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# OSCILLATION CRITERIA FOR THIRD-ORDER NONLINEAR DIFFERENCE EQUATIONS WITH AN UNBOUNDED SUBLINEAR NEUTRAL TERM

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Abstract. New sufficient conditions are obtained for the oscillation of all solutions of third-order difference equations with a sublinear neutral term. We derive these criteria by a reduction of order, reducing the third-order equations into first-order delay difference equations with a known oscillatory behavior. We provide examples, illustrating these results.

## 1. Introduction

In this paper, we investigate the oscillatory behavior of the solutions of the third-order nonlinear difference equation with a sublinear neutral term

$$\Delta\left(\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta}\right)+\xi(\ell)\nu^{\lambda}(\ell-\sigma)=0,\ \ell\geq\ell_0 \tag{1}$$

where $\chi(\ell)=\nu(\ell)+\mu(\ell)\nu^{\alpha}(\ell-\tau)$ and $\ell_0$ is a positive integer.

Throughout the paper, we assume that:

($E_1$) Each of the $\alpha,\ \beta$ and $\lambda$ is a ratio of odd positive integers with $0<\alpha\leq 1$;
($E_2$) $\{\varphi(\ell)\},\ \{\psi(\ell)\},\ \{\mu(\ell)\}$ and $\{\xi(\ell)\}$ are positive real sequences with $\mu(\ell)\geq 1$ and $\mu(\ell)\to\infty$ as $\ell\to\infty$;
($E_3$) $\sigma$ and $\tau$ are positive integers with $\tau<\sigma$;
($E_4$) $\sum_{\ell=\ell_0}^{\infty}\varphi^{-\frac{1}{\beta}}(\ell)=\sum_{\ell=\ell_0}^{\infty}\frac{1}{\psi(\ell)}=\infty$.

By a solution of (1), we mean a nontrivial sequence $\{\nu(\ell)\}$ defined for all $\ell\geq\ell_0-\theta$, where $\theta=\max\{\tau,\ \sigma\}$, which satisfies (1) for all $\ell\geq\ell_0$. A solution of (1) is called *oscillatory* if it is neither eventually positive nor eventually negative. Otherwise, it is *nonoscillatory*. A difference equation is called oscillatory if all its solutions are oscillatory.

In recent years, there has been a great interest in the study of the oscillatory and asymptotic behavior of the solutions to different classes of third-order neutral difference equations. We refer the reader to the papers in [1, 3, 6, 7, 4, 10, 8, 9, 11, 12, 14, 18, 19, 20, 21, 22, 24, 25] and the references cited therein. However, the majority of the criteria thus far, for this type of equations, are limited equations having a linear neutral term, that is, $\alpha=1$. Only a few results have been reported

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on equations with a nonlinear neutral term, that is, $\alpha \neq 1$; see [17] for sublinear neutral term ($\alpha < 1$) and [27] for a superlinear neutral term ($\alpha > 1$).

To the best of authors knowledge, there have been no results on third-order difference equations with a sublinear neutral term except [17], where equation (1) was studied in the case where $\psi(\ell) = 1$, $\beta = 1$ and $0 < \mu(\ell) \leq \mu < 1$.

In view of the above observations, the main objective in this work is to derive sufficient conditions which ensure that all solutions of (1) are oscillatory. We obtain these results by a comparison method and a summation averaging criterion. We reduce the third-order equations into first-order delay difference equations with known oscillatory characteristics. Examples illustrating the main results are included in the text of the paper.

## 2. Main Results

We begin with the following lemmas that will paly an important part in establishing our main results. For easy of reference, we let,

$$B(\ell) = \sum_{s=\ell_1}^{\ell-1} \varphi^{-\frac{1}{\alpha}}(s),\ D(\ell) = \sum_{s=\ell_2}^{\ell-1} \frac{B(s)}{\psi(s)}$$

for $\ell \geq \ell_2 \geq \ell_1 \geq \ell_0$, and there exist integers $\eta$ and $\zeta$ such that $\ell + \tau - \sigma < \ell - \eta < \ell - \zeta$. Let

$$R(\ell) = \left( \sum_{t=\ell+\tau-\sigma}^{\ell-\eta-1} \frac{1}{\psi(t)} \right) \left( \sum_{t=\ell-\eta}^{\ell-\zeta-1} \varphi^{\frac{1}{\beta}}(t) \right).$$

Furthermore, we assume that for every positive constants $\gamma$ and $\delta$

$(E_5)$ $\Phi(\ell) = \frac{1}{\mu(\ell+\tau)} \left[ 1 - \frac{D^{\frac{1}{\alpha}}(\ell+2\tau)}{D(\ell+\tau)} \frac{\gamma^{\frac{1}{\alpha}-1}}{\mu^{\frac{1}{\alpha}}(\ell+2\tau)} \right] > 0$

and

$(E_6)$ $\Psi(\ell) = \frac{1}{\mu(\ell+\tau)} \left[ 1 - \frac{\delta^{\frac{1}{\alpha}-1}}{\mu^{\frac{1}{\alpha}}(\ell+2\tau)} \right] > 0$

for all sufficiently large $\ell$.

**Lemma 2.1.** *Assume that $(E_1) - (E_4)$ hold and $\{\nu(\ell)\}$ is an eventually positive solution of (1). Then there exists an integer $\ell_1 \geq \ell_0$ such that the corresponding sequence $\{\chi(\ell)\}$, satisfies one of the following cases:*

(A) $\chi(\ell) > 0,\ \Delta\chi(\ell) > 0,\ \Delta(\psi(\ell)\Delta\chi(\ell)) > 0,$ *and* $\Delta\left(\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta}\right) < 0$;

(B) $\chi(\ell) > 0,\ \Delta\chi(\ell) < 0, \Delta(\psi(\ell)\Delta\chi(\ell)) > 0,$ *and* $\Delta\left(\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta}\right) < 0$

*for all $\ell \geq \ell_1$.*

**Proof.** The proof is not difficult and thus, the details are omitted. □

**Lemma 2.2.** *Assume that $(E_1){-}(E_4)$ hold and $\{\chi(\ell)\}$ satisfies Case (A) of Lemma 2.1 for all $\ell \geq \ell_1$. Then*

$$\Delta\chi(\ell) \geq \frac{B(\ell)}{\psi(\ell)} \varphi^{\frac{1}{\beta}}(\ell) \Delta(\psi(\ell)\Delta\chi(\ell)), \tag{2}$$

$$\chi(\ell) \geq D(\ell) \varphi^{\frac{1}{\beta}}(\ell) \Delta(\psi(\ell)\Delta\chi(\ell)), \tag{3}$$

$$\chi(\ell) \geq \frac{D(\ell)}{B(\ell)}\psi(\ell)\Delta\chi(\ell), \tag{4}$$

*and*

$$\left\{\frac{\chi(\ell)}{D(\ell)}\right\} \text{ is nonincreasing for all } \ell \geq \ell_1. \tag{5}$$

