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A TWO-POINTS NONLOCAL PROBLEM OF AN IMPLICIT DELAY FUNCTIONAL INTEGRO-DIFFERENTIAL EQUATION

MALAK. M. S. BA-ALI

Abstract. In this work, we have discussed the existence of solution for a two-points nonlocal boundary value problem of an implicit delay functional integro-differential equation in two classes, we use the technique of the Banach fixed point theorem. Moreover, we study the continuous dependence of the solution on the initial condition and on the delay function. The anti-periodic boundary value problem will be considered as an application.

1. Introduction

It is well known that the nonlinear boundary value problems create an important branch of nonlinear analysis and have numerous applications in describing of miscellaneous real world problems. For papers studying such kind of problems (see [1, 6, 12, 13]) and therein.

Implicit differential equations have been studied in many papers and monographs [2, 3, 4, 5, 6, 7, 8] and [14, 15, 16, 17, 18]. Here, we are concerning with the delay implicit functional integro-differential equation

$$\frac{dx}{dt} = f\left(t, \frac{dx}{dt}, \int_0^{\phi(t)} g(t,s,\frac{dx}{ds})ds\right),\ t \in (0,1) \tag{1}$$

with the two-points nonlocal condition

$$\alpha x(\tau) + \beta x(\xi) = x_0,\ \tau \in [0,1),\ \xi \in (0,1],\ \alpha + \beta \neq 0. \tag{2}$$

Let $\frac{dx}{dt} = y(t)$, then the solution of the problem (1)-(2) can be given by

$$x(t) = x(0) + \int_0^t y(s)ds,\ t \in (0,1) \tag{3}$$

where y is the solution of the functional integral equation

$$y(t) = f\left(t, y(t), \int_0^{\phi(t)} g(t,s,y(s)ds)\right),\ t \in (0,1). \tag{4}$$

Firstly, we study the existence of the solution of the functional integral equation (4) in the two classes $C[0,1]$ and $L_1[0,1]$. Then we deduce the solution of the boundary value problem (1)-(2) in the two classes $C^1[0,1]$ and $AC[0,1]$. The continuous dependence of the solution on the parameters α, β, x_0 and on the functions g and ϕ will be studied. The anti-periodic boundary value problem will be considered as an application of the results.
Now, we have the following lemma.

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Lemma 1. *If the solution of the problem (1)-(2) exists, then it can be presented by the integral equation*

$$x(t) = \frac{1}{\alpha+\beta}\left[x_0 - \alpha\int_0^\tau y(s)ds - \beta\int_0^\xi y(s)ds\right] + \int_0^t y(s)ds$$

where y is the solution of the functional integral equation

$$y(t) = f\left(t, y(t), \int_0^{\phi(t)} g(t,s,y(s))ds\right).$$

Proof. Let the boundary value problem (1)-(2) be satisfied, integrating equation (1), we obtain

$$x(t) = x(0) + \int_0^t y(s)ds.$$

For $t = \tau$, we get

$$x(\tau) = x(0) + \int_0^\tau y(s)ds$$

and

$$\alpha x(\tau) = \alpha x(0) + \alpha\int_0^\tau y(s)ds$$

and for $t = \xi$, we get

$$x(\xi) = x(0) + \int_0^\xi y(s)ds$$

and

$$\beta x(\xi) = \beta x(0) + \beta\int_0^\xi y(s)ds,$$

then

$$x_0 = \alpha x(\tau) + \beta x(\xi) = (\alpha+\beta)x(0) + \alpha\int_0^\tau y(s)ds + \beta\int_0^\xi y(s)ds$$

$$x(t) = \frac{1}{\alpha+\beta}\left[x_0 - \alpha\int_0^\tau y(s)ds - \beta\int_0^\xi y(s)ds\right] + \int_0^t y(s)ds.$$

□

2. The functional integral equation

2.1. **Existence of continuous solution.** Let I=[0, 1], consider the functional integral equation (4) under the following assumptions:

(i) $\phi : I \to I$, $\phi(t) \le t$ is continuous and increasing.

(ii) $f : I \times R \times R \to R$ is continuous and satisfies Lipschitz condition,

$$|f(t,x,y) - f(t,x_1,y_1)| \le b_1\left(|x-x_1| + |y-y_1|\right), \ t \in I, \forall x,y,x_1,y_1 \in R.$$

where b_1 is a positive constant.

