, z) + d(x, z)]^p,$$
for all $x, y, z \in X$ and $p > 1$ [22]. Let us consider $X = \mathbb{R}^3$, the metric d be a usual metric with $d(x, y) = |x - y|$ and $p = 3$. If we take $x_0 = (1, 1, 1)$ and $r = 40$, then we obtain the circle $C_{x_0,r}^{S_b}$ as
$$\begin{aligned} C_{x_0,r}^{S_b} &= \left\{x \in \mathbb{R}^3 : S_b(x, x, x_0) = 40\right\} \\ &= \left\{x \in \mathbb{R}^3 : |x-1|^3 + |y-1|^3 + |z-1|^3 = 5\right\} \end{aligned}$$
and the disc $D_{x_0,r}^{S_b}$ as
$$\begin{aligned} D_{x_0,r}^{S_b} &= \left\{x \in \mathbb{R}^3 : S_b(x, x, x_0) \leq 40\right\} \\ &= \left\{x \in \mathbb{R}^3 : |x-1|^3 + |y-1|^3 + |z-1|^3 \leq 5\right\}. \end{aligned}$$

(A) The circle $C_{x_0,r}^{S_b}$ (B) The disc $D_{x_0,r}^{S_b}$

FIGURE 1. The geometric figures of the circle and the disc

Also, if we take $x_1=(1,1,1)$, $x_1=(-1,-1,-1)$ *and* $r=400$, *then we obtain the ellipse* $E_r^{S_b}(x_1,x_2)$ *as*

$$\begin{aligned} E_r^{S_b}(x_1,x_2) &= \left\{x\in\mathbb{R}^3 : S_b(x,x,x_1)+S_b(x,x,x_2)=400\right\} \\ &= \left\{\begin{array}{c} x\in\mathbb{R}^3 : (|x-1|+|x+1|)^3+(|y-1|+|y+1|)^3 \\ +(|z-1|+|z+1|)^3\leq 50 \end{array}\right\}. \end{aligned}$$

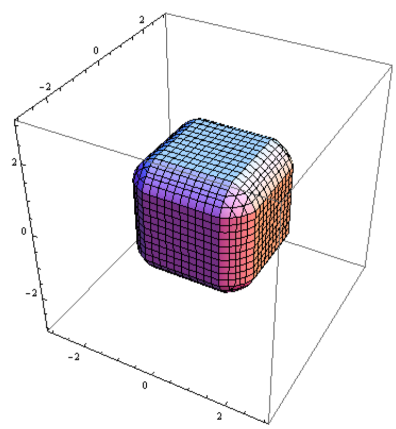


FIGURE 2. The ellipse $E_r^{S_b}(x_1,x_2)$

If we take $x_1=(1,1,1)$, $x_1=(-1,-1,-1)$ *and* $r=40$, *then we obtain the hyperbola* $H_r^{S_b}(x_1,x_2)$ *as*

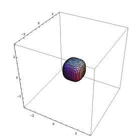
$$\begin{aligned} H_r^{S_b}(x_1,x_2) &= \left\{x\in\mathbb{R}^3 : |S_b(x,x,x_1)-S_b(x,x,x_2)|=40\right\} \\ &= \left\{\begin{array}{c} x\in\mathbb{R}^3 : ||x-1|-|x+1||^3+||y-1|-|y+1||^3 \\ +||z-1|-|z+1||^3\leq 5 \end{array}\right\}, \end{aligned}$$

the Cassini curve $C_r^{S_b}(x_1,x_2)$ *as*

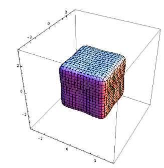
$$\begin{aligned} C_r^{S_b}(x_1,x_2) &= \left\{x\in\mathbb{R}^3 : S_b(x,x,x_1)\,S_b(x,x,x_2)=40\right\} \\ &= \left\{\begin{array}{c} x\in\mathbb{R}^3 : (|x-1|\,|x+1|)^3+(|y-1|\,|y+1|)^3 \\ +(|z-1|\,|z+1|)^3\leq 5 \end{array}\right\} \end{aligned}$$

and the Apollonius circle $A_r^{S_b}(x_1, x_2)$ *as*

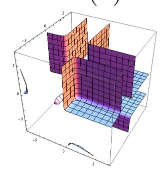
$$\begin{aligned} A_r^{S_b}(x_1, x_2) &= \left\{ x \in \mathbb{R}^3 : \frac{S_b(x, x, x_1)}{S_b(x, x, x_2)} = 40 \right\} \\ &= \left\{ x \in \mathbb{R}^3 : \left(\frac{|x-1|}{|x+1|} \right)^3 + \left(\frac{|y-1|}{|y+1|} \right)^3 + \left(\frac{|z-1|}{|z+1|} \right)^3 \leq 5 \right\}. \end{aligned}$$



(A) The hyperbola $H_r^{S_b}(x_1, x_2)$



(B) The Cassini curve $C_r^{S_b}(x_1, x_2)$



(C) The Apollonius circle $A_r^{S_b}(x_1, x_2)$

FIGURE 3. The geometric figures of the hyperbola, Cassini curve and Apollonius circle

We give the following definitions of new notions to obtain some fixed-figure results.