**Proof.** Since $\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^\beta > 0$ and nonincreasing, we have

$$\begin{aligned}\psi(\ell)\Delta\chi(\ell) &= \psi(\ell_1)\Delta\chi(\ell_1) + \sum_{s=\ell_1}^{\ell-1} \frac{\varphi^{\frac{1}{\beta}}(s)\Delta(\psi(s)\Delta\chi(s))}{\varphi^{\frac{1}{\beta}}(s)} \\ &\geq B(\ell)\varphi^{\frac{1}{\beta}}(\ell)\Delta(\psi(\ell)\Delta\chi(\ell)),\end{aligned}$$

which proves (2). Dividing the last inequality by $\psi(\ell)$ and summing up the resulting inequality from $\ell_1$ to $\ell-1$, we obtain

$$\chi(\ell) \geq D(\ell)\varphi^{\frac{1}{\beta}}(\ell)\Delta(\psi(\ell)\Delta\chi(\ell))$$

which proves (3). From (2), it is easy to see that

$$\frac{\psi(\ell)\Delta\chi(\ell)}{B(\ell)} \text{ is nonincreasing.} \tag{6}$$

Therefore, from (6), it follows that

$$\begin{aligned}\chi(\ell) &= \chi(\ell_1) + \sum_{s=\ell_1}^{\ell-1} \frac{\psi(s)\Delta\chi(s)}{B(s)}\frac{B(s)}{\psi(s)} \\ &\geq D(\ell)\frac{\psi(\ell)\Delta\chi(\ell)}{B(\ell)},\end{aligned}$$

which proves (4). Again from the last inequality, we see that

$$\Delta\left(\frac{\chi(\ell)}{D(\ell)}\right) = \frac{D(\ell)\psi(\ell)\Delta\chi(\ell) - \chi(\ell)B(\ell)}{\psi(\ell)D(\ell)D(\ell+1)} \leq 0$$

which implies that $\left\{\frac{\chi(\ell)}{D(\ell)}\right\}$ is nonincreasing. The proof of the lemma is complete. □

**Lemma 2.3.** *Assume that* $(E_1)-(E_5)$ *hold and* $\{\nu(\ell)\}$ *is an eventually positive solution of* (1) *with the corresponding sequence* $\{\chi(\ell)\}$ *satisfying Case (A) of Lemma 2.1 for all* $\ell \geq \ell_1$. *Then* $\{\chi(\ell)\}$ *satisfies the inequality*

$$\Delta\left(\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^\beta\right) + \xi(\ell)\Phi^{\frac{\lambda}{\alpha}}(\ell-\sigma)\chi^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma) \leq 0 \tag{7}$$

*for large* $\ell$.

**Proof.** Assume $\{\nu(\ell)\}$ is an eventually positive solution of (1). Thus, $\nu(\ell) > 0$, $\nu(\ell-\tau) > 0$ and $\nu(\ell-\sigma) > 0$ for $\ell \geq \ell_1$ for some integer $\ell_1 \geq \ell_0$. From the definition of $\chi(\ell)$, we have

$$\nu^\alpha(\ell) \geq \frac{1}{\mu(\ell+\tau)}\left(\chi(\ell+\tau) - \frac{\chi^{\frac{1}{\alpha}}(\ell+2\tau)}{\mu^{\frac{1}{\alpha}}(\ell+2\tau)}\right),\ \ell \geq \ell_1. \tag{8}$$

Since $\frac{\chi(\ell)}{D(\ell)}$ is nonincreasing, (8) becomes

$$\nu^\alpha(\ell) \geq \frac{1}{\mu(\ell+\tau)}\left(\chi(\ell+\tau) - \frac{D^{\frac{1}{\alpha}}(\ell+2\tau)}{D^{\frac{1}{\alpha}}(\ell+\tau)}\frac{\chi^{\frac{1}{\alpha}}(\ell+\tau)}{\mu^{\frac{1}{\alpha}}(\ell+2\tau)}\right). \tag{9}$$

Moreover, there exists a constant $\gamma > 0$ so that $\frac{\chi(\ell+\tau)}{D(\ell+\tau)} \leq \gamma$. From this and (9), we obtain

$$\begin{aligned} \nu^{\alpha}(\ell) &\geq \frac{\chi(\ell+\tau)}{\mu(\ell+\tau)}\left(1-\frac{D^{\frac{1}{\alpha}}(\ell+2\tau)}{D^{\frac{1}{\alpha}}(\ell+\tau)}\frac{\gamma^{\frac{1}{\alpha}-1}}{\mu^{\frac{1}{\alpha}}(\ell+2\tau)}\right) \\ &= \Phi(\ell)\chi(\ell+\tau). \end{aligned}$$

Using the last inequality in (1) we obtain (7). This completes the proof. $\square$

**Lemma 2.4.** *Assume that* $(E_1)-(E_4)$ *and* $(E_6)$ *hold. If* $\{\nu(\ell)\}$ *is an eventually positive solution of* (1) *with the corresponding sequence* $\{\chi(\ell)\}$ *satisfying Case (B) of Lemma 2.1 for all* $\ell \geq \ell_1 \geq \ell_0$*, then* $\{\chi(\ell)\}$ *satisfies the inequality*

$$\Delta\left(\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta}\right)+\xi(\ell)\Psi^{\frac{\lambda}{\alpha}}(\ell-\sigma)\chi^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)\leq 0 \tag{10}$$

*for large* $\ell$.

