Electronic Journal of Mathematical Analysis and Applications
Vol. 11(2) July 2023, No. 1
ISSN: 2090-729X(online).
https://ejmaa.journals.ekb.eg/

ON THE DYNAMICS OF A RICCATI DIFFERENTIAL EQUATION WITH PERTURBED DELAY

A. M. A. EL-SAYED, S. M. SALMAN, A. A. F. ABDELFATTAH

Abstract. Within the scope of this study, we discuss the new concept of a perturbed delay. As a simple example, we will focus on a Riccati differential equation with a perturbed delay to illustrate this concept. We look at both the solution's existence and its continuous dependence on the initial conditions. Analyses of Hopf bifurcations and the local stability of the fixed points are presented. In order to solve the delay differential equation with piecewise constant arguments, we adopt a discretization procedure. We do an analysis of the local stability of the discrete system. We use numerical simulations to draw out the results, like bifurcation diagrams, Lyapunov exponents, and phase diagrams. This helps us confirm our research and unearth more complex dynamics. We contrast the results of theoretical studies of the delayed Riccati differential equation and its perturbed equation. Our results show that, under certain conditions, the Riccati differential equation with perturbed delay is equivalent to the Riccati differential equation with the same dynamical properties.

1. Introduction

Many phenomena in domains as diverse as the economy, chemistry, physics, engineering, and biology exhibit both time and space change [1-3]. Modeling and interpreting these phenomena may be greatly aided by studying dynamical systems. Synchronization and chaos control, secure communications, brain research, machine learning, electrical circuits, cryptography, and image encryption are just a few of the numerous fields that benefit from the study of dynamical systems [4-9].

Applications of Riccati differential equations are widespread throughout classical and contemporary science and engineering, including diffusion problems, stochastic realisation theory, network synthesis, optimal filtering, controls, financial mathematics, robust

2020 *Mathematics Subject Classification.* 34A12, 34K20, 34K27, 37C25, 37G15.
Key words and phrases. Riccati equation, Perturbed delay, Stability analysis, Bifurcation, Chaos.
Submitted February 13, 2023. Revised March 24, 2023.

stabilisation, random processes, and variational calculus [10-13]. Another important model in physics, the Riccati differential equation is related to the Schrodinger equation of one dimension [14].

Differential equations with a delay are equations that use the derivatives of an unknown function at a particular time that is determined by the function's values at earlier periods [15-16]. In addition, the delay differential equation may be used to characterise the dynamics of physiological systems as well as electrochemical intercalation [17-23]. In addition, there are certain systems that are not stable with only one delay, but if a second delay is introduced to the system, the system is able to maintain its stability [24].

The delayed Riccati differential equation reads

$$\begin{aligned} &\frac{dx}{dt} = 1 - \rho x(t)x(t-r), \qquad t \in (0,T], \\ &x(t) = x_o, \qquad t \leq 0, \end{aligned} \tag{1.1}$$

where $\rho, r > 0$.

The problem (1.1) can be rewritten as follows

$$\begin{aligned} &\frac{dx}{dt} = 1 - \rho x(t)y(t), \qquad t \in (0,T], \\ &y(t) = x(t-r), \\ &x(t) = x_o, \quad y(t) = y_o, \qquad t \leq 0. \end{aligned} \tag{1.2}$$

Let there exists a perturbed delay as

$$y(t) = ax(t-r) + \epsilon x(t-2r),$$

where $0 < a, \epsilon < 1$.

The Riccati differential equation with perturbed delay can be considered as

$$\begin{aligned} &\frac{dx}{dt} = 1 - \rho x(t)y(t), \qquad t \in (0,T], \\ &y(t) = ax(t-r) + \epsilon x(t-2r), \\ &x(t) = x_o, \quad y(t) = y_o, \qquad t \leq 0. \end{aligned} \tag{1.3}$$

The structure of this article is as shown. The existence of a Riccati differential equation's solution with perturbed delay is discussed in Subsection (2.1). In Subsection (2.2) the continuous dependence of the solution on the initial conditions is studied. Local stability of the Riccati differential equation with perturbed delay is studied in Subsection (2.3). The Hopf bifurcation analysis is performed in Subsection (2.4). The method of discretization of the Riccati differential equation with a perturbed delay is presented in Subsection (2.5). Local stability of the discrete system is performed in Subsection (2.6). In Subsection (2.7), we confirm the obtained results with numerical simulations. The work's summary and knowledge discussion are included in Section (3).