(iii) $g : I \times I \times R \to R$ is continuous in $t \in I$, measurable in $s \in I$ and satisfies Lipschitz condition,

$$|g(t,s,x) - g(t,s,x_1)| \le b_2|x-x_1|, \ t \in I, \forall x,x_1 \in R.$$

where b_2 is a positive constant.

(iv) $b_1 + b_1 b_2 < 1$.

Remark 1. *From the assumption (ii), we have*

$$|f(t,x,y)| - |f(t,0,0)| \le |f(t,x,y) - f(t,0,0)| \le b_1(|x| + |y|),$$

$$|f(t,x,y)| \le |f(t,0,0)| + b_1(|x| + |y|)$$

and

$$|f(t,x,y)| \le m_1 + b_1(|x| + |y|), \text{ where } m_1 = \sup_{t\in I} |f(t,0,0)|.$$

Similarly, from the assumption (iii), we get

$$|g(t,s,x)| - |g(t,s,0)| \le |g(t,s,x) - g(t,s,0)| \le b_2|x|,$$

$$|g(t,s,x)| \le |g(t,s,0)| + b_2|x|$$

and

$$|g(t,s,x)| \le m_2(t,s) + b_2|x|, \text{ where } m_2(t,s) = \sup_{t,s\in I} |g(t,s,0)|.$$

Now, we have the following theorem.

Theorem 1. *Let the assumptions (i)-(iv) be satisfied, then the functional integral equation (4) has a unique solution $y \in C(I)$.*

Proof. Define the operator F by

$$Fy(t) = f_1\left(t, y(t), \int_0^{\phi(t)} g(t,s,y(s))ds\right), \ y \in C(I).$$

Now, let $y \in C(I)$ and $t_1, t_2 \in I$, $t_1 < t_2$, $|t_2 - t_1| < \delta$ and denote
$\theta_1(\delta) = \sup_{x,y\in C(I)} \{|f(t_2, x(t), y(t)) - f(t_1, x(t), y(t))| : t_1, t_2 \in I, \ t_1 < t_2, \ |t_2 - t_1| < \delta\}$,
$\theta_2(\delta) = \sup_{s,y\in C(I)} \{|f(t_2, s, y(t)) - f(t_1, s, y(t))| : t_1, t_2 \in I, \ t_1 < t_2, \ |t_2 - t_1| < \delta\}$
then we have

$$\begin{aligned}
|Fy(t_2) - Fy(t_1)| &= \left|f\left(t_2, y(t_2), \int_0^{\phi(t_2)} g(t_2,s,y(s))ds\right) - f\left(t_1, y(t_1), \int_0^{\phi(t_1)} g(t_1,s,y(s))ds\right)\right| \\
&\le \left|f\left(t_2, y(t_2), \int_0^{\phi(t_2)} g(t_2,s,y(s))ds\right) - f\left(t_1, y(t_2), \int_0^{\phi(t_2)} g(t_2,s,y(s))ds\right)\right. \\
&+ f\left(t_1, y(t_2), \int_0^{\phi(t_2)} g(t_2,s,y(s))ds\right) - f\left(t_1, y(t_1), \int_0^{\phi(t_2)} g(t_2,s,y(s))ds\right) \\
&+ f\left(t_1, y(t_1), \int_0^{\phi(t_2)} g(t_2,s,y(s))ds\right) - f\left(t_1, y(t_1), \int_0^{\phi(t_2)} g(t_1,s,y(s))ds\right) \\
&+ \left. f\left(t_1, y(t_1), \int_0^{\phi(t_2)} g(t_1,s,y(s))ds\right) - f\left(t_1, y(t_1), \int_0^{\phi(t_1)} g(t_1,s,y(s))ds\right)\right| \\
&\le \theta_1(\delta) + b_1|y(t_2) - y(t_1)| + b_1\theta_2(\delta) + b_1 \int_{\phi(t_1)}^{\phi(t_2)} |g(t_1,s,y(s))|ds \\
&\le \theta_1(\delta) + b_1|y(t_2) - y(t_1)| + b_1\theta_2(\delta) + b_1 \int_{\phi(t_1)}^{\phi(t_2)} |m_2(t,s)|ds \\
&+ b_1 b_2 |y(t)||t_2 - t_1|.
\end{aligned}$$