**Proof.** Assume $\{\nu(\ell)\}$ is an eventually positive solution of (1) such that $\nu(\ell) > 0$, $\nu(\ell-\tau) > 0$ and $\nu(\ell-\sigma) > 0$ for $\ell \geq \ell_1$ for some integer $\ell_1 \geq \ell_0$. Proceeding as in the proof of Lemma 2.3, we see that (8) holds for all $\ell \geq \ell_1$. Since $\{\chi(\ell)\}$ is decreasing, it follows from (8) that

$$\nu^{\alpha}(\ell) \geq \frac{\chi(\ell+\tau)}{\mu(\ell+\tau)}\left(1-\frac{\chi^{\frac{1}{\alpha}-1}(\ell+\tau)}{\mu^{\frac{1}{\alpha}}(\ell+2\tau)}\right). \tag{11}$$

Also there exists a constant $\delta > 0$ such that $\chi(\ell) \leq \delta$ and $\frac{1}{\alpha} > 1$. We see that $\chi^{\frac{1}{\alpha}-1}(\ell+\tau) \leq \delta^{\frac{1}{\alpha}-1}$ for all $\ell \geq \ell_1$. Using this in (11), we obtain

$$\nu^{\alpha}(\ell) \geq \Psi(\ell)\chi(\ell+\tau). \tag{12}$$

Combining (12) with (1) yields (10). This completes the proof. $\square$

**Theorem 2.5.** *Assume that* $(E_1)-(E_6)$ *hold. If for all sufficiently large integer* $\ell_1 \geq \ell_0$*, and for some integer* $\ell_2 \geq \ell_1$*,*

$$\sum_{\ell=\ell_2}^{\infty}\xi(\ell)\Phi^{\frac{\lambda}{\alpha}}(\ell-\sigma)=\infty \tag{13}$$

*and*

$$\sum_{\ell=\ell_1}^{\infty}\xi(\ell)\Psi^{\frac{\lambda}{\alpha}}(\ell-\sigma)=\infty, \tag{14}$$

*then every solution of* (1) *is either oscillatory or* $\lim_{\ell\to\infty}\nu(\ell)=0$.

**Proof.** Assume that $\{\nu(\ell)\}$ is a nonoscillatory solution of (1). Without loss of generality, we can assume $\nu(\ell) > 0$, $\nu(\ell-\tau) > 0$ and $\nu(\ell-\sigma) > 0$ for $\ell \geq \ell_1$ for some integer $\ell_1 \geq \ell_0$. Then, in view of Lemma 2.1, $\{\chi(\ell)\}$ satisfies either Case (A) or Case (B) for all $\ell \geq \ell_1$.

**Case (A)**. From Lemma 2.3, we see that inequality (7) holds for all $\ell \geq \ell_1$. Since $\{\chi(\ell)\}$ is increasing, there exists a constant $M > 0$ and an integer $\ell_2 \geq \ell_1$ such that $\chi(\ell) \geq M$ for all $\ell \geq \ell_2$. Using this in (7) and then summing the resulting inequality from $\ell_2$ to $\ell-1$ yields

$$\varphi(\ell)\left(\Delta(\psi(\ell)\Delta\chi(\ell))\right)^{\beta} \leq \varphi(\ell_2)\left(\Delta(\psi(\ell_2)\Delta\chi(\ell_2))\right)^{\beta} - M^{\frac{\lambda}{\alpha}}\sum_{s=\ell_2}^{\ell-1}\xi(s)\Phi^{\frac{\lambda}{\alpha}}(s-\sigma)$$

which tends to $\to -\infty$ as $\ell \to \infty$. This contradiction eliminates Case (A).

Next, consider **Case (B)**. Since $\{\chi(\ell)\}$ is decreasing there exists a constant $c$ such that $\lim_{\ell\to\infty}\chi(\ell) = c \geq 0$. If $c > 0$, then $\chi(\ell) \geq c$ for all $\ell \geq \ell_1 \geq \ell_0$. Using this in (10), we have

$$\Delta\left(\varphi(\ell)\left(\Delta(\psi(\ell)\Delta\chi(\ell))\right)^{\beta}\right) \leq -c^{\frac{\lambda}{\alpha}}\xi(\ell)\Psi^{\frac{\lambda}{\alpha}}(\ell-\sigma),\ \ell \geq \ell_1.$$

Summing up the last inequality from $\ell_1$ to $\ell - 1$ yields

$$\varphi(\ell)\left(\Delta(\psi(\ell)\Delta\chi(\ell))\right)^{\beta} \leq \varphi(\ell_1)\left(\Delta(\psi(\ell_1)\Delta\chi(\ell_1))\right)^{\beta} - c^{\frac{\lambda}{\alpha}}\sum_{s=\ell_1}^{\ell-1}\xi(s)\Psi^{\frac{\lambda}{\alpha}}(s-\sigma)$$

which tends to $\to -\infty$ as $\ell \to \infty$, which is a contradiction.

If $c = 0$, then $\nu(\ell) \leq \chi(\ell)$ implies that $\nu(\ell) \to 0$ as $\ell \to \infty$. This completes the proof of the theorem. □

Next, we present a new criteria for the oscillation of all solutions of (1) using a comparison with first-order delay difference equations whose oscillatory properties are known.

Before presenting our results, let us assume that

$(E_7)$ $\lim_{\ell\to\infty}\frac{D^{\frac{1}{\alpha}}(\ell+2\tau)}{D(\ell+\tau)\mu^{\frac{1}{\alpha}}(\ell+2\tau)} = 0.$

Since $\{D(\ell)\}$ is positive and increasing and $\frac{1}{\alpha} > 1$, we can easily see that $(E_7)$ implies

$$\lim_{\ell\to\infty}\frac{1}{\mu^{\frac{1}{\alpha}}(\ell+2\tau)} = 0. \tag{15}$$

We can use these conditions to simplify $\Phi(\ell)$ and $\Psi(\ell)$ as follows.

In view of $(E_7)$, there exists $\epsilon \in (0,1)$ and an integer $\ell_1 \geq \ell_0$ such that

$$\frac{D^{\frac{1}{\alpha}}(\ell+2\tau)}{D(\ell+\tau)\mu^{\frac{1}{\alpha}}(\ell+2\tau)} \leq \gamma^{1-\frac{1}{\alpha}}(1-\epsilon).$$

Using this with $\Phi(\ell)$, we obtain

$$\Phi(\ell) \geq \frac{\epsilon}{\mu(\ell+\tau)},\ \ell \geq \ell_1. \tag{16}$$

Again from (15), there exists $\epsilon_1 \in (0,1)$ and an integer $\ell_2 \geq \ell_1$ such that

$$\frac{1}{\mu^{\frac{1}{\alpha}}(\ell+2\tau)} \leq \gamma^{1-\frac{1}{\alpha}}(1-\epsilon_1).$$

Using this with $\Psi(\ell)$, we have

$$\Psi(\ell) \geq \frac{\epsilon_1}{\mu(\ell+\tau)},\ \ell \geq \ell_2. \tag{17}$$

**Theorem 2.6.** *Assume that $(E_1) - (E_4)$ and $(E_7)$ hold. Moreover, suppose that there exist positive integers $\eta$ and $\zeta$ such that $\ell + \tau - \sigma \leq \ell - \eta \leq \ell - \zeta$ for $\ell \geq \ell_0$. If the first-order delay difference equations*

$$\Delta X(\ell) + \frac{\epsilon^{\frac{\lambda}{\alpha}}\xi(\ell)}{\mu^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}D^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)X^{\frac{\lambda}{\alpha\beta}}(\ell+\tau-\sigma) = 0 \tag{18}$$