2. Main Results

It is possible to rewrite the problem (1.3) as

$$\begin{aligned} &\frac{dx}{dt} = 1 - \rho x(t)\Big[ax(t-r) + \epsilon x(t-2r)\Big], \qquad t \in (0, T], \\ &x(t) = x_o, \qquad t \le 0. \end{aligned} \tag{2.1}$$

2.1. Existence and uniqueness.

Theorem 2.1. *If* $\rho < \dfrac{1}{(3a+4\epsilon)T}$ *, then the problem* (2.1) *has a unique solution* $x \in C[0,T]$, $0 \le x(t) \le 1$.

Proof. Assume the following operator $F : C[0,T] \to C[0,T]$, defined by

$$Fx(t) = x_o + \int_0^t \Big(1 - \rho x(s)\Big[ax(s-r) + \epsilon x(s-2r)\Big]\Big)ds,$$

$$\begin{aligned} &= x_o + \int_0^r \Big(1 - \rho x(s)\Big[ax(s-r) + \epsilon x(s-2r)\Big]\Big)ds + \int_r^{2r} \Big(1 - \rho x(s)\Big[ax(s-r) + \epsilon x(s-2r)\Big]\Big)ds \\ &+ \int_{2r}^t \Big(1 - \rho x(s)\Big[ax(s-r) + \epsilon x(s-2r)\Big]\Big)ds, \end{aligned}$$

$$\begin{aligned} &= x_o + \int_0^r \Big(1 - \rho x(s)\Big[ax_o + \epsilon x_o\Big]\Big)ds + \int_r^{2r} \Big(1 - \rho x(s)\Big[ax(s-r) + \epsilon x_o\Big]\Big)ds \\ &+ \int_{2r}^t \Big(1 - \rho x(s)\Big[ax(s-r) + \epsilon x(s-2r)\Big]\Big)ds. \end{aligned}$$

We can deduce for each $x,\ y \in C[0,T]$

$$\begin{aligned} |Fx - Fy| &\le \rho(a+\epsilon)x_o \int_0^r |x(s) - y(s)|ds + \rho\epsilon x_o \int_r^{2r} |x(s) - y(s)|ds \\ &+ \rho a \int_r^{2r} |x(s)x(s-r) - y(s)y(s-r)|ds + \rho a \int_{2r}^t |x(s)x(s-r) - y(s)y(s-r)|ds \\ &+ \rho\epsilon \int_{2r}^t |x(s)x(s-2r) - y(s)y(s-2r)|ds, \\ &\le \rho(a+\epsilon)x_o \int_0^r |x(s) - y(s)|ds + \rho\epsilon x_o \int_r^{2r} |x(s) - y(s)|ds \\ &+ \rho a \int_r^t \Big|\Big[x(s) - y(s)\Big]y(s-r) + x(s)\Big[x(s-r) - y(s-r)\Big]\Big|ds \\ &+ \rho\epsilon \int_{2r}^t \Big|\Big[x(s) - y(s)\Big]y(s-2r) + x(s)\Big[x(s-2r) - y(s-2r)\Big]\Big|ds, \end{aligned}$$

then,

$$\begin{aligned}||Fx-Fy|| &\leq \rho r(a+\epsilon)x_o\ ||x-y|| + \rho\epsilon r x_o\ ||x-y|| + \rho a(t-r)\ ||x-y||\ ||y|| + \rho a(t-r)\ ||x-y||\ ||x|| \\ &+ \rho\epsilon(t-2r)\ ||x-y||\ ||y|| + \rho\epsilon(t-2r)\ ||x-y||\ ||x||, \\ &\leq \rho(a+\epsilon)T\ ||x-y|| + \rho\epsilon T\ ||x-y|| + \rho a T\ ||x-y|| + \rho a T\ ||x-y|| \\ &+ \rho\epsilon T\ ||x-y|| + \rho\epsilon T\ ||x-y||, \\ &\leq \rho(3a+4\epsilon)T\ ||x-y||.\end{aligned}$$

If $\rho < \dfrac{1}{(3a+4\epsilon)T}$, then F is contraction, and the problem (2.1) has a unique solution $x \in C[0,T]$. □

2.2. **Continuous dependence.**

Definition 2.1. *The solution of the problem* (2.1) *depends continuously on the initial value* x_o *if* $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ *such that* $|x_o - x_o^*| \leq \delta$ *implies that* $||x - x^*|| \leq \epsilon$ *where* x^* *is the solution of the problem*

$$\begin{aligned}&\frac{dx}{dt} = 1 - \rho x(t)\Big[ax(t-r) + \epsilon x(t-2r)\Big], \qquad t \in (0,T], \\ &x(t) = x_o^*, \qquad t \leq 0.\end{aligned} \tag{2.2}$$

Theorem 2.2. *If* $\rho(3a+4\epsilon)T \neq 1$, *then the unique solution of the problem* (2.1) *depends continuously on the initial value* x_o.