This means that $F : C(I) \to C(I)$. Now let $y_1, y_2 \in C(I)$, then

$$\begin{aligned} |Fy_2(t) - Fy_1(t)| &= \left| f(t, y_2(t), \int_0^{\phi(t)} g(t,s,y_2(s))ds) - f(t, y_1(t), \int_0^{\phi(t)} g(t,s,y_1(s))ds) \right| \\ &\leq \left| f(t, y_2(t), \int_0^{\phi(t)} g(t,s,y_2(s))ds) - f(t, y_2(t), \int_0^{\phi(t)} g(t,s,y_1(s))ds) \right. \\ &+ \left. f(t, y_2(t), \int_0^{\phi(t)} g(t,s,y_1(s))ds) - f(t, y_1(t), \int_0^{\phi(t)} g(t,s,y_1(s))ds) \right| \\ &\leq b_1 b_2 \int_0^{\phi(t)} |y_2(s) - y_1(s)| ds + b_1 |y_2(t) - y_1(t)| \\ &\leq b_1 b_2 \|y_2 - y_1\| + b_1 \|y_2 - y_1\|, \end{aligned}$$

then

$$\begin{aligned} \|Fy_2 - Fy_1\| &\leq b_1 b_2 \|y_2 - y_1\| + b_1 \|y_2 - y_1\| \\ &\leq (b_1 + b_1 b_2) \|y_2 - y_1\|. \end{aligned}$$

Since $b_1 + b_1 b_2 < 1$, then F is a contraction and by using the Banach fixed point theorem [11] there exists a unique solution $y \in C(I)$ of the equation (4). □

2.1.1. *Continuous dependence.*

Theorem 2. *Let the assumptions of Theorem 1 be satisfied, then the functional integral equation (4) depends continuously on the function g.*

Proof. Let $\delta > 0$ be given such that $|g(t, x(t), y(t)) - g^*(t, x(t), y(t))| \leq \delta$ and let x^* be the solution of (1)-(2) corresponding to $g^*(t, x(t), y(t))$, then

$$\begin{aligned} |y(t) - y^*(t)| &= \left| f(t, y(t), \int_0^{\phi(t)} g(t,s,y(s))ds) - f(t, y^*(t), \int_0^{\phi(t)} g^*(t,s,y^*(s))ds) \right| \\ &\leq \left| f(t, y(t), \int_0^{\phi(t)} g(t,s,y(s))ds) - f(t, y(t), \int_0^{\phi(t)} g^*(t,s,y(s))ds) \right. \\ &+ \left. f(t, y(t), \int_0^{\phi(t)} g^*(t,s,y(s))ds) - f(t, y^*(t), \int_0^{\phi(t)} g^*(t,s,y^*(s))ds) \right| \\ &\leq b_1 \int_0^{\phi(t)} \left| g(t,s,y(s)) - g^*(t,s,y(s)) \right| ds + b_1 |y(t) - y^*(t)| \\ &+ b_1 \int_0^{\phi(t)} \left| g^*(t,s,y(s))ds) - g^*(t,s,y^*(s)) \right| ds \\ &\leq b_1 \delta + b_1 \|y - y^*\| + b_1 b_2 |y(t) - y^*(t)| \\ &\leq b_1 \delta + b_1 \|y - y^*\| + b_1 b_2 \|y - y^*\|. \end{aligned}$$

Then

$$\|y - y^*\| \leq b_1 \delta + b_1 \|y - y^*\| + b_1 b_2 \|y - y^*\|.$$

Hence

$$\|y - y^*\| (1 - (b_1 + b_1 b_2)) \leq b_1 \delta,$$

then

$$\|y - y^*\|_{C(I)} \leq \frac{b_1 \delta}{1 - (b_1 + b_1 b_2)} = \frac{\epsilon}{2}. \tag{5}$$

□

Theorem 3. *Let the assumptions of Theorem 1 be satisfied, then the functional integral equation (4) depends continuously on the function ϕ.*

Proof. Let $\delta > 0$ be given such that $|\phi(t) - \phi^*(t)| \leq \delta$ and let x^* be the solution of (1)-(2) corresponding to $\phi^*(t)$, then

