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ROOTS OF PELL POLYNOMIALS

F. BIROL, . KORUOGLU AND B.DEMIR

Abstract. In this paper we consider the Pell polynomials. We express these polynomials as complex hyperbolic functions. Using this we obtain roots of Pell polynomials. Further we give some interesting identities about images of roots of a polynomial under another member of the family.

1. Introduction

Fibonacci, Lucas and Pell polynomials are the families of orthogonal polynomials, and they are expressed recursively. These polynomials are widely used in the study of many topics such as number theory, combinatorics, algebra, approximation theory, geometry, graph theory (see [15] and [16]). The ratio of two consecutive polynomials of Fibonacci and Lucas families converges to the Golden Ratio which appears in many fields in the literature. For example; nature, art, architecture, biology, physics, chemistry, cosmos, theology, finance and so on (see, e.g., [9], [11], [15], [17], [19] and [20]). Furthermore, the ratio of two consecutive polynomials of Pell family converges to Silver Mean. The ratio is another member of the class of metallic means defined by Spinadel, apart from the Golden Mean. Other metallic means with special naming are Bronze Mean and Cooper Mean (see [24]). There are many interesting studies on different aspects related to the number sequences, polynomials and metallic means mentioned above (see [1], [2], [3], [5], [6], [7], [8], [10], [14], [18], [21], [23] and [25] for more details). Pell sequence is another phenomennon in mathematics like Fibonaci and Lucas numbers. These sequences fascinate mathematical society with their beauty, ubiquity and applicability. Pell numbers can be thougt as a member of extended Fibonacci family and share interesting numerous properties. Recursive formula of Pell sequence is

$$P_n = 2P_{n-1} + P_{n-2} \tag{1}$$

for $n \geq 2$ with initial conditions $P_0 = 0$ and $P_1 = 1$ (see [16]). Ratio of consequtive two Pell numbers converges to Silver Ratio $\phi_2 = 1 + \sqrt{2}$. Pell numbers can also be calculated by Binet formula

$$P_n = \frac{\gamma^n - \delta^n}{\gamma - \delta} \tag{2}$$

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where $\gamma = 1+\sqrt{2}$ and $\delta = 1-\sqrt{2}$. By the inspiration of the recursive definition of Pell sequence, in [14] Horadam defined Pell polynomials as

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x) \tag{3}$$

where $n \geq 2$ and $P_0(x) = 0$, $P_1(x) = 1$. Pell numbers are some values of these polynomials. For $\gamma(x) = x+\sqrt{1+x^2}$ and $\delta(x) = x-\sqrt{1+x^2}$ Binet formula of Pell polynomials is obtained:

$$P_n(x) = \frac{\gamma^n(x) - \delta^n(x)}{\gamma(x) - \delta(x)}. \tag{4}$$

Also there is a Cassini like identity for Pell polynomials as

$$P_{n+1}(x) - P_{n-1}(x) - P_n^2(x) = (-1)^n.$$

In this study we focus on the roots of these polynomials. Using the Binet formula of Pell polynomials and the theory of complex functions is critical significant to obtain the root formula of Pell polynomials. The roots of some classes of polynomials with recursive relation were obtained by this approach in [13]. In Section 2 we give some background about Fibonacci and Pell numbers and polynomials. After we express Pell polynomials in terms of complex hyperbolic functions in Section 3. Then we obtain roots of Pell Polynomials. Finally we investigate the image of a root of a polynomial under another member of the family.

2. Motivation and Background

Fibonacci numbers are the most compelling sequence in mathematics. They enamoured not only mathematicians but also people who interested in numerical sciences. First two terms of Fibonacci sequence are $F_0 = 0$ and $F_1 = 1$ then other terms can be calculated by the recurance $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.

E.C. Catalan defined Fibonacci polynomials for $n \geq 3$ an integer

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x) \tag{5}$$

and $F_1(x) = 1$, $F_2(x) = x$ [15]. Relations between Fibonacci polynomials and the diagonal of the Pascal's triange are generalized in [12] by Hoggat and Bicknell in 1973. In the same year they expressed Fibonacci polynomials as complex hyperbolic functions and from this point they obtained general root formula for these polynomials in [13]. Derivatives of Fibonacci and Lucas polynomials studied in [26]. And zeros of derivative Fibonacci polynomials are obtained in [22].

In 1963 P. F. Byrd studied on hyperbolic function represents of Pell polynomials as follows [4]:

Theorem 1 Let $x = \sinh z$ then

$$P_{2n}(x) = \frac{e^{2nz} - (-1)^{2n}e^{-2nz}}{e^z + e^{-z}} = \frac{\sinh 2nz}{\cosh z} \tag{6}$$

$$P_{2n+1}(x) = \frac{e^{(2n+1)z} - (-1)^{2n+1}e^{-(2n+1)z}}{e^z + e^{-z}} = \frac{\cosh{(2n+1)z}}{\cosh z} \tag{7}$$

It is known from W. N. H. Abel that an algebraic equation of degree five or more that cannot be solved by radicals. Also considering the D'Alembert-Gauss theorem, we can interpret that the general root formulas for the polynomials are very valuable. At that point finding a general root formula for Pell polynomials is striking and important.

