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# EXTENDED $G_b$-METRIC SPACES AND SOME FIXED POINT RESULTS WITH AN APPLICATION TO FREDHOLM INTEGRAL EQUATION

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Abstract. In this study, our aim is to demonstrate the formation of generalization of $G_b$-metric space, that is, extended $G_b$-metric space by means of the results of Kamran et al. [19]. We prove an analogue of Banach contraction principle and introduce outcomes in the framework of extended $G_b$-metric space. Various illustrations are given to describe our results. To show the usability, an application associated to our main result is also presented.

## 1. Introduction

In recent times, metric fixed point theory has been very dynamic research area because of its applications in various fields [9]. In 1906, M. Frechet contributed the concept of metric space and after that various authors proved numerous results. Banach contraction principle by S. Banach [7] is utmost attractive outcome in the area of fixed point theory. Bakhtin [6] popularized the theory of $b$-metric spaces and Czerwik [8] extended the results of $b$-metric spaces. For some recent works on $b$-metric spaces, see [14, 17, 18]. Mustafa and Sims [22] familiarized an upgraded version of the generalized metric, that is, $G$-metric space. Several authors [1, 4, 5, 23, 24] demonstrated many fixed point consequences in $G$-metric spaces. Aghajani et al. [2] presented the idea of $G_b$-metric spaces. Afterward, a number of fascinating consequences in $G_b$-metric spaces which have been described in [3, 10, 11, 12, 13, 15, 16, 20, 21, 25, 26, 27].

## 2. Preliminaries

In this part, we gather several applicable definitions and essential results of $G_b$-metric space for further use.

**Definition 2.1.** [2] *A mapping $G_b : S \times S \times S \to \mathbb{R}^+$ with $s \geq 1$, defined as a generalized $b$-metric if it satisfies:*
($Gb1$) $G_b(\rho, \tau, \sigma) = 0$ *if* $\rho = \tau = \sigma$,
($Gb2$) $0 < G_b(\rho, \rho, \tau)$ *for all* $\rho, \tau \in S$ *with* $\rho \neq \tau$,

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*($Gb3$) $G_b(\rho,\rho,\tau) \le G_b(\rho,\tau,\sigma)$ for all $\rho,\tau,\sigma \in S$ with $\rho \neq \tau$,*
*($Gb4$) $G_b(\rho,\tau,\sigma) = G_b\{P(\rho,\sigma,\tau)\}$, where $P$ is the permutation for all $\rho,\tau,\sigma \in S$,*
*($Gb5$) $G_b(\rho,\tau,\sigma) \le s\{G_b(\rho,x,x) + G_b(x,\tau,\sigma)\}$ for every $\rho,\tau,\sigma \in S$.*

*The pair $(S, G_b)$ is named as $G_b$-metric space. Every $G_b$-metric space is a $G_b$-metric space for $s = 1$, but the next illustration shows that $G_b$-metric space is not inevitably a $G$-metric space.*

**Example 2.1.** [2] *Mapping defined by*

$$G(\rho,\tau,\sigma) = \frac{1}{9}(|\rho-\tau| + |\tau-\sigma| + |\sigma-\rho|)^2,$$

*is a $G_b$-metric on $S = \mathbb{R}$ with $s = 2$. Now check for $G$-metric space, take $\rho = 2$, $\tau = 4$, $\sigma = 6$ and $x = \frac{5}{2}$. Then we get*

$$G_2(2,4,6) = \frac{64}{9}, \quad G_2(2,\frac{5}{2},\frac{5}{2}) = \frac{1}{9}, \quad G_2(\frac{5}{2},4,6) = \frac{49}{9},$$

*so*

$$G_2(2,4,6) = \frac{64}{9} \ge \frac{50}{9} = G_2(2,\frac{5}{2},\frac{5}{2}) + G_2(\frac{5}{2},4,6).$$

*Here, the rectangle inequality is not satisfied. Therefore, it is not a $G$-metric space.*