$$\begin{aligned}
|y(t) - y^*(t)| &= \left| f(t, y(t), \int_0^{\phi(t)} g(t,s,y(s))ds) - f(t, y^*(t), \int_0^{\phi^*(t)} g(t,s,y^*(s))ds) \right| \\
&\leq \left| f(t, y(t), \int_0^{\phi(t)} g(t,s,y(s))ds) - f(t, y(t), \int_0^{\phi^*(t)} g(t,s,y^*(s))ds) \right. \\
&+ \left. f(t, y(t), \int_0^{\phi^*(t)} g(t,s,y^*(s))ds) - f(t, y^*(t), \int_0^{\phi^*(t)} g(t,s,y^*(s))ds) \right| \\
&\leq b_1 \left| \int_0^{\phi(t)} g(t,s,y(s))ds - \int_0^{\phi^*(t)} g(t,s,y^*(s))ds \right| + b_1 |y(t) - y^*(t)| \\
&\leq b_1 \left| \int_0^{\phi(t)} g(t,s,y(s))ds - \int_0^{\phi(t)} g(t,s,y^*(s))ds \right. \\
&+ \left. \int_0^{\phi(t)} g(t,s,y^*(s))ds - \int_0^{\phi^*(t)} g(t,s,y^*(s))ds \right| + b_1 \|y - y^*\| \\
&\leq b_1 \left(\int_0^{\phi(t)} b_2 |y(s) - y^*(s)| ds + \int_{\phi^*(t)}^{\phi(t)} |g(t,s,y^*(s))| ds \right) + b_1 \|y - y^*\| \\
&\leq b_1 b_2 \|y - y^*\| + b_1 (m_2(t,s) + b_2 |y^*(s)|) |\phi - \phi^*| + b_1 \|y - y^*\| \\
&\leq b_1 b_2 \|y - y^*\| + (b_1 m_2(t,s) + b_1 b_2 \|y^*\|) \delta + b_1 \|y - y^*\|.
\end{aligned}$$

Hence

$$\|y - y^*\| (1 - (b_1 + b_1 b_2)) \leq (b_1 m_2(t,s) + b_1 b_2 \|y^*\|) \delta,$$

then

$$\|y - y^*\|_{C(I)} \leq \frac{(b_1 m_2(t,s) + b_1 b_2 \|y^*\|) \delta}{1 - (b_1 + b_1 b_2)} = \frac{\epsilon}{2}. \tag{6}$$

□

2.2. **Existence of integrable solution.** Consider the functional integral equation (4) under the assumption $(i), (iii)$ and (iv) and the following assumption:

$(ii)^*$ $f : I \times R \times R \to R$ is measurable in $t \in I$ and satisfies Lipschitz condition,

$$|f(t,x,y) - f(t,x_1,y_1)| \leq b_1 \Big(|x - x_1| + |y - y_1| \Big), \; t \in I, \; x, y, x_1, y_1 \in R.$$

where b_1 is a positive constant.

Remark 2. *From the assumption $(ii)^*$ and as done before*

$$|f(t,x,y)| \leq m_1 + b_1(|x| + |y|), \; \textit{where } m_1 = \int_0^1 |f(t,0,0)| dt.$$

Also

$$|g(t,s,x)| \leq m_2(t,s) + b_2 |x|, \; \textit{where } m_2(t,s) = \int_0^1 |g(t,s,0)| dt.$$

Theorem 4. *Let the assumptions $(i), (ii)^*, (iii)$ and (iv) be satisfied, then the functional integral equation (4) has a unique solution $y \in L_1(I)$.*

Proof. Define the operator F by

$$Fy(t) = f_2 \Big(t, y(t), \int_0^{\phi(t)} g(t,s,y(s)) ds \Big).$$

Now, let $y \in L_1(I)$, then

$$\begin{aligned} |Fy(t)| &= \left|f\left(t, y(t), \int_0^{\phi(t)} g(t,s,y(s))\right)ds\right| \\ &\leq |m_1(t)| + b_1\left(|y(t)| + |\int_0^{\phi(t)} g(t,s,y(s))ds|\right) \\ &\leq |m_1(t)| + b_1|y(t)| + b_1 \int_0^{\phi(t)} (|m_2(t,s)| + b_2|y(s)|)ds \\ &\leq |m_1(t)| + b_1|y(t)| + b_1 \int_0^{\phi(t)} |m_2(t,s)|ds + b_1b_2 \int_0^{\phi(t)} |y(s)|ds \end{aligned}$$

and

$$\begin{aligned} \int_0^1 |Fy(t)|dt &\leq \int_0^1 |m_1(t)|dt + b_1 \int_0^1 |y(t)|dt + b_1 \int_0^1 \int_0^{\phi(t)} |m_2(t,s)|dsdt \\ &+ b_1b_2 \int_0^1 \int_0^{\phi(t)} |y(s)|ds. \end{aligned}$$