Proof. Let x and x^* are the solutions of the problems (2.1) and (2.2) respectively, then

$$\begin{aligned}||x(t)-x^*(t)|| &\leq |x_o - x_o^*| + \rho(a+\epsilon)\int_0^r |x(s)x_o - x^*(s)x_o^*|ds + \rho\epsilon\int_r^{2r} |x(s)x_o - x^*(s)x_o^*|ds \\ &+ \rho a\int_r^{2r} |x(s)x(s-r) - x^*(s)x^*(s-r)|ds + \rho a\int_{2r}^t |x(s)x(s-r) - x^*(s)x^*(s-r)|ds \\ &+ \rho\epsilon\int_{2r}^t |x(s)x(s-2r) - x^*(s)x^*(s-2r)|ds, \\ &\leq |x_o - x_o^*| + \rho(a+\epsilon)\int_0^r |[x(s)-x^*(s)]x_o + [x_o - x_o^*]x^*(s)|ds \\ &+ \rho\epsilon\int_r^{2r} |[x(s)-x^*(s)]x_o + [x_o - x_o^*]x^*(s)|ds + \rho a\int_r^t |x(s)x(s-r) - x^*(s)x^*(s-r)|ds \\ &+ \rho\epsilon\int_{2r}^t |x(s)x(s-2r) - x^*(s)x^*(s-2r)|ds, \\ &\leq |x_o - x_o^*| + \rho(a+\epsilon)||x-x^*||\ |x_o|\int_0^r ds + \rho(a+\epsilon)|x_o - x_o^*|\ ||x^*||\int_0^r ds \\ &+ \rho\epsilon||x-x^*||\ |x_o|\int_r^{2r} ds + \rho\epsilon|x_o - x_o^*|\ ||x^*||\int_r^{2r} ds \\ &+ \rho a\int_r^t \Big|\Big[x(s)-x^*(s)\Big]x^*(s-r) + x(s)\Big[x(s-r) - x^*(s-r)\Big]\Big|ds \\ &+ \rho\epsilon\int_{2r}^t \Big|\Big[x(s)-x^*(s)\Big]x^*(s-2r) + x(s)\Big[x(s-2r) - x^*(s-2r)\Big]\Big|ds,\end{aligned}$$

then,

$$\begin{aligned}||x-x^*|| &\leq |x_o - x_o^*| + \rho(a+\epsilon)r||x-x^*||\ |x_o| + \rho(a+\epsilon)r|x_o - x_o^*|\ ||x^*|| + \rho\epsilon r||x-x^*||\ |x_o| \\ &+ \rho\epsilon r|x_o - x_o^*|\ ||x^*|| + \rho a(t-r)||x-x^*||\ ||x^*|| + \rho a(t-r)||x||\ ||x-x^*|| \\ &+ \rho\epsilon(t-2r)||x-x^*||\ ||x^*|| + \rho\epsilon(t-2r)||x||\ ||x-x^*||,\end{aligned}$$

which implies

$$||x - x^*|| \leq \frac{1 + \rho(a + 2\epsilon)T}{1 - \rho(3a + 4\epsilon)T} \; |x_o - x_o^*|,$$

which proves that

$$|x_o - x_o^*| \leq \delta \quad \Longrightarrow \quad ||x - x^*|| \leq \frac{1 + \rho(a + 2\epsilon)T}{1 - \rho(3a + 4\epsilon)T}\delta = \epsilon^*.$$

□

2.3. **The local stability of problem (2.1).** The local stability of the equilibrium points of (2.1) will be studied. Specifically, $x^*_{1,2} = \pm\dfrac{1}{\sqrt{\rho(a+\epsilon)}}$, are solutions to the equation $1 - \rho x[ax + \epsilon x] = 0$.

We get the linearized equation as

$$\frac{dy}{dt} = \mp\sqrt{\rho(a+\epsilon)}y(t) \mp \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}y(t-r) \mp \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}y(t-2r).$$

The characteristic equation is given by

$$\lambda \pm \sqrt{\rho(a+\epsilon)} \pm \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}} \; e^{-r\lambda} \pm \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}} \; e^{-2r\lambda} = 0. \tag{2.3}$$

A following corollary is a helpful tool that may be used to estimate the local stability of equation (2.3) at the points of equilibrium $x^*_{1,2}$.

Corollary 2.1. *[25] The scalar equation*

$$\dot{x}(t) = a_o \; x(t) + \sum_{k=1}^{N} a_k \; x(t - r_k)$$

is asymptotically stable if and only if $\sum_{k=0}^{N} a_k \neq 0$, $\sum_{k=1}^{N} |a_k| \leq |a_o|$ *and* $a_o < 0$.