*and*

$$\Delta\omega(\ell) + \frac{\epsilon_1^{\frac{\lambda}{\alpha}}\xi(\ell)}{\mu^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)} R^{\frac{\lambda}{\alpha}}(\ell)\omega^{\frac{\lambda}{\alpha\beta}}(\ell-\zeta) = 0 \tag{19}$$

*are oscillatory, then* (1) *is oscillatory.*

**Proof.** Let $\{\nu(\ell)\}$ be a nonoscillatory solution of (1), say $\nu(\ell) > 0$, $\nu(\ell-\tau) > 0$ and $\nu(\ell-\sigma) > 0$ for all $\ell \geq \ell_1$ for some integer $\ell_1 \geq \ell_0$. Then from Lemma 2.1, $\{\chi(\ell)\}$ satisfies either Case (A) or Case (B) for $\ell \geq \ell_1$.

**Case(A).** Proceeding as in the proof of Lemma 2.3, we arrive at (7). Using (16) in (7), we obtain

$$\Delta\left(\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta}\right) + \frac{\epsilon^{\frac{\lambda}{\alpha}}\xi(\ell)}{\mu^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}\chi^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma) \leq 0, \ \ell \geq \ell_1. \tag{20}$$

Using (3) in (20) and letting $X(\ell) = \varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta} > 0$, we see that $\{X(\ell)\}$ is a positive solution of the first-order delay difference inequality

$$\Delta X(\ell) + \frac{\epsilon^{\frac{\lambda}{\alpha}}\xi(\ell)}{\mu^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)} D^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma) X^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma) \leq 0.$$

Hence by Lemma 1.3 in [2] and Lemma 6.2.2 in [3], we conclude that equation (18) has a positive solution, which is a contradiction.

**Case (B).** For $s \geq m \geq \ell_2 \geq \ell_1$, we have

$$\chi(m) = \chi(s) + \sum_{t=m}^{s-1} \frac{\psi(t)\Delta\chi(t)}{\psi(t)} \geq -\psi(s)\Delta\chi(s)\sum_{t=m}^{s-1}\frac{1}{\psi(t)}. \tag{21}$$

Setting $m = \ell+\tau-\sigma$ and $s = \ell-\eta$ in (21), we obtain

$$\chi(\ell+\tau-\sigma) \geq \left(\sum_{t=\ell+\tau-\sigma}^{\ell-\eta-1}\frac{1}{\psi(t)}\right)(-\psi(\ell-\eta)\Delta\chi(\ell-\eta)). \tag{22}$$

Since $\psi(\ell)\Delta\chi(\ell) < 0$ and $\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta} > 0$ is decreasing, we have

$$\begin{aligned} -\psi(m)\Delta\chi(m) &\geq \psi(s)\Delta\chi(s) - \psi(m)\Delta\chi(m) \\ &= \sum_{t=m}^{s-1}\varphi^{-\frac{1}{\beta}}(t)\left(\varphi^{\frac{1}{\beta}}(t)\Delta(\psi(t)\Delta\chi(t))\right) \\ &\geq \left(\varphi(s)(\Delta(\psi(s)\Delta\chi(s)))^{\beta}\right)^{\frac{1}{\beta}}\sum_{t=m}^{s-1}\varphi^{-\frac{1}{\beta}}(t). \end{aligned}$$

Hence,

$$-\psi(m)\Delta\chi(m) \geq \left(\varphi(s)(\Delta(\psi(s)\Delta\chi(s)))^{\beta}\right)^{\frac{1}{\beta}}\sum_{t=m}^{s-1}\varphi^{-\frac{1}{\beta}}(t).$$

Letting $m = \ell-\eta$ and $s = \ell-\zeta$ in the last inequality, we have

$$-\psi(\ell-\eta)\Delta\chi(\ell-\eta) \geq \left(\sum_{t=\ell-\eta}^{\ell-\zeta-1}\varphi^{-\frac{1}{\beta}}(t)\right)\left(\varphi(\ell-\zeta)(\Delta(\psi(\ell-\zeta)\Delta\chi(\ell-\zeta)))^{\beta}\right)^{\frac{1}{\beta}}. \tag{23}$$

Combining (22) and (23), we get

$$\chi(\ell+\tau-\sigma) \geq R(\ell)\left(\varphi(\ell-\zeta)(\Delta(\psi(\ell-\zeta)\Delta\chi(\ell-\zeta)))^{\beta}\right)^{\frac{1}{\beta}}. \tag{24}$$

Using (24) and (17) in (10), we obtain

$$\Delta\left(\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta}\right) + \frac{\epsilon_1^{\frac{\lambda}{\alpha}}\xi(\ell)}{\mu^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)} R^{\frac{\lambda}{\alpha}}(\ell)\left(\varphi(\ell-\zeta)(\Delta(\psi(\ell-\zeta)\Delta\chi(\ell-\zeta)))^{\beta}\right)^{\frac{1}{\beta}} \le 0.$$

Set $\omega(\ell) = \varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta}$. Clearly $\omega(\ell) > 0$. In view of this, the above inequality takes the form

$$\Delta\omega(\ell) + \frac{\epsilon_1^{\frac{\lambda}{\alpha}}\xi(\ell)}{\mu^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)} R^{\frac{\lambda}{\alpha}}(\ell)\omega^{\frac{\lambda}{\alpha\beta}}(\ell-\zeta) \le 0.$$

The rest of the proof is similar to Case (A) and hence is omitted. The proof of the theorem is complete. □

Next, for any $\epsilon,\ \epsilon_1 \in (0,1)$, let

$$Q(\ell) \le \min\left\{\epsilon^{\frac{\lambda}{\alpha}} D^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma), \epsilon_1^{\frac{\lambda}{\alpha}} R^{\frac{\lambda}{\alpha}}(\ell)\right\}.$$

We see that Theorem 2.6 is equivalent to:

**Theorem 2.7.** *Assume that $(E_1)-(E_4)$ and $(E_7)$ hold. Moreover, suppose that there exist positive integers $\eta$ and $\zeta$ such that $\ell+\tau-\sigma < \ell-\eta < \ell-\zeta$ for $\ell \ge \ell_0$. If the first-order delay difference equations*

$$\Delta Z(\ell) + \xi(\ell)Q(\ell)\mu^{-\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)Z^{\frac{\lambda}{\alpha\beta}}(\ell+\tau-\sigma) = 0 \tag{25}$$