Definition 2.2. *Let* (X, S_b) *be an* S_b*-metric space with* $b \geq 1$ *and* $f : X \to X$ *be a self-mapping. A geometric figure* $\mathcal{F}$ *contained in the fixed point set* $Fix(f)$ *is called a fixed figure of the self-mapping* f*.*

Definition 2.3. *Let* (X, S_b) *be an* S_b*-metric space and* $f : X \to X$ *a self mapping. If there exists* $x_0 \in X$ *such that*

$$S_b(x, x, fx) > 0 \Rightarrow \varphi(S_b(x, x, fx)) \leq [\varphi(S_b(x, x, x_0))]^\alpha$$

for all $x \in X$ *where* $\alpha \in (0, 1)$ *and the function* $\varphi : (0, \infty) \to (1, \infty)$ *is such that* φ *is non-decreasing, then* f *is called Jleli-Samet type* D_{x_0}*-*S_b*-contraction.*

Theorem 2.1. *Let* (X, S_b) *be an* S_b*-metric space and* $f : X \to X$ *Jleli-Samet type* D_{x_0}*-*S_b*-contraction with* $x_0 \in X$ *and the number* r *defined as*

$$r = \inf \{S_b(x, x, fx) : x \neq fx, x \in X\}. \tag{1}$$

Then f *fixes the disc* $D_{x_0,r}^{S_b}$*.*

Proof. At first, we show $fx_0 = x_0$. On the contrary, let $fx_0 \neq x_0$. Using the Jleli-Samet type D_{x_0}-S_b-contraction hypothesis, we get

$$\begin{aligned} \varphi(S_b(x_0, x_0, fx_0)) &\leq [\varphi(S_b(x_0, x_0, x_0))]^\alpha \\ &= [\varphi(0)]^\alpha, \end{aligned}$$

a contradiction. So we get

$$fx_0 = x_0. \tag{2}$$

To show that f fixes the disc $D_{x_0,r}^{S_b}$, we consider the following cases:

Case 1: Let $r=0$. Then we have $D_{x_0,r}^{S_b}=\{x_0\}$ and by the equality (2), we get $fx_0=x_0$.

Case 2: Let $r>0$ and $x\in D_{x_0,r}^{S_b}$ be any point such that $x\neq fx$. Using the hypothesis, we obtain

$$\begin{aligned}\varphi\left(S_b\left(x,x,fx\right)\right) &\leq \left[\varphi\left(S_b\left(x,x,x_0\right)\right)\right]^{\alpha}\\ &\leq \left[\varphi\left(r\right)\right]^{\alpha}\\ &\leq \left[\varphi\left(S_b\left(x,x,fx\right)\right)\right]^{\alpha}\end{aligned}$$

a contradiction with $\alpha\in(0,1)$. Hence, it should be $fx=x$. Consequently f fixes the disc $D_{x_0,r}^{S_b}$. □

Now we give the following corollary:

Corollary 2.1. *If we take $b=1$, then we get Theorem 2.2 in* [24].

Definition 2.4. *Let (X,S_b) be an S_b-metric space and $f:X\to X$ a self mapping. If there exists $x_1,x_2\in X$ such that*

$$S_b\left(x,x,fx\right)>0\Rightarrow\varphi\left(S_b\left(x,x,fx\right)\right)\leq\left[\varphi\left(S_b\left(x,x,x_1\right)+S_b\left(x,x,x_2\right)\right)\right]^{\alpha}$$

for all $x\in X\setminus\{x_1,x_2\}$ where $\alpha\in(0,1)$ and the function $\varphi:(0,\infty)\to(1,\infty)$ is such that φ is non-decreasing, then f is called Jleli-Samet type E_{x_1,x_2}-S_b-contraction.

Theorem 2.2. *Let (X,S_b) be an S_b-metric space and $f:X\to X$ Jleli-Samet type E_{x_1,x_2}-S_b-contraction with $x_1,x_2\in X$ and the number r defined as (1). If $fx_1=x_1$ and $fx_2=x_2$, then f fixes the ellipse $E_r^{S_b}(x_1,x_2)$.*

Proof. We consider the following cases:

Case 1: Let $r=0$. Then we have $x_1=x_2$ and $E_r^{S_b}(x_1,x_2)=\{x_1\}=\{x_2\}$. Using the hypothesis we have $fx_1=x_1$ and $fx_2=x_2$.