Now let $a_o = \sqrt{\rho(a+\epsilon)}$, $a_1 = \dfrac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}$, $a_2 = \dfrac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}$ and using above corollary we get the following results.

Proposition 2.1.

1. *The equilibrium point* $x^*_1 = \dfrac{1}{\sqrt{\rho(a+\epsilon)}}$ *is always stable.*
2. *The equilibrium point* $x^*_2 = \dfrac{-1}{\sqrt{\rho(a+\epsilon)}}$ *is always unstable.*

2.4. **Hopf bifurcation.** Following is a discussion of the Hopf bifurcation that we cover in this part.

Theorem 2.3. *If* $\left.\dfrac{d(Re(\lambda))}{d\epsilon}\right|_{\epsilon=\epsilon_*} = \left.\dfrac{dk}{d\epsilon}\right|_{k=0,\omega=\omega_o,\epsilon=\epsilon_*} \neq 0$, $\epsilon_* = \dfrac{-a - a\cos(r\omega)}{2\cos^2(r\omega)}$,

$\omega_o = \tan(2r\omega_o)\Big(-\sqrt{\rho(a+\epsilon)} - \dfrac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\cos(r\omega_o)\Big) + \dfrac{a\sqrt{\rho}}{\sqrt{a+\epsilon}} \; \sin(r\omega_o)$, *then there* *is a Hopf bifurcation when* $\epsilon = \epsilon_*$ *at the equilibrium* x_1^*, *where*

$$\frac{dk}{d\epsilon}\Big|_{k=0,\omega=\omega_o,\epsilon=\epsilon_*} = \Bigg[\Big(\frac{\rho}{2\sqrt{\rho}} - \frac{a\sqrt{\rho}}{2(a+\epsilon_*)}\cos(r\omega_o) + \sqrt{\rho}\cos(2r\omega_o) - \frac{\epsilon_*\sqrt{\rho}}{2(a+\epsilon_*)}\cos(2r\omega_o)\Big)$$
$$\Big(ar\sqrt{\rho}\cos(r\omega_o) + 2\epsilon_* r\sqrt{\rho}\cos(2r\omega_o) - \sqrt{a+\epsilon_*}\Big) - \Big(\frac{a\sqrt{\rho}}{2(a+\epsilon_*)}\sin(r\omega_o)$$
$$-\sqrt{\rho}\sin(2r\omega_o) + \frac{\epsilon_*\sqrt{\rho}}{2(a+\epsilon_*)}\sin(2r\omega_o)\Big)\Big(ar\sqrt{\rho}\sin(r\omega_o) + 2\epsilon_* r\sqrt{\rho}\sin(2r\omega_o)\Big)\Bigg]$$
$$\div\Bigg[\Big(ar\sqrt{\rho}\sin(r\omega_o) + 2\epsilon_* r\sqrt{\rho}\sin(2r\omega_o)\Big)^2 + \Big(ar\sqrt{\rho}\cos(r\omega_o) + 2\epsilon_* r\sqrt{\rho}\cos(2r\omega_o)$$
$$-\sqrt{a+\epsilon_*}\Big)^2\Bigg].$$

Proof. Suppose that equation (2.3) has a pure imaginary solution $\lambda = i\omega_o,\ \omega_o \in R^+$ for a given value of a parameter $\epsilon = \epsilon_*$. Therefore, we get the following equation

$$i\omega_o + \sqrt{\rho(a+\epsilon)} + \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-ri\omega_o} + \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-2ri\omega_o} = 0.$$

We can rephrase that by

$$i\omega_o + \sqrt{\rho(a+\epsilon)} + \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ \Big(\cos(r\omega_o) - i\sin(r\omega_o)\Big) + \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ \Big(\cos(2r\omega_o) - i\sin(2r\omega_o)\Big) = 0.$$

This complex equation is equivalent to the two real equations

$$\sqrt{\rho(a+\epsilon)} + \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\cos(r\omega_o) + \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ \cos(2r\omega_o) = 0, \tag{2.4}$$

$$\omega_o - \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ \sin(r\omega_o) - \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ \sin(2r\omega_o) = 0. \tag{2.5}$$

Following the resolution of equation (2.4) and equation (2.5), we get

$$\epsilon_* = \frac{-a - a\cos(r\omega_o)}{2\cos^2(r\omega_o)},$$
$$\omega_o = \tan(2r\omega_o)\Big(-\sqrt{\rho(a+\epsilon)} - \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\cos(r\omega_o)\Big) + \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ \sin(r\omega_o).$$