*and*

$$\Delta Y(\ell) + \xi(\ell)Q(\ell)\mu^{-\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)Y^{\frac{\lambda}{\alpha\beta}}(\ell-\zeta) = 0 \tag{26}$$

*are oscillatory, then equation* (1) *is oscillatory.*

**Corollary 2.8.** *Assume that $(E_1)-(E_4)$ and $(E_7)$ hold. Moreover, suppose that there exist positive integers $\eta$ and $\zeta$ such that $\ell+\tau-\sigma < \ell-\eta < \ell-\zeta$ for all $\ell \ge \ell_0$. If $\lambda = \alpha\beta$, and*

$$\liminf_{\ell\to\infty} \sum_{s=\ell+\tau-\sigma}^{\ell-1} \xi(s)Q(s)\mu^{\frac{-\lambda}{\alpha}}(s+\tau-\sigma) > \left(\frac{\sigma-\tau}{\sigma-\tau+1}\right)^{\sigma-\tau+1} \tag{27}$$

*and*

$$\liminf_{\ell\to\infty} \sum_{s=\ell-\zeta}^{\ell-1} \xi(s)Q(s)\mu^{\frac{-\lambda}{\alpha}}(s+\tau-\sigma) > \left(\frac{\zeta}{\zeta+1}\right)^{\zeta+1} \tag{28}$$

*then equation* (1) *is oscillatory.*

**Proof.** Conditions (27), (28) and Theorem 7.6.1 of [15] imply that equations (25) and (26) are oscillatory. This conclusion follows from Theorem 2.7. This completes the proof. □

**Corollary 2.9.** *Let the assumptions of Corollary 2.8 hold. If $\lambda < \alpha\beta$ and*

$$\sum_{\ell=\ell_0}^{\infty} \xi(\ell)Q(\ell)\mu^{-\frac{\lambda}{\alpha}}(\ell+\tau-\sigma) = \infty \tag{29}$$

*then equation* (1) *is oscillatory.*

**Proof.** Condition (29) and Theorem 1 of [23] imply that equations (25) and (26) are oscillatory. This conclusion follows from Theorem 2.7. Thus, the proof is complete. □

**Corollary 2.10.** *Let the assumptions of Corollary 2.8 hold. If $\lambda > \alpha\beta$ and there exist $a > \frac{1}{\sigma-\tau}\ln\left(\frac{\lambda}{\alpha\beta}\right)$ and $a_1 > \frac{1}{\zeta}\ln\left(\frac{\lambda}{\alpha\beta}\right)$ such that*

$$\lim_{\ell\to\infty}\inf\left[\xi(\ell)Q(\ell)\mu^{\frac{-\lambda}{\alpha}}(\ell+\tau-\sigma)\exp\left(-e^{a\ell}\right)\right] > 0 \tag{30}$$

*and*

$$\lim_{\ell\to\infty}\inf\left[\xi(\ell)Q(\ell)\mu^{\frac{-\lambda}{\alpha}}(\ell+\tau-\sigma)\exp\left(-e^{a_1\ell}\right)\right] > 0 \tag{31}$$

*then equation* (1) *is oscillatory.*

**Proof.** In view of conditions (30), (31) and Theorem 2 in [23], we see that equations (25) and (26) are oscillatory. This conclusion follows from Theorem 2.7 and the proof is complete. □

The following result is of interest.

**Theorem 2.11.** *Assume that $(E_1)-(E_4)$ and $(E_7)$ hold. If $\lambda = \alpha\beta$*

$$\lim_{\ell\to\infty}\sup\sum_{s=N}^{\ell}\left[\xi(s)\mu^{-\frac{\lambda}{\alpha}}(s+\tau-\sigma)\frac{D^{\frac{\lambda}{\alpha}}(s+\tau-\sigma)}{B^{\frac{\lambda}{\alpha}}(s)}\right] = \infty \tag{32}$$

*and*

$$\lim_{\ell\to\infty}\sup\sum_{t=\ell}^{\ell+\sigma-\tau}\left[\frac{1}{\psi(t)}\sum_{s=\ell}^{t}\left(\frac{1}{\varphi(s)}\sum_{i=s}^{t}\xi(i)\mu^{-\frac{\lambda}{\alpha}}(i+\tau-\sigma)\right)^{\frac{1}{\beta}}\right] > \frac{1}{\epsilon_1} \tag{33}$$

*for all $\ell \geq N \geq \ell_0$, then equation* (1) *is oscillatory.*

**Proof.** Let $\{\nu(\ell)\}$ be a nonoscillatory solution of (1). Without loss of generality, there exists an integer $\ell_1 \geq \ell_0$ such that $\nu(\ell) > 0$, $\nu(\ell-\tau) > 0$ and $\nu(\ell-\sigma) > 0$ for all $\ell \geq \ell_1$. Then by Lemma 2.1, $\{\chi(\ell)\}$ satisfies either Case (A) or Case (B) for $\ell \geq \ell_1$.

**Case(A).** Define

$$Z(\ell) = \frac{\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta}}{(\psi(\ell)\Delta\chi(\ell))^{\beta}},\ \ell \geq \ell_1. \tag{34}$$

Then $Z(\ell) > 0$ for all $\ell \geq \ell_1$ and from (34), we have

$$\begin{aligned}\Delta Z(\ell) &= \frac{\Delta(\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta})}{(\psi(\ell)\Delta\chi(\ell))^{\beta}}\\ &\quad -\frac{\varphi(\ell+1)(\Delta(\psi(\ell+1)\Delta\chi(\ell+1)))^{\beta}\Delta(\psi(\ell)\Delta\chi(\ell))^{\beta}}{(\psi(\ell)\Delta\chi(\ell))^{\beta}(\psi(\ell+1)\Delta\chi(\ell+1))^{\beta}}\\ &\leq -\epsilon^{\frac{\lambda}{\alpha}}\xi(\ell)\mu^{-\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)\frac{\chi^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}{(\psi(\ell)\Delta\chi(\ell))^{\beta}},\ \ell \geq \ell_1,\end{aligned} \tag{35}$$

where we have used (20). From (6) and (4), we have

$$\begin{aligned}\chi(\ell+\tau-\sigma) &\geq \frac{D(\ell+\tau-\sigma)}{B(\ell+\tau-\sigma)}\psi(\ell+\tau-\sigma)\Delta\chi(\ell+\tau-\sigma)\\ &\geq D(\ell+\tau-\sigma)\frac{\psi(\ell)\Delta\chi(\ell)}{B(\ell)},\ \ell \geq \ell_1.\end{aligned}$$

Using this in (35), we obtain

$$\Delta Z(\ell) \le -\epsilon^{\frac{\lambda}{\alpha}} \mu^{-\frac{\lambda}{\alpha}}(\ell+\tau-\sigma) \frac{\xi(\ell) D^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}{B^{\frac{\lambda}{\sigma}}(\ell)},\ \ell \ge \ell_1.$$

Summing up the last inequality from $\ell_1$ to $\ell$, we have

$$\epsilon^{\frac{\lambda}{\alpha}} \sum_{s=\ell_1}^{\ell} \frac{\xi(s) D^{\frac{\lambda}{\alpha}}(s+\tau-\sigma)}{\mu^{\frac{\lambda}{\alpha}}(s+\tau-\sigma) B^{\frac{\lambda}{\sigma}}(s)} \le Z(\ell_1) < \infty$$

which contradicts (32).