Case 2: Let $r>0$ and $x\in E_r^{S_b}(x_1,x_2)$ be any point such that $x\neq fx$. Using the hypothesis we get

$$\begin{aligned}\varphi\left(S_b\left(x,x,fx\right)\right) &\leq \left[\varphi\left(S_b\left(x,x,x_1\right)+S_b\left(x,x,x_2\right)\right)\right]^{\alpha}\\ &\leq \left[\varphi\left(r\right)\right]^{\alpha}\\ &\leq \left[\varphi\left(S_b\left(x,x,fx\right)\right)\right]^{\alpha}\end{aligned}$$

a contradiction with $\alpha\in(0,1)$. Hence it should be $fx=x$. Consequently f fixes the ellipse $E_r^{S_b}(x_1,x_2)$.

□

Corollary 2.2. *If we take $b=1$, then we get fixed ellipse results on an S-metric space.*

Definition 2.5. *Let (X,S_b) be an S_b-metric space and $f:X\to X$ a self mapping. If there exists $x_1,x_2\in X$ such that*

$$S_b\left(x,x,fx\right)>0\Rightarrow\varphi\left(S_b\left(x,x,fx\right)\right)\leq\left[\varphi\left(\left|S_b\left(x,x,x_1\right)-S_b\left(x,x,x_2\right)\right|\right)\right]^{\alpha}$$

for all $x\in X\setminus\{x_1,x_2\}$ where $\alpha\in(0,1)$ and the function $\varphi:(0,\infty)\to(1,\infty)$ is such that φ is non-decreasing, then f is called Jleli-Samet type H_{x_1,x_2}-S_b-contraction.

Theorem 2.3. *Let (X,S_b) be an S_b-metric space and $f:X\to X$ Jleli-Samet type H_{x_1,x_2}-S_b-contraction with $x_1,x_2\in X$ and the number r defined as (1). If $fx_1=x_1$ and $fx_2=x_2$ and $r>0$, then f fixes the hyperbola $H_r^{S_b}(x_1,x_2)$.*

Proof. Let $x \in H_r^{S_b}(x_1, x_2)$ be any point such that $x \neq fx$. Using the hypothesis we get

$$\begin{aligned} \varphi(S_b(x,x,fx)) &\leq [\varphi(|S_b(x,x,x_1) - S_b(x,x,x_2)|)]^\alpha \\ &\leq [\varphi(r)]^\alpha \\ &\leq [\varphi(S_b(x,x,fx))]^\alpha \end{aligned}$$

a contradiction with $\alpha \in (0,1)$. Hence it should be $fx = x$. Consequently f fixes the hyperbola $H_r^{S_b}(x_1, x_2)$. □

Corollary 2.3. *If we take $b = 1$, then we get fixed hyperbola results on an S-metric space*

Definition 2.6. *Let (X, S_b) be an S_b-metric space and $f : X \to X$ a self mapping. If there exists $x_1, x_2 \in X$ such that*

$$S_b(x,x,fx) > 0 \Rightarrow \varphi(S_b(x,x,fx)) \leq [\varphi(S_b(x,x,x_1)\,S_b(x,x,x_2))]^\alpha$$

for all $x \in X \setminus \{x_1, x_2\}$ where $\alpha \in (0,1)$ and the function $\varphi : (0,\infty) \to (1,\infty)$ is such that φ is non-decreasing, then f is called Jleli-Samet type C_{x_1,x_2}-S_b-contraction.

Theorem 2.4. *Let (X, S_b) be an S_b-metric space and $f : X \to X$ Jleli-Samet type C_{x_1,x_2}-S_b-contraction with $x_1, x_2 \in X$ and the number r defined as (1). If $fx_1 = x_1$ and $fx_2 = x_2$, then f fixes the Cassini curve $C_r^{S_b}(x_1, x_2)$.*

Proof. We consider the following cases:

Case 1: Let $r = 0$. Then we have $x_1 = x_2$ and $C_r^{S_b}(x_1, x_2) = \{x_1\} = \{x_2\}$. Using the hypothesis we have $fx_1 = x_1$ and $fx_2 = x_2$.

Case 2: Let $r > 0$ and $x \in C_r^{S_b}(x_1, x_2)$ be any point such that $x \neq fx$. Using the hypothesis we get

$$\begin{aligned} \varphi(S_b(x,x,fx)) &\leq [\varphi(S_b(x,x,x_1)\,S_b(x,x,x_2))]^\alpha \\ &\leq [\varphi(r)]^\alpha \\ &\leq [\varphi(S_b(x,x,fx))]^\alpha \end{aligned}$$

a contradiction with $\alpha \in (0,1)$. Hence it should be $fx = x$. Consequently f fixes the Cassini curve $C_r^{S_b}(x_1, x_2)$. □

Corollary 2.4. *If we take $b = 1$, then we get fixed Cassini curve results on an S-metric space.*

Definition 2.7. *Let (X, S_b) be an S_b-metric space and $f : X \to X$ a self mapping. If there exists $x_1, x_2 \in X$ such that*

$$S_b(x,x,fx) > 0 \Rightarrow \varphi(S_b(x,x,fx)) \leq \left[\varphi\left(\frac{S_b(x,x,x_1)}{S_b(x,x,x_2)}\right)\right]^\alpha$$

for all $x \in X \setminus \{x_1, x_2\}$ where $\alpha \in (0,1)$ and the function $\varphi : (0,\infty) \to (1,\infty)$ is such that φ is non-decreasing, then f is called Jleli-Samet type A_{x_1,x_2}-S_b-contraction.