This means that $F : L_1(I) \to L_1(I)$. Now let $y_1, y_2 \in L_1(I)$, then

$$\begin{aligned} |Fy_2(t) - Fy_1(t)| &= \left|f(t, y_2(t), \int_0^{\phi(t)} g(t,s,y_2(s))ds) - f(t, y_1(t), \int_0^{\phi(t)} g(t,s,y_1(s))ds)\right| \\ &\leq \left|f(t, y_2(t), \int_0^{\phi(t)} g(t,s,y_2(s))ds) - f(t, y_2(t), \int_0^{\phi(t)} g(t,s,y_1(s))ds)\right. \\ &+ \left. f(t, y_2(t), \int_0^{\phi(t)} g(t,s,y_1(s))ds) - f(t, y_1(t), \int_0^{\phi(t)} g(t,s,y_1(s))ds)\right| \\ &\leq b_1b_2 \int_0^{\phi(t)} |y_2(s) - y_1(s)|ds + b_1|y_2(t) - y_1(t)| \\ &\leq b_1b_2\|y_2 - y_1\| + b_1|y_2(t) - y_1(t)|. \end{aligned}$$

Then

$$\int_0^1 |Fy_2(t) - Fy_1(t)|dt \leq b_1b_2\|y_2 - y_1\| \int_0^1 dt + b_1 \int_0^1 |y_2(t) - y_1(t)|dt$$

and

$$\begin{aligned} \|Fy_2 - Fy_1\|_{L_1(I)} &\leq b_1b_2\|y_2 - y_1\| + b_1\|y_2 - y_1\| \\ &\leq (b_1 + b_1b_2)\|y_2 - y_1\|_{L_1(I)}. \end{aligned}$$

Since $b_1 + b_1b_2 < 1$, then F is a contraction, then by using Banach fixed point theorem [11] there exists a unique solution $y \in L_1(I)$ of the equation (4). □

2.2.1. *Continuous dependence.*

Theorem 5. *Let the assumptions of Theorem 4 be satisfied, then the functional integral equation (4) depends continuously on the function g.*

Proof. Let $\delta > 0$ be given such that $|g(t, x(t), y(t)) - g^*(t, x(t), y(t))| \leq \delta$ and let x^* be the solution of (1)-(2) corresponding to $g^*(t, x(t), y(t))$, then

$$\begin{aligned} |y(t) - y^*(t)| &= \Big| f(t, y(t), \int_0^{\phi(t)} g(t, s, y(s))ds) - f(t, y^*(t), \int_0^{\phi(t)} g^*(t, s, y^*(s))ds) \Big| \\ &\leq \Big| f(t, y(t), \int_0^{\phi(t)} g(t, s, y(s))ds) - f(t, y(t), \int_0^{\phi(t)} g^*(t, s, y(s))ds) \\ &+ f(t, y(t), \int_0^{\phi(t)} g^*(t, s, y(s))ds) - f(t, y^*(t), \int_0^{\phi(t)} g^*(t, s, y^*(s))ds) \Big| \\ &\leq b_1 \int_0^{\phi(t)} \Big| g(t, s, y(s)) - g^*(t, s, y(s)) \Big| ds + b_1 |y(t) - y^*(t)| \\ &+ b_1 \int_0^{\phi(t)} \Big| g^*(t, s, y(s))ds) - g^*(t, s, y^*(s)) \Big| ds \\ &\leq b_1 \delta + b_1 |y(t) - y^*(t)| + b_1 b_2 \int_0^{\phi(t)} |y(t) - y^*(t)|. \end{aligned}$$

Then

$$\int_0^t |y(t) - y^*(t)|dt = b_1 \delta \int_0^t dt + b_1 \int_0^t |y(t) - y^*(t)|dt + b_1 b_2 \int_0^t \int_0^{\phi(t)} |y(t) - y^*(t)|dt.$$

Hence

$$\|y - y^*\|(1 - (b_1 + b_1 b_2)) \quad \leq \quad b_1 \delta,$$

then

$$\|y - y^*\|_{L_1(I)} \quad \leq \quad \frac{b_1 \delta}{1 - (b_1 + b_1 b_2)} = \frac{\epsilon}{2}. \tag{7}$$

□

Theorem 6. *Let the assumptions of Theorem 4 be satisfied, then the functional integral equation (4) depends continuously on the function ϕ.*