**Definition 2.2.** [2] *A sequence $\{\rho_n\}$ is $G_b$-convergent to a point $\rho \in S$ if for each $\varepsilon > 0$, there always be an integer $n_1$ such that for each $m, n \ge n_1$, $G(\rho_m, \rho_n, \rho) < \varepsilon$.*

**Definition 2.3.** [2] *A sequence $\{\rho_n\}$ in $S$ is named as $G_b$-Cauchy if for any $\varepsilon > 0$, there always be an integer $n_1$ such that for every $l, m, n \ge n_1$, $G(\rho_l, \rho_m, \rho_n) < \varepsilon$.*

**Definition 2.4.** [2] *If every $G_b$-Cauchy sequence is $G_b$-convergent, then $(S, G_b)$ is complete $G_b$-metric space.*

In recent times, Kamran et al. [19] introduced an expansion of $b$-metric space known as extended $b$-metric space.

**Definition 2.5.** [19] *Consider a nonempty set $S$ and a mapping $\phi : S \times S \to [1, +\infty)$. A mapping $d_\phi : S \times S \to [0, +\infty)$ is known to be an extended b-metric space if it satisfies the succeeding assumptions:*
*($d_\phi 1$) $d_\phi(\rho,\tau)$ if $\rho = \tau$,*
*($d_\phi 2$) $d_\phi(\rho,\tau) = d_\phi(\tau,\rho)$,*
*($d_\phi 3$) $d_\phi(\rho,\tau) \le \phi(\rho,\tau)\{d_\phi(\rho,x) + d_\phi(x,\tau)\}$ for every $\rho,\tau,x \in S$.*
*$(S, d_\phi)$ is known as an extended b-metric space.*

**Remark 2.1.** *When $\phi(\rho,\tau) = s$ for $s \ge 1$, then perception of extended b-metric space agree with b-metric space.*

## 3. Main results

In this section, we present an extension of $G_b$-metric space, that is, extended $G_b$-metric space along with its topology.

**Definition 3.1.** *Consider a nonempty set $S$ and a mapping $\phi : S \times S \times S \to [1, +\infty)$. A mapping $G_\phi : S \times S \times S \to [0, +\infty)$ is said to be an extended $G_b$-metric if it satisfies the following conditions:*
*($G_\phi 1$) $G_\phi(\rho,\tau,\sigma) = 0$ if $\rho = \tau = \sigma$,*
*($G_\phi 2$) $0 < G_\phi(\rho,\rho,\tau)$ for every $\rho,\tau \in S$ with $\rho \neq \tau$,*
*($G_\phi 3$) $G_\phi(\rho,\rho,\tau) \le G_\phi(\rho,\tau,\sigma)$ for all $\rho,\tau,\sigma \in S$ with $\rho \neq \tau$,*

*($G_\phi 4$) $G_\phi(\rho,\tau,\sigma) = P\{G_\phi(\rho,\sigma,\tau)\}$, where $P$ is the permutation,*
*($G_\phi 5$) $G_\phi(\rho,\tau,\sigma) \le \phi(\rho,\tau,\sigma)\{G_\phi(\rho,x,x) + G_\phi(x,\tau,\sigma)\}$ for every $\rho,\tau,\sigma,x \in S$.*
*The pair $(S, G_\phi)$ is called as an extended $G_b$-metric space.*

Now, we give the following example.

**Example 3.1.** *Let $S = \{4,5,6,7\}$. Define $G_\phi : S\times S\times S \to \mathbb{R}^+$ and $\phi : S\times S\times S \to \mathbb{R}^+$ as*

$$\phi(\rho,\tau,\sigma) = 1+\rho+\tau+\sigma,$$
$$G_\phi(4,4,4) = G_\phi(5,5,5) = G_\phi(6,6,6) = G_\phi(7,7,7) = 0,$$
$$G_\phi(4,5,7) = G_\phi\{P(4,5,7)\} = 80,$$
$$G_\phi(4,5,5) = G_\phi(4,6,6) = G_\phi(4,7,7) = G_\phi(5,4,4) = G_\phi(5,6,6) = \ldots = 50,$$
$$G_\phi(4,4,5) = G_\phi(4,4,6) = G_\phi(4,4,7) = G_\phi(5,5,4) = G_\phi(5,5,6) = \ldots = 40.$$