In what follows, we show that condition $\frac{d(Re(\lambda))}{d\epsilon}\Big|_{\epsilon=\epsilon_*} \neq 0$ is investigated. Put $\lambda = k(\epsilon) + i\omega(\epsilon)$ and use equation (2.3), we have

$$k + i\omega + \sqrt{\rho(a+\epsilon)} + \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-r(k+i\omega)} + \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-2r(k+i\omega)} = 0,$$

then,

$$k + \sqrt{\rho(a+\epsilon)} + \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-rk}\cos(r\omega) + \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-2rk}\cos(2r\omega) = 0, \tag{2.6}$$

$$\omega - \frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-rk}\sin(r\omega) - \frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-2rk}\sin(2r\omega) = 0. \tag{2.7}$$

By differentiating equation (2.6) and equation (2.7) with respect to ϵ, we obtain

$$\begin{aligned}&\frac{dk}{d\epsilon}+\frac{\rho}{2\sqrt{\rho(a+\epsilon)}}-\frac{a\sqrt{\rho}}{2\,(a+\epsilon)^{\frac{3}{2}}}\ e^{-rk}\cos(r\omega)-\frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ re^{-rk}\cos(r\omega)\frac{dk}{d\epsilon}\\&-\frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-rk}r\sin(r\omega)\frac{d\omega}{d\epsilon}+\frac{\sqrt{\rho}}{\sqrt{a+\epsilon}}e^{-2rk}\cos(2r\omega)-\frac{\epsilon\sqrt{\rho}}{2\,(a+\epsilon)^{\frac{3}{2}}}e^{-2rk}\cos(2r\omega)\\&-2\frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ re^{-2rk}\cos(2r\omega)\frac{dk}{d\epsilon}-2\frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-2rk}r\sin(2r\omega)\frac{d\omega}{d\epsilon}=0,\end{aligned}\tag{2.8}$$

$$\begin{aligned}&\frac{d\omega}{d\epsilon}+\frac{a\sqrt{\rho}}{2\,(a+\epsilon)^{\frac{3}{2}}}\ e^{-rk}\sin(r\omega)+\frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ re^{-rk}\sin(r\omega)\frac{dk}{d\epsilon}-\frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-rk}r\cos(r\omega)\frac{d\omega}{d\epsilon}\\&-\frac{\sqrt{\rho}}{\sqrt{a+\epsilon}}e^{-2rk}\sin(2r\omega)+\frac{\epsilon\sqrt{\rho}}{2\,(a+\epsilon)^{\frac{3}{2}}}e^{-2rk}\sin(2r\omega)+2\frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ re^{-2rk}\sin(2r\omega)\frac{dk}{d\epsilon}\\&-2\frac{\epsilon\sqrt{\rho}}{\sqrt{a+\epsilon}}\ e^{-2rk}r\cos(2r\omega)\frac{d\omega}{d\epsilon}=0.\end{aligned}\tag{2.9}$$

Following the resolution of equation (2.8) and equation (2.9), we get

$$\begin{aligned}\frac{dk}{d\epsilon}\bigg|_{k=0,\omega=\omega_o,\epsilon=\epsilon_*}=&\Bigg[\bigg(\frac{\rho}{2\sqrt{\rho}}-\frac{a\sqrt{\rho}}{2(a+\epsilon_*)}\cos(r\omega_o)+\sqrt{\rho}\cos(2r\omega_o)-\frac{\epsilon_*\sqrt{\rho}}{2(a+\epsilon_*)}\cos(2r\omega_o)\bigg)\\&\bigg(ar\sqrt{\rho}\cos(r\omega_o)+2\epsilon_*r\sqrt{\rho}\cos(2r\omega_o)-\sqrt{a+\epsilon_*}\bigg)-\bigg(\frac{a\sqrt{\rho}}{2(a+\epsilon_*)}\sin(r\omega_o)\\&-\sqrt{\rho}\sin(2r\omega_o)+\frac{\epsilon_*\sqrt{\rho}}{2(a+\epsilon_*)}\sin(2r\omega_o)\bigg)\bigg(ar\sqrt{\rho}\sin(r\omega_o)+2\epsilon_*r\sqrt{\rho}\sin(2r\omega_o)\bigg)\Bigg]\\&\div\Bigg[\bigg(ar\sqrt{\rho}\sin(r\omega_o)+2\epsilon_*r\sqrt{\rho}\sin(2r\omega_o)\bigg)^2+\bigg(ar\sqrt{\rho}\cos(r\omega_o)+2\epsilon_*r\sqrt{\rho}\cos(2r\omega_o)\\&-\sqrt{a+\epsilon_*}\bigg)^2\Bigg].\end{aligned}$$

If $\frac{d(Re(\lambda))}{d\epsilon}\bigg|_{\epsilon=\epsilon_*}=\frac{dk}{d\epsilon}\bigg|_{k=0,\omega=\omega_o,\epsilon=\epsilon_*}\neq 0$, hence when the parameter ϵ crosses a certain critical value
$\epsilon=\epsilon_*=\frac{-a-a\cos(r\omega)}{2\cos^2(r\omega)}$, $\omega_o=\tan(2r\omega_o)\Big(-\sqrt{\rho(a+\epsilon)}-\frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\cos(r\omega_o)\Big)+\frac{a\sqrt{\rho}}{\sqrt{a+\epsilon}}\sin(r\omega_o)$,
the equilibrium point x_1^* undergoes Hopf bifurcation. □

Likewise, we can illustrate that the equilibrium point x_2^* undergoes Hopf bifurcation.