**Case (B).** Let $\ell \ge N \ge \ell_0$ be fixed. Using (17) in (10), we obtain

$$\Delta\left(\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta}\right) + \epsilon_1^{\frac{\lambda}{\alpha}} \xi(\ell) \mu^{-\frac{\lambda}{\alpha}}(\ell+\tau-\sigma) \chi^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma) \le 0,\ \ell \ge N. \quad (36)$$

Summing up the above inequality from $\ell$ to $j$, we have

$$\begin{aligned}\varphi(j+1)(\Delta(\psi(j+1)\Delta\chi(j+1)))^{\beta} \quad - \quad & \varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta} \\ & + \epsilon_1^{\frac{\lambda}{\alpha}} \sum_{t=\ell}^{j} \xi(t) \mu^{-\frac{\lambda}{\alpha}}(t+\tau-\sigma) \chi^{\frac{\lambda}{\alpha}}(t+\tau-\sigma) \le 0.\end{aligned}$$

Since $\{\varphi(\ell)(\Delta(\psi(\ell)\Delta\chi(\ell)))^{\beta}\}$ is positive and decreasing, the above inequality that, as $j \to \infty$,

$$-\Delta(\psi(\ell)\Delta\chi(\ell)) + \epsilon_1 \left[\frac{1}{\varphi(\ell)} \sum_{t=\ell}^{\infty} \xi(t) \mu^{-\frac{\lambda}{\alpha}}(t+\tau-\sigma) \chi^{\frac{\lambda}{\alpha}}(t+\tau-\sigma)\right]^{\frac{1}{\beta}} \le 0.$$

Summing up from $n$ to $j$ and rearranging, we have

$$\begin{aligned}& -\psi(j+1)\Delta\chi(j+1) + \psi(\ell)\Delta\chi(\ell)) \\ & + \epsilon_1 \sum_{t=\ell}^{j} \left[\frac{1}{\varphi(\ell)} \sum_{s=\ell}^{t} \xi(s) \mu^{-\frac{\lambda}{\alpha}}(s+\tau-\sigma)\right]^{\frac{1}{\beta}} \chi(t+\tau-\sigma) \le 0.\end{aligned}$$

Since $\{\psi(j)\Delta\chi(j)\}$ is negative and increasing, as $j \to \infty$, we have

$$\Delta\chi(\ell) + \epsilon_1 \frac{1}{\psi(\ell)} \sum_{t=\ell}^{\infty} \left[\frac{1}{\varphi(\ell)} \sum_{s=\ell}^{t} \xi(s) \mu^{-\frac{\lambda}{\alpha}}(s+\tau-\sigma)\right]^{\frac{1}{\beta}} \chi(t+\tau-\sigma) \le 0.$$

Summing the above inequality from $\ell$ to $j$ and rearranging, we have

$$\chi(j+1) - \chi(\ell) + \epsilon_1 \sum_{t=\ell}^{j} \left(\frac{1}{\psi(t)} \sum_{s=\ell}^{t} \left(\frac{1}{\varphi(s)} \sum_{i=s}^{t} \xi(i) \mu^{-\frac{\lambda}{\alpha}}(i+\tau-\sigma)\right)^{\frac{1}{\beta}}\right) \chi(t+\tau-\sigma) \le 0.$$

Since $\{\chi(\ell)\}$ is positive and decreasing, we have from the last inequality, as $j \to \infty$, we have that

$$\sum_{t=\ell}^{\infty} \left(\frac{1}{\psi(t)} \sum_{s=\ell}^{t} \left(\frac{1}{\varphi(s)} \sum_{i=s}^{t} \xi(i) \mu^{-\frac{\lambda}{\alpha}}(i+\tau-\sigma)\right)^{\frac{1}{\beta}}\right) \chi(t+\tau-\sigma) \le \frac{1}{\epsilon_1} \chi(\ell)$$

or

$$\sum_{t=\ell}^{\ell+\sigma-\tau} \left(\frac{1}{\psi(t)} \sum_{s=\ell}^{t} \left(\frac{1}{\varphi(s)} \sum_{i=s}^{t} \xi(i) \mu^{-\frac{\lambda}{\alpha}}(i+\tau-\sigma)\right)^{\frac{1}{\beta}}\right) \le \frac{1}{\epsilon_1}$$

which contradicts (33) as $\ell \to \infty$. This completes the proof. $\square$

Next, we have the following result.

**Theorem 2.12.** *Assume that* $(E_1)-(E_4)$ *and* $(E_7)$ *hold. Moreover, suppose there exists an integer* $\eta$ *such that* $\ell+\tau-\sigma<\ell-\eta$ *for all* $\ell\geq\ell_0$. *If the second-order difference inequality*

$$\Delta\left(\varphi(\ell)(\Delta Y(\ell))^{\beta}\right)+\epsilon^{\frac{\lambda}{\alpha}}\xi(\ell)\left(\frac{D(\ell+\tau-\sigma)}{\mu(\ell+\tau-\sigma)B(\ell+\tau-\sigma)}\right)^{\frac{\lambda}{\alpha}}Y^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)\leq 0 \quad (37)$$

*has no positive increasing solution and the second-order difference inequality*

$$\Delta\left(\varphi(\ell)(\Delta Z(\ell))^{\beta}\right)-\epsilon_1^{\frac{\lambda}{\alpha}}\xi(\ell)\left(\frac{1}{\mu(\ell+\tau-\sigma)}\sum_{t=\ell+\tau-\sigma}^{\ell-\eta-1}\frac{1}{\psi(s)}\right)^{\frac{\lambda}{\alpha}}Z^{\frac{\lambda}{\alpha}}(\ell-\eta)\geq 0 \quad (38)$$