*($G_\phi 1$) to ($G_\phi 4$) hold trivially. We will check only condition ($G_\phi 5$).*

$$G_\phi(4,5,7) = 80, \quad \phi(4,5,7)\{G_\phi(4,6,6) + G_\phi(6,5,7)\} = 17(50+80) = 2210.$$

*We can check the same calculations for all other terms. Therefore, for all $\rho,\tau,\sigma \in S$,*

$$G_\phi(\rho,\tau,\sigma) \le \phi(\rho,\tau,\sigma)\{G_\phi(\rho,x,x) + G_\phi(x,\tau,\sigma)\}.$$

*Hence $(S, G_\phi)$ is an extended $G_b$-metric space.*

**Example 3.2.** *Consider $S = C[\alpha,\beta]$ and*

$$G_\phi(\rho,\tau,\sigma) = \sup_{t\in[\alpha,\beta]} |\max\{\rho(t), \tau(t) - \sigma(t)\}|^2$$

*and $\phi(\rho,\tau,\sigma) = |\rho(t)+\tau(t)+\sigma(t)| + 2$, where $\phi : S\times S\times S \to [1,+\infty)$. Clearly, $(S, G_\phi)$ is an extended $G_b$-metric space.*

**Remark 3.1.** *Every $G_b$-metric space is an extended $G_b$-metric space when*

$$\phi(\rho,\tau,\sigma) = s \ge 1$$

*but its converse is not true.*

**Definition 3.2.** *Let $\{\rho_n\}$ be a sequence in extended $G_b$-metric space $(S, G_\phi)$. The sequence $\{\rho_n\}$ is said to be convergent to a point $\rho \in S$ if for any $\varepsilon > 0$, there always be a positive integer $n_1$ such that for all $m,n \ge n_1$, $G_\phi(\rho_n,\rho_m,\rho) < \varepsilon$.*

**Definition 3.3.** *A sequence $\{\rho_n\}$ in extended $G_b$-metric space $(S, G_\phi)$ is said to be Cauchy if for any $\varepsilon > 0$, there always be a positive integer $n_1$ such that for every $m,n,l \ge n_1$, $G_\phi(\rho_n,\rho_m,\rho_l) < \varepsilon$.*

**Definition 3.4.** *An extended $G_b$-metric space $(S, G_\phi)$ is named as complete if every Cauchy sequence is convergent in it.*

**Definition 3.5.** *An extended $G_b$-metric space $(S, G_\phi)$ is known to be symmetric if $G_\phi(\rho,\tau,\tau) = G_\phi(\tau,\rho,\rho)$ for every $\rho,\tau \in S$.*

**Definition 3.6.** *Let $(S, G_\phi)$ be an extended $G_b$-metric space, then for any $x \in S$ and $\varepsilon > 0$, we define*

$$B(x,\varepsilon) = \{\rho \in S \mid G_\phi(x,\rho,\rho) < \varepsilon\}$$

*as a ball with center $x$ and radius $\varepsilon$.*

**Remark 3.2.** *An extended $G_b$-metric is not continuous functional because $G_b$-metric is not continuous.*

**Lemma 3.1.** *In an extended $G_b$-metric space $(S, G_\phi)$, unique limit of every convergent sequence exists only if $G_\phi$ is continuous.*

**Theorem 3.1.** *In a complete extended $G_b$-metric space $(S, G_\phi)$, the function $G_\phi$ is continuous and a self-mapping $A$ on $S$ satisfies:*

$$G_\phi(A\rho, A\tau, A\sigma) \leq qG_\phi(\rho, \tau, \sigma), \quad \text{for every } \rho, \tau, \sigma \in S, \tag{1}$$