2.5. **The discrete system.** Dynamical systems generated by piecewise constant arguments have been studied in [26-29].

Consider the problem (1.3) with piecewise constant arguments as follows.

$$\begin{aligned}\frac{dx}{dt}&=1-\rho x(r[\frac{t}{r}])y(r[\frac{t}{r}]),\qquad t\in(0,T],\\y(r[\frac{t}{r}])&=ax(r[\frac{t}{r}]-r)+\epsilon x(r[\frac{t}{r}]-2r),\\ \\x(t)&=x_o,\quad y(t)=y_o,\qquad t\leq 0,\end{aligned}\tag{2.10}$$

where [.] denotes the greatest integer function and r is a constant argument.

Let $t \in [nr,\ (n+1)r)$ and $n = 0, 1, 2, \dots$. The procedure for discretization is as given below.

1) Let $t \in [0, r)$, then $[\frac{t}{r}] = 0$ and the solution of problem (2.10) is given by

$$\frac{dx}{dt} = 1 - \rho x_o y_o$$
$$x(t) - x(0) = \Big(1 - \rho x_o y_o\Big) \int_0^t 1\ ds$$
$$x(t) = x_o + t\Big(1 - \rho x_o y_o\Big),$$
$$y_o = a x_o + \epsilon x_o.$$

When $t \to r$ and $x(r) = x_1$ we get

$$x_1 = x_o + r(1 - \rho x_o y_o),$$
$$y_o = a x_o + \epsilon x_o.$$

2) Let $t \in [r, 2r)$, then $[\frac{t}{r}] = 1$ and the solution of problem (2.10) is given by

$$\frac{dx}{dt} = 1 - \rho x_1 y(r)$$
$$x(t) - x(r) = \Big(1 - \rho x_1 y(r)\Big) \int_r^t 1\ ds$$
$$x(t) = x_1 + (t - r)\Big(1 - \rho x_1 y(r)\Big),$$
$$y(r) = a x_o + \epsilon x_o.$$

When $t \to 2r$, $x(2r) = x_2$ and $y(r) = y_1$ we get

$$x_2 = x_1 + r(1 - \rho x_1 y_1),$$
$$y_1 = a x_o + \epsilon x_o.$$

3) Let $t \in [2r, 3r)$, then $[\frac{t}{r}] = 2$ and the solution of problem (2.10) is given by

$$\frac{dx}{dt} = 1 - \rho x_2 y(2r)$$
$$x(t) - x(2r) = \Big(1 - \rho x_2 y(2r)\Big) \int_{2r}^t 1\ ds$$
$$x(t) = x_2 + (t - 2r)\Big(1 - \rho x_2 y(2r)\Big),$$
$$y(2r) = a x_1 + \epsilon x_o.$$

When $t \to 3r$, $x(3r) = x_3$ and $y(2r) = y_2$ we get

$$x_3 = x_2 + r(1 - \rho x_2 y_2),$$
$$y_2 = a x_1 + \epsilon x_o.$$

We can conclude from iterating the procedure that the presented is a solution to problem (2.10)

$$\begin{aligned} x_{n+1} &= x_n + r(1 - \rho x_n y_n), \\ y_{n+1} &= a x_n + \epsilon x_{n-1}. \end{aligned} \tag{2.11}$$

2.6. **The local stability of the discrete system.** System (2.11) can be rephrased as below

$$\begin{aligned} x_{n+1} &= x_n + r(1 - \rho x_n y_n), \\ y_{n+1} &= a x_n + \epsilon z_n, \\ z_{n+1} &= x_n. \end{aligned} \tag{2.12}$$