*has no positive decreasing solution, then equation* (1) *is oscillatory.*

**Proof.** Let $\{\nu(\ell)\}$ be a nonoscillatory solution of (1), say $\nu(\ell)>0,\ \nu(\ell-\tau)>0$ and $\nu(\ell-\sigma)>0$ for $\ell\geq\ell_1$ for some $\ell_1\geq\ell_0$. As in Lemma 2.1, we consider two cases, namely Cases (A) and (B) for $\ell\geq\ell_1$.
**Case (A).** Proceeding as in Case (A) of Theorem 2.6, we have (20). Now using (4) in (10), we obtain

$$\begin{aligned}\Delta\left(\varphi(\ell)\left(\Delta(\psi(\ell)\Delta\chi(\ell))\right)^{\beta}\right) \quad + \quad & \frac{\epsilon^{\frac{\lambda}{\alpha}}\xi(\ell)}{\mu^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}\frac{D^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}{B^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}\\ & (\psi(\ell+\tau-\sigma)\Delta\chi(\ell+\tau-\sigma))^{\frac{\lambda}{\alpha}}\leq 0. \quad (39)\end{aligned}$$

Let $Y(\ell)=\psi(\ell)\Delta\chi(\ell)>0$ for $\ell\geq\ell_1$. Then from (39), we see that

$$\Delta\left(\varphi(\ell)\left(\Delta Y(\ell)\right)^{\beta}\right)+\frac{\epsilon^{\frac{\lambda}{\alpha}}\xi(\ell)}{\mu^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}\frac{D^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}{B^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}Y^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)\leq 0$$

has a positive increasing solution, which is a contradiction.
**Case (B).** In this case as in Case (B), in the proof of Theorem 2.11, we have that inequality (36) holds. Now from (22) and (36), we obtain the following

$$\begin{aligned}\Delta\left(\varphi(\ell)\left(\Delta(\psi(\ell)\Delta\chi(\ell))\right)^{\beta}\right) \quad + \quad & \frac{\epsilon_1^{\frac{\lambda}{\alpha}}\xi(\ell)}{\mu^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}\\ & \left(\sum_{t=\ell+\tau-\sigma}^{\ell-\eta-1}\frac{1}{\psi(t)}\right)^{\frac{\lambda}{\alpha}}(-\psi(\ell-\tau)\Delta\chi(\ell-\tau))^{\frac{\lambda}{\alpha}}\leq 0.\end{aligned}$$

Define $Z(\ell)=-\psi(\ell)\Delta\chi(\ell)>0$ for $\ell\geq\ell_1\geq N\geq\ell_0$, then we see that

$$\Delta(\varphi(\ell)(\Delta Z(\ell))^{\beta})-\frac{\epsilon_1^{\frac{\lambda}{\alpha}}\xi(\ell)}{\mu^{\frac{\lambda}{\alpha}}(\ell+\tau-\sigma)}\left(\sum_{t=\ell+\tau-\sigma}^{\ell-\eta-1}\frac{1}{\psi(t)}\right)^{\frac{\lambda}{\alpha}}Z^{\frac{\lambda}{\alpha}}(\ell-\eta)\geq 0$$

has a positive decreasing solution which constitutes a contradiction. This completes the proof. □

**Corollary 2.13.** *Assume that* $(E_1)-(E_4)$ *and* $(E_7)$ *hold. Moreover suppose there exists an integer* $\eta$ *such that* $\ell+\tau-\sigma<\ell-\eta$ *for all* $\ell\geq\ell_0$. *If* $\lambda=\alpha\beta$,

$$\lim_{\ell\to\infty}\sup\sum_{s=N}^{\ell}\xi(s)\left(\frac{D(s+\tau-\sigma)}{\mu(s+\tau-\sigma)B(s+\tau-\sigma)}\right)^{\beta}=\infty \quad (40)$$

*and*

$$\lim_{\ell\to\infty}\sup\sum_{s=\ell-\eta}^{\ell-1}\frac{1}{\varphi(s)}\left(\sum_{t=s}^{\ell-1}Q(t)\right)^{\frac{1}{\beta}}>1 \tag{41}$$

*where* $Q(\ell)=\epsilon_1^{\beta}\xi(\ell)\left(\frac{1}{\mu(\ell+\tau-\sigma)}\sum_{t=\ell+\tau-\sigma}^{\ell-\eta-1}\frac{1}{\psi(t)}\right)^{\beta}$, *for all* $N\geq\ell_0$, *then equation* (1) *is oscillatory.*

**Proof.** In view of Theorem 2.1 in [13], condition (40) ensures that inequality (37) has no positive increasing solution. By an extension of Theorem 2.1 of [26], condition (41) ensures that inequality (38) has no positive decreasing solution. This conclusion follows from Theorem 2.12. The proof of the corollary is complete. □

## 3. Examples

In this section, we present some examples to illustrate the main results.

**Example 3.1.** *Consider the third-order neutral difference equation with a sublinear neutral term*

$$\Delta\left(\frac{1}{\ell^3}(\Delta(\ell\Delta\chi(\ell)))^3\right)+\ell^2\nu^{\frac{1}{3}}(\ell-4)=0,\ \ell\geq 1, \tag{42}$$

*where* $\chi(\ell)=\nu(\ell)+\ell^2\nu^{\frac{1}{3}}(\ell-3)$. *Here* $\varphi(\ell)=\frac{1}{\ell^3},\ \psi(\ell)=\ell,\ \mu(\ell)=\ell^2,\ \xi(\ell)=\ell^2,\ \tau=3,\ \sigma=4,\ \alpha=\frac{1}{3},\ \beta=3,$ *and* $\lambda=\frac{1}{3}$. *It is easy to see that* $(E_1)-(E_4)$ *hold. A simple calculation shows that* $B(\ell)=\frac{\ell-1}{2},\ D(\ell)=\frac{(\ell-1)(\ell-2)}{4},\ \Phi(\ell)\geq\frac{1}{(\ell+3)^2}\left(1-\frac{\gamma^2}{16(\ell+1)(\ell+2)}\right)>0,$ *and* $\Psi(\ell)=\frac{1}{(\ell+3)^2}\left(1-\frac{\delta^2}{(\ell+6)^2}\right)>0$, *that is,* $(E_5)$ *and* $(E_6)$ *are satisfied. It follows from* (13) *and* (14) *that*

$$\sum_{\ell=4}^{\infty}\frac{\ell^2}{(\ell-1)^2}\left(1-\frac{\gamma^2}{16(\ell-3)(\ell-2)}\right)=\infty$$

*and*

$$\sum_{\ell=4}^{\infty}\frac{\ell^2}{(\ell-1)^2}\left(1-\frac{\delta^2}{(\ell+2)^2}\right)=\infty,$$

*that is,* (13) *and* (14) *hold. Hence by Theorem 2.5, every solution* $\{\nu(\ell)\}$ *of* (42) *is either oscillatory or tends to zero, as* $\ell\to\infty$.