*where $0 < q < 1$ and for every $\rho_0 \in S$, there is*

$$\lim_{n,m\to+\infty} \phi(A^n\rho, A^m\tau, A^m\sigma) < \frac{1}{q},$$

*then there exists a unique fixed point $c$ for the mapping $A$. Moreover, for every $a \in S$, we have*

$$\lim_{n\to+\infty} A^n a = c.$$

*Proof.* Consider an arbitrary point $\rho_0 \in S$, and a sequence $\{\rho_n\}$ such that

$$\rho_1 = A\rho_0, \quad \rho_2 = A\rho_1 = AA\rho_0 = A^2\rho_0, \dots, \ \rho_n = A^n\rho_0.$$

Continuously repeating the inequality (1), we obtain

$$\begin{aligned} G_\phi(\rho_n, \rho_{n+1}, \rho_{n+1}) &\leq qG_\phi(\rho_{n-1}, \rho_n, \rho_n) \\ &\leq q^2 G_\phi(\rho_{n-2}, \rho_{n-1}, \rho_{n-1}) \\ &\ \vdots \\ &\leq q^n G_\phi(\rho_0, \rho_1, \rho_1). \end{aligned} \tag{2}$$

For every $n, m \in \mathbb{N}$ and $n < m$, we get the followings:

$$\begin{aligned} G_\phi(\rho_n, \rho_m, \rho_m) \leq& \phi(\rho_n, \rho_m, \rho_m) q^n G_\phi(\rho_0, \rho_1, \rho_1) \\ &+ \phi(\rho_n, \rho_m, \rho_m)\phi(\rho_{n+1}, \rho_m, \rho_m) q^{n+1} G_\phi(\rho_0, \rho_1, \rho_1) + \dots \\ &+ \phi(\rho_n, \rho_m, \rho_m) \dots \phi(\rho_{m-1}, \rho_m, \rho_m) q^{m-1} G_\phi(\rho_0, \rho_1, \rho_1) \\ \leq& G_\phi(\rho_0, \rho_1, \rho_1)[\phi(\rho_1, \rho_m, \rho_m) \dots \phi(\rho_{n-1}, \rho_m, \rho_m)\phi(\rho_n, \rho_m, \rho_m) q^n \\ &+ \phi(\rho_1, \rho_m, \rho_m) \dots \phi(\rho_n, \rho_m, \rho_m)\phi(\rho_{n+1}, \rho_m, \rho_m) q^{n+1} + \dots \\ &+ \phi(\rho_1, \rho_m, \rho_m) \dots \phi(\rho_{m-2}, \rho_m, \rho_m)\phi(\rho_{m-1}, \rho_m, \rho_m) q^{m-1}]. \end{aligned} \tag{3}$$

But it is given that $\lim_{n,m\to+\infty} \phi(\rho_n, \rho_m, \rho_m) q < 1$. Therefore, the series

$$\sum_{n=1}^{+\infty} q^n \prod_{j=1}^{n} \phi(\rho_j, \rho_m, \rho_m)$$

is convergent by ratio test. Let

$$B = \sum_{n=1}^{+\infty} q^n \prod_{j=1}^{n} \phi(\rho_j, \rho_m, \rho_m) \quad \text{and} \quad B_n = \sum_{k=1}^{n} q^k \prod_{j=1}^{k} \phi(\rho_j, \rho_m, \rho_m). \tag{4}$$

Now using equation (4) in equation (3), we have

$$G_\phi(\rho_n, \rho_m, \rho_m) \leq G_\phi(\rho_0, \rho_1, \rho_1)[B_{m-1} - B_{n-1}].$$

Proceeding $m, n \to +\infty$, and this indicates that the sequence $\{\rho_n\}$ is Cauchy. Due to the completeness property of $S$, we can able to find a point $c \in S$ such that $\{\rho_n\}$ converges to $c$.