This system has two fixed points (x_1^*, y_1^*, z_1^*) and (x_2^*, y_2^*, z_2^*) where

$$x_1^* = \frac{1}{\sqrt{\rho(a+\epsilon)}}, \quad y_1^* = \frac{a+\epsilon}{\sqrt{\rho(a+\epsilon)}}, \quad z_1^* = \frac{1}{\sqrt{\rho(a+\epsilon)}},$$

$$x_2^* = \frac{-1}{\sqrt{\rho(a+\epsilon)}}, \quad y_2^* = \frac{-(a+\epsilon)}{\sqrt{\rho(a+\epsilon)}}, \quad z_2^* = \frac{-1}{\sqrt{\rho(a+\epsilon)}},$$

and $(a+\epsilon) \neq 0$, which are solutions to the next algebraic system

$$\begin{aligned} x &= x + r(1 - \rho x y), \\ y &= a x + \epsilon z, \\ z &= x. \end{aligned}$$

The system's (2.12) associated Jacobian matrix reads

$$J(x, y, z) = \begin{bmatrix} 1 - r\rho y & -r\rho x & 0 \\ a & 0 & \epsilon \\ 1 & 0 & 0 \end{bmatrix}.$$

What follows is an analysis of fixed points' stability.

2.6.1. *Stability analysis at* (x_1^*, y_1^*, z_1^*). Jacobian matrix at (x_1^*, y_1^*, z_1^*) represents

$$J(x_1^*, y_1^*, z_1^*) = \begin{bmatrix} 1 - \dfrac{r\sqrt{\rho}(a+\epsilon)}{\sqrt{a+\epsilon}} & -\dfrac{r\sqrt{\rho}}{\sqrt{a+\epsilon}} & 0 \\ a & 0 & \epsilon \\ 1 & 0 & 0 \end{bmatrix}.$$

$J(x_1^*, y_1^*, z_1^*)$ has a characteristic equation given by

$$P(\lambda) \equiv \lambda^3 + \left(\frac{r\sqrt{\rho}(a+\epsilon)}{\sqrt{a+\epsilon}} - 1\right)\lambda^2 + \frac{ar\sqrt{\rho}}{\sqrt{a+\epsilon}}\lambda + \frac{\epsilon r\sqrt{\rho}}{\sqrt{a+\epsilon}} = 0.$$

The Jury test described in [30] is used to establish whether or not system (2.12), at the fixed point (x_1^*, y_1^*, z_1^*), is locally stable. We find the following.

Proposition 2.2. *The fixed point* (x_1^*, y_1^*, z_1^*) *is stable if* $0 < \rho < \dfrac{a+\epsilon}{a^2\ r^2}$ *and unstable if* $\rho > \dfrac{a+\epsilon}{a^2\ r^2}$.

2.6.2. *Stability analysis at* (x_2^*, y_2^*, z_2^*). Jacobian matrix at (x_2^*, y_2^*, z_2^*) represents

$$J(x_2^*, y_2^*, z_2^*) = \begin{bmatrix} 1 + \dfrac{r\sqrt{\rho}(a+\epsilon)}{\sqrt{a+\epsilon}} & \dfrac{r\sqrt{\rho}}{\sqrt{a+\epsilon}} & 0 \\ a & 0 & \epsilon \\ 1 & 0 & 0 \end{bmatrix}.$$

$J(x_2^*, y_2^*, z_2^*)$ has a characteristic equation of the form

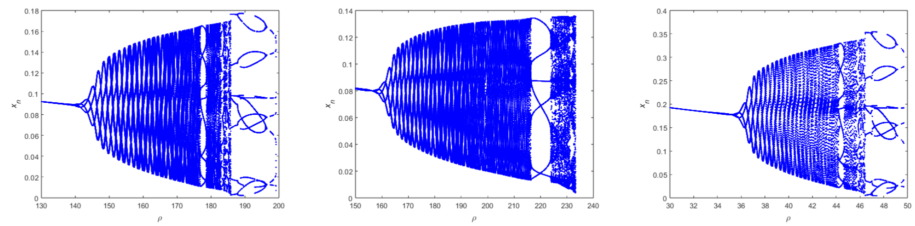
$$P(\lambda) \equiv \lambda^3 - (\frac{r\sqrt{\rho}(a+\epsilon)}{\sqrt{a+\epsilon}} + 1)\lambda^2 - \frac{ar\sqrt{\rho}}{\sqrt{a+\epsilon}}\lambda - \frac{\epsilon r\sqrt{\rho}}{\sqrt{a+\epsilon}} = 0.$$

Using the Jury test, the second condition is not satisfied and we find the following.