Theorem 2.5. *Let (X, S_b) be an S_b-metric space and $f : X \to X$ Jleli-Samet type A_{x_1,x_2}-S_b-contraction with $x_1, x_2 \in X$ and the number r defined as (1). If $fx_1 = x_1$ and $fx_2 = x_2$, then f fixes the Apollonius circle $A_r^{S_b}(x_1, x_2)$.*

Proof. We consider the following cases:

Case 1: Let $r=0$. Then we have $x_1=x_2$ and $A_r^{S_b}(x_1,x_2)=\{x_1\}=\{x_2\}$. Using the hypothesis we have $fx_1=x_1$ and $fx_2=x_2$.

Case 2: Let $r>0$ and $x\in A_r^{S_b}(x_1,x_2)$ be any point such that $x\neq fx$. Using the hypothesis we get

$$\begin{aligned}\varphi(S_b(x,x,fx)) &\leq \left[\varphi\left(\frac{S_b(x,x,x_1)}{S_b(x,x,x_2)}\right)\right]^{\alpha}\\ &\leq [\varphi(r)]^{\alpha}\\ &\leq [\varphi(S_b(x,x,fx))]^{\alpha}\end{aligned}$$

a contradiction with $\alpha\in(0,1)$. Hence it should be $fx=x$. Consequently f fixes the Apollonius circle $C_r^{S_b}(x_1,x_2)$. □

Corollary 2.5. *If we take $b=1$, then we get fixed Apollonius circle results on an S-metric space.*

Finally we give the following illustrative example.

Example 2.2. *Let $X=[-1,1]\cup\left\{-7,-\sqrt{2},\sqrt{2},\frac{7}{3},7,8,21\right\}$ and the S-metric defined as*

$$S(x,y,z)=|x-z|+|x+z-2y|,$$

for all $x,y,z\in\mathbb{R}$ [10]. This S-metric is also an S_b-metric with $b=1$. Let us define the function $f:X\to X$ as

$$fx=\begin{cases} x & , \quad X-\{8\} \\ 7 & , \quad x=8 \end{cases},$$

for all $x\in X$ and the function $\varphi:(0,\infty)\to(1,\infty)$ as

$$\varphi(t)=t+1,$$

for all $t>0$ with $r=2$. Then,

▷ *The function f is Jleli-Samet type D_{x_0}-S_b-contraction with $\alpha=0.5, x_0=0$. Consequently, f fixes the disc $D_{0,2}^{S_b}=[-1,1]$.*

▷ *The function f is Jleli-Samet type E_{x_1,x_2}-S_b-contraction with $x_1=-\frac{1}{2}$, $x_2=\frac{1}{2}$ and $\alpha=0.5$. Consequently, f fixes the ellipse $E_2^{S_b}\left(-\frac{1}{2},\frac{1}{2}\right)=\left[-\frac{1}{2},\frac{1}{2}\right]$.*

▷ *The function f is Jleli-Samet type H_{x_1,x_2}-S_b-contraction with $x_1=-1, x_2=1$ and $\alpha=0.9$. Consequently, f fixes the hyperbola $H_2^{S_b}(-1,1)=\left\{-\frac{1}{2},\frac{1}{2}\right\}$.*

▷ *The function f is Jleli-Samet type C_{x_1,x_2}-S_b-contraction with $x_1=-1, x_2=1$ and $\alpha=0.5$. Consequently, f fixes the Cassini curve $C_2^{S_b}(-1,1)=\left\{-\sqrt{2},0,\sqrt{2}\right\}$.*

▷ *The function f is Jleli-Samet type A_{x_1,x_2}-S_b-contraction with $x_1=-7, x_2=7$ and $\alpha=0.5$. Consequently, f fixes the Apollonius circle $A_2^{S_b}(-7,7)=\left\{\frac{7}{3},21\right\}$.*

3. Conclusion

In this paper, we present some new contractions and some fixed-figure results on an S_b-metric space. The obtained results can be considered as some geometric consequences of fixed-point theory. Using these approaches, new geometric generalizations of known fixed-point theorems can be studied on metric and generalized metric spaces.

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