$$\begin{aligned} G_\phi(c, Ac, Ac) &\leq \phi(c, Ac, Ac)\{G_\phi(c, \rho_n, \rho_n) + G_\phi(\rho_n, Ac, Ac)\} \\ &\leq \phi(c, Ac, Ac)\{G_\phi(c, \rho_n, \rho_n) + qG_\phi(\rho_{n-1}, c, c)\}. \end{aligned} \tag{5}$$

Taking the limit as $n \to +\infty$ in (5), we have $G_\phi(c, Ac, Ac) = 0$. This implies that function $A$ has $c$ as its fixed point. Therefore, $Ac = c$. Let if possible, there is another point $c_1 \in S$ such that $Ac_1 = c_1$. Then

$$G_\phi(c_1, c, c) = G_\phi(Ac_1, Ac, Ac) \leq qG_\phi(c_1, c, c),$$

but this is not possible. Hence $c$ is the unique fixed point for the function $A$. □

**Definition 3.7.** *Consider a mapping $A : S \to S$ and $\rho_0 \in S$,*

$$O(\rho_0) = \{\rho_0, g\rho_0, g^2\rho_0, \ldots\}$$

*orbit of $\rho_0$. An into mapping $h : S \to \mathbb{R}$ is called $A$-orbitally lower semi-continuous at $a \in S$ if $\{\rho_n\} \subset O(\rho_0)$ and $\rho_n \to a$ denotes $h(a) \leq \lim_{n\to+\infty} \inf h(\rho_n)$.*

**Theorem 3.2.** *In complete extended $G_b$-metric space, function $G_\phi$ is continuous and a self- mapping $A$ on $S$ satisfies the assumption:*

$$G_\phi(A\rho, A^2\rho, A^2\rho) \leq qG_\phi(\rho, A\rho, A\rho) \quad \text{for every } \rho \in O(\rho_0) \tag{6}$$

*where $0 \leq q < 1$ and $\lim_{n,m\to+\infty} \phi(\rho_n, \rho_m, \rho_m) < \frac{1}{q}$ for every $\rho_0 \in S$. Then, $A^n\rho_0 \to c \in S$. Moreover, the mapping $A$ has a fixed point $c$ if and only if $h(\rho) = G_\phi(\rho, A\rho, A\rho)$ is $A$-orbitally lower semi-continuous at $c$.*

*Proof.* Consider an arbitrary point $\rho_0 \in S$ and a sequence $\{\rho_n\}$ such that

$$\rho_1 = A\rho_0, \ \rho_2 = A\rho_1 = AA\rho_0 = A^2\rho_0, \ \rho_n = A^n\rho_0, \ldots$$

Continuously repeating equation (6), we get

$$G_\phi(A^n\rho_0, A^{n+1}\rho_0, A^{n+1}\rho_0) = qG_\phi(\rho_n, \rho_{n+1}, \rho_{n+1}) \leq \ldots \leq q^n G_\phi(\rho_0, \rho_1, \rho_1). \tag{7}$$

For every $n, m \in \mathbb{N}$ and $n < m$, we have

$$\begin{aligned} G_\phi(\rho_n, \rho_m, \rho_m) \leq& \phi(\rho_n, \rho_m, \rho_m)q^n G_\phi(\rho_0, \rho_1, \rho_1) \\ &+ \phi(\rho_n, \rho_m, \rho_m)\phi(\rho_{n+1}, \rho_m, \rho_m)q^{n+1}G_\phi(\rho_0, \rho_1, \rho_1) + \ldots \\ &+ \phi(\rho_n, \rho_m, \rho_m)\ldots\phi(\rho_{m-1}, \rho_m, \rho_m)q^{m-1}G_\phi(\rho_0, \rho_1, \rho_1) \\ \leq& G_\phi(\rho_0, \rho_1, \rho_1)[\phi(\rho_1, \rho_m, \rho_m)\ldots\phi(\rho_{n-1}, \rho_m, \rho_m)\phi(\rho_n, \rho_m, \rho_m)q^n \\ &+ \phi(\rho_1, \rho_m, \rho_m)\ldots\phi(\rho_n, \rho_m, \rho_m)\phi(\rho_{n+1}, \rho_m, \rho_m)q^{n+1} + \ldots \\ &+ \phi(\rho_1, \rho_m, \rho_m)\ldots\phi(\rho_{m-2}, \rho_m, \rho_m)\phi(\rho_{m-1}, \rho_m, \rho_m)q^{m-1}]. \end{aligned} \tag{8}$$