Proof. Let $\delta > 0$ be given such that $|\phi(t) - \phi^*(t)| \leq \delta$ and let x^* be the solution of (1)-(2) corresponding to $\phi^*(t)$, then

$$\begin{aligned} |y(t) - y^*(t)| &= \Big| f(t, y(t), \int_0^{\phi(t)} g(t, s, y(s))ds) - f(t, y^*(t), \int_0^{\phi^*(t)} g(t, s, y^*(s))ds) \Big| \\ &\leq \Big| f(t, y(t), \int_0^{\phi(t)} g(t, s, y(s))ds) - f(t, y(t), \int_0^{\phi^*(t)} g(t, s, y^*(s))ds) \\ &+ f(t, y(t), \int_0^{\phi^*(t)} g(t, s, y^*(s))ds) - f(t, y^*(t), \int_0^{\phi^*(t)} g(t, s, y^*(s))ds) \Big| \\ &\leq b_1 \Big| \int_0^{\phi(t)} g(t, s, y(s))ds - \int_0^{\phi^*(t)} g(t, s, y^*(s))ds \Big| + b_1 |y(t) - y^*(t)| \\ &\leq b_1 \Big| \int_0^{\phi(t)} g(t, s, y(s))ds - \int_0^{\phi(t)} g(t, s, y^*(s))ds \\ &+ \int_0^{\phi(t)} g(t, s, y^*(s))ds - \int_0^{\phi^*(t)} g(t, s, y^*(s))ds \Big| + b_1 |y(t) - y^*(t)| \\ &\leq b_1 \Big(\int_0^{\phi(t)} b_2 |y(s) - y^*(s)|ds + \int_{\phi^*(t)}^{\phi(t)} |g(t, s, y^*(s))|ds \Big) + b_1 |y(t) - y^*(t)| \\ &\leq b_1 b_2 \int_0^{\phi(t)} |y(t) - y^*(t)| + b_1 (m_2(t, s) + b_2 |y^*(s)|) |\phi - \phi^*| + b_1 |y(t) - y^*(t)| \\ &\leq b_1 b_2 \|y - y^*\| + (b_1 m_2(t, s) + b_1 b_2 |y^*(s)|) \delta + b_1 |y(t) - y^*(t)|. \end{aligned}$$

Hence

$$\int_0^t |y(t)-y^*(t)|dt \quad \leq \quad b_1b_2\|y-y^*\| \int_0^t dt + (b_1m_2(t,s)+b_1b_2|y^*(s)|)\delta \int_0^t dt + b_1 \int_0^t |y(t)-y^*(t)|dt,$$

then

$$\|y-y^*\|_{L_1(I)} \quad \leq \quad \frac{(b_1m_2(t,s)+b_1b_2|y^*(s)|)\delta}{1-(b_1+b_1b_2)} = \frac{\epsilon}{2} \tag{8}$$

□

3. The two-points nonlocal problem

Theorem 7. *Let the assumptions (i)-(iv) be satisfied, then the problem (1)-(2) has a unique solution $x \in C^1(I)$.*

Proof. From Theorem 1 the solution $y \in C(I)$ of the functional integral equation (4) exists and from equation (3), we obtain

$$\frac{dx}{dt} = y(t) \in C(I)$$

then $x \in C^1(I)$. □

Theorem 8. *Let the assumptions $(i), (ii)^*, (iii)$ and (iv) be satisfied, then the problem (1)-(2) has a unique solution $x \in AC(I)$.*

Proof. From Theorem 4 the solution $y \in L_1(I)$ of the functional integral equation (4) exists and from equation (3), we obtain

$$\frac{dx}{dt} = y(t) \in L_1(I)$$

then $x \in AC(I)$. □

3.1. Continuous dependence.

Theorem 9. *Let the assumptions $(i)-(iv)$ be satisfied, then the unique solution of the problem (1)-(2) depends continuously on the parameter x_0.*

Proof. Let $\delta > 0$ be given such that $|x_0 - x_0^*| \leq \delta$ and let x^* be the solution of (1)-(2) corresponding to initial value x_0^*, then

$$\begin{aligned} |x(t)-x^*(t)| &= \Big|\frac{1}{\alpha+\beta}[x_0 - \alpha\int_0^\tau y(s)ds - \beta\int_0^\xi y(s)ds] + \int_0^t y(s)ds \\ &- \frac{1}{\alpha+\beta}[x_0^* - \alpha\int_0^\tau y(s)ds - \beta\int_0^\xi y(s)ds] + \int_0^t y(s)ds\Big| \\ &\leq \frac{1}{\alpha+\beta}|x_0 - x_0^*|. \end{aligned}$$