**Example 3.2.** *Consider the third-order difference equation with a sublinear neutral term*

$$\Delta\left(\frac{1}{\ell}\Delta(\ell\Delta\chi(\ell))\right)+\ell^6\exp(e^{2\ell})\nu(\ell-4)=0,\ \ell\geq 1, \tag{43}$$

*with* $\chi(\ell)=\nu(\ell)+\ell^2\nu^{\frac{1}{3}}(\ell-1)$. *Here* $\varphi(\ell)=\frac{1}{\ell},\ \psi(\ell)=\ell,\ \mu(\ell)=\ell^2,\ \xi(\ell)=\ell^6\exp(e^{2\ell}),\ \tau=1,\ \sigma=4,\ \alpha=\frac{1}{3},\ \beta=\lambda=1$. *It is easy to see that* $(E_1)-(E_4)$ *hold, and* $\alpha\beta<\lambda$. *A simple calculation shows that* $B(\ell)=\frac{\ell-1}{2},\ D(\ell)=\frac{(\ell-1)(\ell-2)}{4}$, *and* $\lim_{\ell\to\infty}\frac{D^{\frac{1}{\alpha}}(\ell+2\tau)}{D(\ell+\tau)\mu^{\frac{1}{\alpha}}(\ell+2\tau)}=\lim_{\ell\to\infty}\frac{\ell^3(\ell+1)^3}{\ell(\ell-1)(\ell+2)^6}=0$ *and so* $(E_7)$ *is holds. Also by taking* $\eta=2$, *then* $\zeta=1$, *we see that*

$$R(\ell)=\left(\sum_{t=\ell-3}^{\ell-3}\frac{1}{t}\right)\left(\sum_{t=\ell-2}^{\ell-2}t\right)=\left(\frac{\ell-2}{\ell-3}\right)\ \text{for}\ \ell\geq 4.$$

*Now* $Q(\ell) = \epsilon_1^3 \left(\frac{\ell-2}{\ell-3}\right)^3$. *By taking* $a = 1$, *and* $a_1 = 2$, *we see that* $1 > \frac{1}{3}\ln(3)$, $2 > \ln(3)$. *Then* (30) *and* (31) *become*

$$\lim_{\ell\to\infty}\inf\left[\ell^6 \exp(e^{2\ell})\epsilon_1^3\left(\frac{\ell-2}{\ell-3}\right)^3\left(\frac{1}{\ell-3}\right)^6\exp(-e^{\ell})\right] = \infty > 0$$

*and*

$$\lim_{\ell\to\infty}\inf\left[\ell^6 \exp(e^{2\ell})\epsilon_1^3\left(\frac{\ell-2}{\ell-3}\right)^3\left(\frac{1}{\ell-3}\right)^6\exp(-e^{2\ell})\right] = \epsilon_1^3 > 0.$$

*Hence by Corollary 2.10, equation* (43) *is oscillatory.*

**Example 3.3.** *Consider the third-order difference equation with a sublinear neutral term*

$$\Delta^2(\ell\Delta\chi(\ell)) + \ell(\ell+3)^2\nu^{\frac{1}{3}}(\ell-4) = 0,\ \ell \geq 1, \tag{44}$$

*where* $\chi(\ell) = \nu(\ell) + \ell\nu^{\frac{1}{3}}(\ell-1)$. *Here* $\varphi(\ell) = 1$, $\psi(\ell) = \mu(\ell) = \ell$, $\xi(\ell) = \ell(\ell+3)^2$, $\tau = 1$, $\sigma = 4$, $\alpha = \frac{1}{3}$, $\beta = 1$, *and* $\lambda = \frac{1}{3}$. *It is easy to see that* $(E_1)-(E_4)$ *hold. A simple calculation shows that* $B(\ell) = \ell$, $D(\ell) = \ell$, *and* $\lim_{\ell\to\infty}\frac{D^{\frac{1}{\alpha}}(\ell+2\tau)}{D(\ell+\tau)\mu^{\frac{1}{\alpha}}(\ell+2\tau)} = \lim_{\ell\to\infty}\frac{(\ell+2)^3}{(\ell+1)(\ell+2)^3} = 0$ *and so* $(E_7)$ *is holds. Choose* $\eta = 2$, *then* $\ell-3 < \ell-2$ *for all* $\ell \geq 1$, *and* $Q(\ell) = \epsilon_1\frac{\ell(\ell+3)^2}{(\ell-3)}\sum_{t=\ell-3}^{\ell-3}\frac{1}{t} = \epsilon_1\frac{\ell(\ell+3)^2}{(\ell-3)}$. *Now conditions* (40) *and* (41) *become*

$$\lim_{\ell\to\infty}\sup\sum_{s=4}^{\ell}s(s+3)^2\frac{(s-3)}{(s-3)(s-3)} = \lim_{\ell\to\infty}\sup\sum_{s=4}^{\ell}\frac{s(s+3)^2}{(s-3)} = \infty$$

*and*

$$\begin{aligned}\lim_{\ell\to\infty}\sup\sum_{s=\ell-2}^{\ell-1}\left(\sum_{t=s}^{\ell-1}\epsilon_1\frac{t(t+3)^2}{(t-3)^2}\right) &\approx \lim_{\ell\to\infty}\sup\sum_{s=\ell-2}^{\ell-1}\epsilon_1\left(\frac{\ell(\ell-1)}{2}-\frac{s(s-1)}{2}\right)\\ &\approx \lim_{\ell\to\infty}\epsilon_1(3\ell-4) = \infty > 1.\end{aligned}$$

*Hence conditions* (40) *and* (41) *are satisfied, and therefore, in view of Corollary 2.13, all solutions of* (44) *are oscillatory.*

## 4. Conclusion

In this paper, we have obtained some new criteria for the solutions of equation (1) to be oscillatory. These criteria can be extended to higher order equation of the form

$$\Delta^{m-1}\left(\varphi(\ell)(\Delta(\nu(\ell)+\mu(\ell)\nu^{\alpha}(\ell-\tau)))^{\beta}\right) + \xi(\ell)\nu^{\lambda}(\ell-\sigma) = 0$$

where $m$ is a positive integer. It will be of interest to study (1) with different values for $\alpha$, $\beta$ and $\lambda$.

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