From the fact that $\lim_{n,m\to+\infty} \phi(\rho_n, \rho_m, \rho_m)q < 1$, the series

$$\sum_{n=1}^{+\infty} q^n \prod_{j=1}^{n} \phi(\rho_j, \rho_m, \rho_m)$$

is convergent by ratio test. Consider the next statements:

$$B = \sum_{n=1}^{+\infty} q^n \prod_{j=1}^{n} \phi(\rho_j, \rho_m, \rho_m) \quad \text{and} \quad B_n = \sum_{k=1}^{n} q^k \prod_{j=1}^{k} \phi(\rho_j, \rho_m, \rho_m). \tag{9}$$

If we substitute equation (9) in equation (8), we find

$$G_\phi(\rho_n,\rho_m,\rho_m) \le G_\phi(\rho_0,\rho_1,\rho_1)[B_{m-1}-B_{n-1}].$$

Taking the limit as $m,n\to+\infty$, we obtain that the sequence $\{\rho_n\}$ is Cauchy. By the completeness property of $S$, we are able to discover a convergent sequence $\{\rho_n\}$ converges to $c\in S$ that is $\rho_n=A^n\rho_0\to c\in S$.

Now consider at $c$, the function $h$ is $A$-orbitally lower semi-continuous, then

$$G_\phi(c,Ac,Ac)\le \lim_{n\to+\infty}\inf G_\phi(\rho_n,\rho_{n+1},\rho_{n+1})\le q^n \lim_{n\to+\infty}\inf G_\phi(\rho_0,\rho_1,\rho_1)=0.$$

Thus $c=Ac$. Conversely, suppose that $c=Ac$ and $\rho_n\in O(\rho_0)$ such that $\rho_n\to c$. Then

$$h(c)=G_\phi(c,Ac,Ac)=0\le \lim_{n\to+\infty}\inf G_\phi(\rho_n,\rho_{n+1},\rho_{n+1})=\lim_{n\to+\infty}\inf h(\rho_n).$$

□

**Example 3.3.** *Let $S=[0,+\infty)$, define $G_\phi: S\times S\times S\to\mathbb{R}^+$ and $\phi: S\times S\times S\to[1,+\infty)$ as follows:*

$$G_\phi(\rho,\tau,\sigma)=(\max\{\rho,\tau\}-\sigma)^2,$$
$$\phi(\rho,\tau,\sigma)=\max\{\rho,\tau\}+\sigma+3.$$

*It is clear that $G_\phi$ is a complete extended $G_b$-metric space on $S$. Define $A:S\to S$ by $Ay=\frac{y}{3}$. Then we obtain*

$$G_\phi(A\rho,A\tau,A\sigma)=\Big(\max\{\frac{\rho}{3},\frac{\tau}{3}\}-\frac{\sigma}{3}\Big)^2\le\frac{1}{4}G_\phi(\rho,\tau,\sigma)=qG_\phi(\rho,\tau,\sigma).$$

*Similarly, if we take for all $y\in S$, we find $A^ny=\frac{y}{3^n}$ and*

$$\lim_{n,m\to+\infty}\phi(A^n\rho,A^m\rho,A^m\rho)=\lim_{n,m\to+\infty}\Big(\frac{\rho}{3^n}+\frac{\rho}{3^m}+3\Big)<4.$$

*All assumptions of Theorem 3.1 are fulfilled. Therefore, the mapping $A$ has a unique fixed point.*

**Example 3.4.** *Consider $S=[0,\frac{1}{4})$, define $G_\phi: S\times S\times S\to\mathbb{R}^+$ and $\phi: S\times S\times S\to[1,+\infty)$ as follows:*

$$G_\phi(\rho,\tau,\sigma)=(\max\{\rho,\tau\}-\sigma)^2,$$
$$\phi(\rho,\tau,\sigma)=\max\{\rho,\tau\}+\sigma+3.$$