Then

$$\|x-x^*\| \quad \leq \quad \frac{\delta}{\alpha+\beta} = \epsilon.$$

□

Theorem 10. *Let the assumptions $(i)-(iv)$ be satisfied, then the unique solution of the problem (1)-(2) depends continuously on the function α, β.*

Proof. Let $\delta_1, \delta_1 > 0$ be given such that $|\alpha - \alpha^*| \leq \delta_1$, $|\beta - \beta^*| \leq \delta_2$ and let x^* be the solution of (1)-(2) corresponding to α^*, β^*, then

$$\begin{aligned}
|x(t) - x^*(t)| &= \Big|\frac{1}{\alpha+\beta}[x_0 - \alpha\int_0^\tau y(s)ds - \beta\int_0^\xi y(s)ds] + \int_0^t y(s)ds \\
&- \frac{1}{\alpha^*+\beta^*}[x_0 - \alpha^*\int_0^\tau y(s)ds - \beta^*\int_0^\xi y(s)ds] + \int_0^t y(s)ds\Big| \\
&\leq \frac{(\alpha^*+\beta^*)-(\alpha+\beta)}{(\alpha+\beta)(\alpha^*+\beta^*)}|x_0| + \frac{\alpha(\alpha^*+\beta^*)-\alpha^*(\alpha+\beta)}{(\alpha^*+\beta^*)(\alpha+\beta)}\int_0^\tau y(s)ds \\
&+ \frac{\beta^*(\alpha+\beta)-\beta(\alpha^*+\beta^*)}{(\alpha^*+\beta^*)(\alpha+\beta)}\int_0^\xi y(s)ds \\
&\leq \frac{\delta_1+\delta_2}{(\alpha+\beta)(\alpha^*+\beta^*)}|x_0| + \frac{(\alpha+\beta)(\alpha^*+\beta^*)(\alpha^*-\alpha)}{(\alpha^*+\beta^*)(\alpha+\beta)}\int_0^\tau y(s)ds \\
&+ \frac{(\alpha+\beta)(\alpha^*+\beta^*)(\beta^*-\beta)}{(\alpha^*+\beta^*)(\alpha+\beta)}\int_0^\xi y(s)ds \\
&\leq \frac{\delta_1+\delta_2}{(\alpha+\beta)(\alpha^*+\beta^*)}|x_0| + \frac{(\alpha+\beta)(\alpha^*+\beta^*)\delta_1}{(\alpha^*+\beta^*)(\alpha+\beta)}\int_0^\tau y(s)ds \\
&+ \frac{(\alpha+\beta)(\alpha^*+\beta^*)\delta_2}{(\alpha^*+\beta^*)(\alpha+\beta)}\int_0^\xi y(s)ds = \epsilon.
\end{aligned}$$

Then

$$\|x - x^*\| \leq \epsilon.$$

□

Theorem 11. *Let the assumptions $(i)-(iv)$ and $(ii)^*$ be satisfied, then the unique solution of the problem (1)-(2) depends continuously on the function g.*

Proof.

$$\begin{aligned}
|x(t) - x^*(t)| &= \Big|\frac{1}{\alpha+\beta}[x_0 - \alpha\int_0^\tau y(s)ds - \beta\int_0^\xi y(s)ds] + \int_0^t y(s)ds \\
&- \frac{1}{\alpha+\beta}[x_0 - \alpha\int_0^\tau y^*(s)ds - \beta\int_0^\xi y^*(s)ds] + \int_0^t y^*(s)ds\Big|
\end{aligned}$$

Then

$$\begin{aligned}
|x(t) - x^*(t)| &\leq \frac{\alpha}{\alpha+\beta}\int_0^\tau |y(s)-y^*(s)|ds \\
&+ \frac{\beta}{\alpha+\beta}\int_0^\xi |y(s)-y^*(s)|ds + \int_0^t |y(s)-y^*(s)|ds. \qquad (9)
\end{aligned}$$