3. Main Results

In this section we give results about zeros of Pell polynomials. We express Pell polynomials as complex hyperbolic functions. Then we obtain interesting identities about images of a zero of a Pell polynomial under another member of the family.

Theorem 2 Zeros of Pell polynomials are

$$P_{2n}(x) = 0 : x = \pm i \sin \frac{k\pi}{2n} \tag{8}$$

$$P_{2n+1}(x) = 0 : x = \pm i \sin \frac{(2k+1)\pi}{(2n+1)2} \tag{9}$$

where $k = 0, 1, ..., n-1$.

Proof.

We first obtain the zeros of the even subscripted Pell polynomials. Consider the Theorem 1. If $P_{2n}(x) = 0$ then $\frac{\sinh 2nz}{\cosh z} = 0$. Which yields $\sinh 2nz = 0$ and $\cosh z \neq 0$. Therefore;

$$\sinh 2nz = \sinh(2na + i2nb) = \sinh 2na \cos 2nb + i \cosh 2na \sin 2nb = 0$$

$$\cosh z = \cosh(a + ib) = \cosh a \cos b + i \sinh a \sin b \neq 0$$

for $z = a + ib$ where $a, b \in \mathbb{R}$. Since $\cosh 2na \geq 1$ for $n \in \mathbb{N}$, $\sin 2nb$ must be zero. So $b = \frac{k\pi}{2n}$ for $0 \leq k \leq 2n-1$. We use this in the real part of the preceding equation in the above line.

$$\sinh 2na \cos 2n \frac{k\pi}{2n} = \sinh 2na \cos k\pi = 0$$

Here $a = 0$ because $\sinh 2na$ must be zero. The error now we have is the Pell polynomial $P_{2n}(x)$ has degree $2n-1$. Hence we must collect at most $2n-1$ zeros. Unlikely we have one value of b which should not be a member. That one is obtained when $k = n$ which is impossible because it leads the denominator $P_{2n}(x) = \frac{\sinh 2nz}{\cosh z}$ to be zero. Therefore we omit it. It can be easily seen that $\cosh a \cos b + i \sinh a \sin b \neq 0$ for other possible values of k. Since $P_{2n}(x)$ is odd function we can restrict k as $0 \leq k \leq n-1$ and give roots as $x = \pm i \sin \frac{k\pi}{2n}$. Zeros of odd subscripted Pell polynomials can be calculated similarly.

It is better to obtain root formula in one identity. Therefore we need to combine the identities given in Theorem 1.

Theorem 3 Let $x = i \cosh z$ then nth member of the Pell polynomials is;

$$P_n(x) = i^{n-1} . \frac{\sinh nz}{\sinh z} . \tag{10}$$

Proof. Observe that if $x = i \cosh z$ we have

$$\gamma(x) = i \cosh z + i \sinh z = ie^{z}$$
$$\delta(x) = i \cosh z - i \sinh z = ie^{-z}$$

After substituting these to ones in (4) we obtain the desired result.
Theorem 4 Zeros of the nth Pell polynomial $P_n(x)$ are $x = i\cos\frac{k\pi}{n}$ for $k = 1, 2, ..., n-1$.
Proof. Let $P_n(x) = i^{n-1}\frac{\sinh nz}{\sinh z} = 0$. Then the numerator $\sinh nz = 0$ and the denominator $\sinh z \neq 0$. For $z = a + ib$ where a and b are real numbers;

$$\sinh nz = \sinh na + inb = \sinh na \cos nb + i \cosh na \sin nb = 0$$

$$\sinh z = \sinh a + ib = \sinh a \cos b + i \cosh a \sin b \neq 0$$

Hence the real numbers a and b must satisfy both of these equalities:

$$\sinh na \cos nb = 0 \tag{11}$$

and

$$\cosh nb \sin nb = 0 \tag{12}$$

Furthermore the real numbers a and b which satisfy the equations above must also satisfy at least one of the followings;

$$\sinh a \cos b \neq 0 \tag{13}$$

or

$$\cosh a \sin b \neq 0 \tag{14}$$

From equation (12) we have $\sin nb = 0$ since $\cosh nb \geq 1$ for all $b \in \mathbb{R}$. Therefore $b = \frac{k\pi}{n}$ for some $k \in \mathbb{Z}$ excluding multiples of n because of (14). In addition one can solve $a = 0$ from (11). As a result we have $P_n(x) = 0$ if and only if $x = i\cos\frac{k\pi}{n}$. Observe that $P_n(x) = P_n(-x) = 0$ where x is a zero of $P_n(x)$. Owing to the fact that Pell polynomial $P_n(x)$ has degree $n-1$, we restrict $k = 1, 2, ..., n-1$.
Now we are ready to calculate images of roots of a Pell polynomial under another member of the family.
Theorem 5 If a is a root of $P_{2n}(x)$ then $P_{2n+1}(a) = P_{2n-1}(a) = \pm 1$.
Proof. First we modify the recurrence relation (3) for odd and even terms:

$$P_{2n+1}(x) = 2xP_{2n}(x) + P_{2n-1}(x)$$

And let $x = a$ be a zero of the polynomial $P_{2n}(x)$ i.e. $P_{2n}(a) = 0$. Then we have;

$$P_{2n+1}(a) = P_{2n-1}(a) \tag{15}$$

On the other hand we consider the Cassini-like identity

$$P_{2n+1}(x)P_{2n-1}(x) - P_{2n}^2(x) = (-1)^{2n}$$

which implies that;

$$P_{2n+1}(a)P_{2n-1}(a) = 1 \tag{16}$$

since $P_{2n}(a) = 0$. Interpreting identities (15) and (16) we complete the proof.
Theorem 6 If a is a zero of the Pell polynomial $P_{2n-1}(x)$, then $P_{2n}(a) = \pm i$ and $P_{2n+1}(a) = \pm 2ai$.
Proof. Let $x = a$ be a root of $P_{2n-1}(x)$. Once more using the Cassini-like identity for Pell polynomials, we get

$$P_{2n}^2(a) = -1$$

Therefore;

$$P_{2n}(a) = \pm i$$

Now we need the image of a under the polynomial $P_{2n+1}(x)$. For doing this we use the recurrence formula;

$$P_{2n+1}(x) = 2xP_{2n}(x) + P_{2n-1}(x)$$

If we put $x = a$ then we obtain; $P_{2n+1}(a) = 2aP_{2n}(a) = \pm 2ai$ which is the proof of the result.

Corollary 1 $P_{2n-1}(a) = 0$, implies $P_{2n}(a).P_{2n+1}(a) = -2a$.

Theorem 7 If a is zero of the Pell polynomial $P_{2n-1}(x)$ then we have $P_{2n}(a)P_{2n+1}(a) \neq 0$. In other words 0 can not be a root of P_{2n-1}.

Proof. We know from Corollary 1 that if $P_{2n-1}(a) = 0$ then $P_{2n}(a).P_{2n+1}(a) = -2a$. Hence if $P_{2n}(a)P_{2n+1}(a) = 0$ then a must be zero. The roots of $P_{2n-1}(x)$ can be calculated by Theorem 4 as

$$a = i \cos \frac{k\pi}{2n-1}$$

for $k = 1, 2, ..., 2n-2$. By considering all possible values of k we have $\frac{k\pi}{2n-1} = \frac{\pi}{2}$. After elementary calculations one has $n - k = \frac{1}{2}$ which is a contradiction.

Following results can be proven using the same techniques. So we leave the proofs to the reader.

Theorem 8 If a is a root of the Pell polynomial $P_{2n+1}(x)$, then $P_{2n}(a) = \mp i$ and $P_{2n-1}(a) = \pm 2ai$.

Corollary 2 $P_{2n+1}(a) = 0$, implies $P_{2n}(a).P_{2n-1}(a) = 2a$.

Theorem 9 If a is root of the Pell polynomial $P_{2n+1}(x)$ then we have $P_{2n}(a)P_{2n-1}(a) \neq 0$. In other words 0 can not be a root of P_{2n+1}.

Now we need some identities proven in [14] by Horadam and Mahon.

$$\sum_{r=1}^{n} P_{2r-1}(x) = \frac{P_{2n}(x)}{2x} \tag{17}$$

$$\sum_{r=1}^{n} P_{2r}(x) = \frac{P_{2n+1}(x) - 1}{2x} \tag{18}$$

$$P_{m+n}(x) = P_{m-1}(x)P_n(x) + P_m(x)P_{n+1}(x). \tag{19}$$

Let us use $x = a$ as a root of $P_{2n+1}(x)$, $P_{2n}(x)$ and $P_n(x)$ in equations (17),(18),(19) respectively. Therefore we have the following theorem.

Theorem 10

(i) $P_{2n}(a) = 0 : P_1(a) + P_3(a) + ... + P_{2n-1}(a) = 0$

(ii) $P_{2n+1}(a) = 0 : P_2(a) + P_4(a) + ... + P_{2n}(a) = \frac{i}{2\cos\frac{k\pi}{2n+1}}$

(iii) $P_n(a) = 0 : P_{m+n}(a) = \pm i^{\lceil \frac{n}{2} - \frac{m}{2} \rceil} P_m(x)$.

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F. Birol
Information Technologies Department, Izmir Katip elebi University, Izmir, Turkey
E-mail address: `furkan.birol@ikc.edu.tr`

. Koruoglu
Department of Secondary Mathematics Education, Balikesir University, Balikesir, Turkey
E-mail address: `ozdenk@balikesir.edu.tr`

B. Demir
Department of Secondary Mathematics Education, Balikesir University, Balikesir, Turkey
E-mail address: `bdemir@balikesir.edu.tr`