*From Example 3.3, we know that $G_\phi$ is a complete extended $G_b$-metric space on $S$. Take $A:S\to S$ by $Ay=y^2$. Then,*

$$\begin{aligned}G_\phi(A\rho,A\tau,A\sigma)&=G_\phi(\rho^2,\tau^2,\sigma^2)\\&=\Big(\max\{\rho^2,\tau^2\}-\sigma^2\Big)^2\\&\le\frac{1}{4}G_\phi(\rho,\tau,\sigma)\\&=qG_\phi(\rho,\tau,\sigma).\end{aligned}$$

*In a similar way, taking all $y\in S$, we find $A^ny=y^{2n}$. As a result,*

$$\lim_{n,m\to+\infty}\phi(A^n\rho,A^m\rho,A^m\rho)=\lim_{n,m\to+\infty}\Big(\rho^{2n}+\rho^{2m}+3\Big)<4.$$

*Since all assumptions of Theorem 3.1 are satisfied, the mapping $A$ has a unique fixed point.*

## 4. An Application to Fredholm Integral Equation

In this section, we obtain the solution of a Fredholm integral equation by an application of our main result.

Consider $S = C([\alpha, \beta], \mathbb{R})$. Define $G_\phi : S \times S \times S \to \mathbb{R}^+$,

$$G_\phi(\rho, \tau, \sigma) = \sup_{z \in [\alpha, \beta]} (\max\{\rho(z), \tau(z)\} - \sigma(z))^2$$

with $\phi : S \times S \times S \to [1, +\infty)$, $\phi(\rho, \tau, \sigma) = \rho(z) + \max(|\tau(z), \sigma(z)|) + 3$.

It is clear that $(S, G_\phi)$ is a complete extended $G_b$-metric space. Now, consider the following Fredholm integral equation:

$$\rho(z) = \int_\alpha^\beta B(z, \theta, \rho(\theta))d\theta + h(z)$$

for $z, \theta \in [\alpha, \beta]$, where $h : [\alpha, \beta] \to \mathbb{R}$ and $B : [\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Let $A : S \to S$ be the operator such that

$$A\rho(z) = \int_\alpha^\beta B(z, \theta, \rho(\theta))d\theta + h(z)$$

for $z, \theta \in [\alpha, \beta]$,where $h : [\alpha, \beta] \to \mathbb{R}$ and $B : [\alpha, \beta] \times [\alpha, \beta] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Also, suppose that the next inequality is satisfied:

$$|B(z, \theta, \rho(\theta)) - B(z, \theta, g\rho(\theta))| \leq \frac{1}{2}|\rho(\theta) - g\rho(\theta)|$$

for $z, \theta \in [\alpha, \beta]$ and $\rho \in S$. Then, the solution of the Fredholm integral equation exists. For $z, \theta \in [\alpha, \beta]$ and $\rho \in S$, we have

$$\begin{aligned} G_\phi(A\rho(z), A(A\rho(z)), A(A\rho(z))) &= |A\rho(z) - A(A\rho(z))|^2 \\ &\leq \Big( \int_\alpha^\beta |B(z, \theta, \rho(\theta)) - B(z, \theta, A\rho(\theta))| \Big)^2 \\ &\leq \frac{1}{4} G_\phi(\rho, A\rho, A\rho). \end{aligned}$$

All the conditions of Theorem 3.2 are fulfilled which enables us to know that a fixed point for function $A$ exists. Thus, the solution of the Fredholm integral equation exists.

## 5. Conclusion

In this paper, we present the notion of extended $G_b$-metric space. Moreover, we prove some fixed point theorems with some supporting examples and interesting application. The obtained results have a huge potential for researchers working in the area of fixed point theory. For example, fixed-circle problem [28, 29, 30] can be studied on this space.

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