From (5) and (9), we have

$$\begin{aligned}
\|x - x^*\|_{C(I)} &\leq \frac{\alpha}{\alpha+\beta}\|y-y^*\|_{C(I)} + \frac{\beta}{\alpha+\beta}\|y-y^*\|_{C(I)} + \|y-y^*\|_{C(I)} \\
&\leq \frac{\alpha}{\alpha+\beta}(\frac{\epsilon}{2}) + \frac{\beta}{\alpha+\beta}(\frac{\epsilon}{2}) + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Also, from(7) and (9), we have

$$\begin{aligned}
\|x - x^*\|_{C(I)} &\leq \frac{\alpha}{\alpha+\beta}\|y-y^*\|_{L_1(I)} + \frac{\beta}{\alpha+\beta}\|y-y^*\|_{L_1(I)} + \|y-y^*\|_{L_1(I)} \\
&\leq \frac{\alpha}{\alpha+\beta}(\frac{\epsilon}{2}) + \frac{\beta}{\alpha+\beta}(\frac{\epsilon}{2}) + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

□

Theorem 12. *Let the assumptions $(i)-(iv)$ and $(ii)^*$ be satisfied, then the unique solution of the problem (1)-(2) depends continuously on the function ϕ.*

Proof.

$$\begin{aligned}|x(t)-x^*(t)| &= \Big|\frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau y(s)ds-\beta\int_0^\xi y(s)ds]+\int_0^t y(s)ds \\ &- \frac{1}{\alpha+\beta}[x_0-\alpha\int_0^\tau y^*(s)ds-\beta\int_0^\xi y^*(s)ds]+\int_0^t y^*(s)ds\Big|.\end{aligned}$$

Then

$$\begin{aligned}|x(t)-x^*(t)| &\leq \frac{\alpha}{\alpha+\beta}\int_0^\tau |y(s)-y^*(s)|ds \\ &+ \frac{\beta}{\alpha+\beta}\int_0^\xi |y(s)-y^*(s)|ds+\int_0^t |y(s)-y^*(s)|ds. \qquad (10)\end{aligned}$$

From (6) and (10), we have

$$\begin{aligned}\|x-x^*\|_{C(I)} &\leq \frac{\alpha}{\alpha+\beta}\|y-y^*\|_{C(I)}+\frac{\beta}{\alpha+\beta}\|y-y^*\|_{C(I)}+\|y-y^*\|_{C(I)} \\ &\leq \frac{\alpha}{\alpha+\beta}(\frac{\epsilon}{2})+\frac{\beta}{\alpha+\beta}(\frac{\epsilon}{2})+\frac{\epsilon}{2}=\epsilon\end{aligned}$$

Also, from(8) and (10), we have

$$\begin{aligned}\|x-x^*\|_{C(I)} &\leq \frac{\alpha}{\alpha+\beta}\|y-y^*\|_{L_1(I)}+\frac{\beta}{\alpha+\beta}\|y-y^*\|_{L_1(I)}+\|y-y^*\|_{L_1(I)} \\ &\leq \frac{\alpha}{\alpha+\beta}(\frac{\epsilon}{2})+\frac{\beta}{\alpha+\beta}(\frac{\epsilon}{2})+\frac{\epsilon}{2}=\epsilon.\end{aligned}$$

□

4. Anti-periodic boundary value problem

Consider the differential equation 1 with the anti-periodic nonlocal condition

$$x(\tau)=-x(1-\tau)\ \tau\in[0,1) \qquad (11)$$

and choices $\alpha=1$, $\beta=1$, $\xi=1-\tau$ and $x_0=0$ in the problem (1)-(2), then the anti-periodic boundary value problem (1)-(11) has the solution $x\in C^1(I)$

$$x(t)=\frac{1}{2}\Big[-\int_0^\tau y(s)ds-\int_0^{1-\tau} y(s)ds\Big]+\int_0^t y(s)ds.$$

Now, let $\tau=\frac{1}{2}$, then

$$\begin{aligned}x(t) &= -\frac{1}{2}\int_0^{\frac{1}{2}} y(s)ds-\frac{1}{2}\int_0^{\frac{1}{2}} y(s)ds+\int_0^t y(s)ds \\ &= \int_0^t y(s)ds-\int_0^{\frac{1}{2}} y(s)ds.\end{aligned}$$

5. Conclusions

In this paper, we have studied a two-points nonlocal problem of an implicit delay functional integro-differential equation. We have investigated the solvability of the problem (1)-(2) by applying Banach fixed point theorem [11] in the two classes $x\in C^1(I)$ and $x\in AC(I)$. Moreover, we have discussed the continuous dependence of the solution on x_0, the function g and on the delay function ϕ. Finally, the anti-periodic boundary value problem will be considered as an application.

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Malak. M. S. Ba-Ali

Faculty of Science, Princess Nourah Bint Abdul Rahman University, Riyadh 11671, Saudi Arabia

Email address: baalimalak1@gmail.com