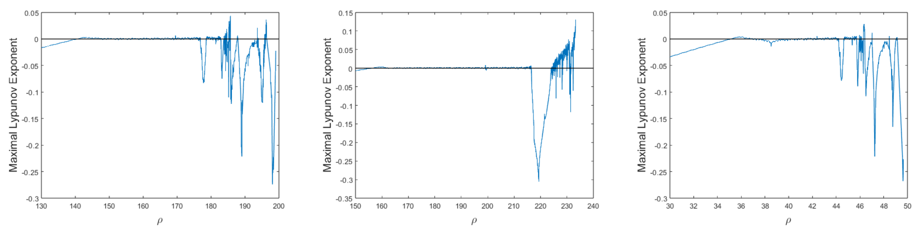
Proposition 2.3. *The fixed point* (x_2^*, y_2^*, z_2^*) *is always unstable.*

2.7. **Numerical simulations.** In this part, to validate our studies we use numerical experiments to draw out the theoretical results and show that changes in r, a, and ϵ affect the dynamical behavior of the dynamical system (2.11). We have been experimenting with different values of r, a and ϵ and then plotting bifurcation diagrams as a function of ρ. Moreover, the maximal Lyapunov exponent corresponding to each bifurcation diagram is introduced below it. In Figure (1a) we start with the initial point $(0.09, 0.08, 0.09)$ at $r = 0.1,\ a = 0.8,\ \epsilon = 0.1$ the system undergoes bifurcation at $\rho \simeq 140.625$. In Figure (1b) we start with the initial point $(0.08, 0.08, 0.08)$ at $r = 0.1,\ a = 0.8,\ \epsilon = 0.2$ the system undergoes bifurcation at $\rho \simeq 156.25$. In Figure (1c) we start with the initial point $(0.1778, 0.16, 0.1778)$ at $r = 0.2,\ a = 0.8,\ \epsilon = 0.1$ the bifurcation occurs in the system at $\rho \simeq 35.156$. Figure (1g) illustrates that the bifurcation occurs in the system at $\rho \simeq 123.457$ with initial point $(0.09, 0.09, 0.09)$ and $r = 0.1,\ a = 0.9,\ \epsilon = 0.1$. Figure (1h) illustrates that the system undergoes bifurcation at $\rho \simeq 100.2$ with initial point $(0.1, 0.1, 0.1)$ and $r = 0.1,\ a = 0.999,\ \epsilon = 0.001$. We noticed that when $a \to 1$ and $\epsilon \to 0$ the Riccati equation with perturbed delay (1.3) will be the Riccati differential equation (1.1) as shown in Figures (1g)-(1h).

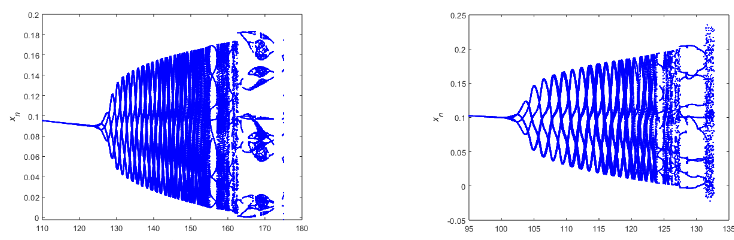
Also, we introduce some phase diagrams by taking $r = 0.1$, $a = 0.8$, $\epsilon = 0.2$, and initial point $= (0.08, 0.08, 0.08)$ as in Figure (2). Through the increase in the value of ρ, the curve rotates clockwise and a period-4 orbit appears and the Lyapunov exponent becomes positive, as shown in Figures (2a)-(2g). The curve turns into an oval with an increase in radius and the Lyapunov exponent changes between negative and positive as in Figures (2h)-(2o). In Figure (2p) the circular curve breaks down and a period-7 orbit appears and the Lyapunov exponent becomes negative again. The curve appears again as in Figure (2q) and the Lyapunov exponent becomes positive. In figure (2r) the curve breaks down again and a period-17 orbit appears and the Lyapunov exponent becomes positive again. Figures (2s)-(2t) show that the circular curve breaks down, appears again then disappears and the Lyapunov exponent changes between negative and positive.



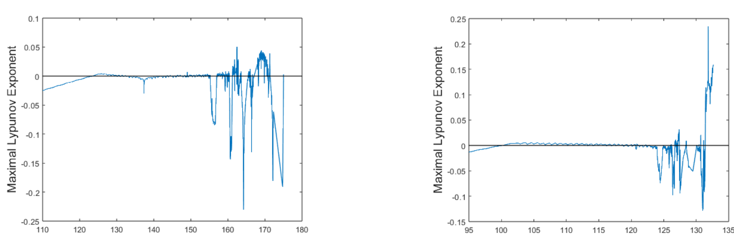
(A) $r = 0.1$, $a = 0.8$, $\epsilon = 0.1$ (B) $r = 0.1$, $a = 0.8$, $\epsilon = 0.2$ (C) $r = 0.2$, $a = 0.8$, $\epsilon = 0.1$



(D) $r = 0.1$, $a = 0.8$, $\epsilon = 0.1$ (E) $r = 0.1$, $a = 0.8$, $\epsilon = 0.2$ (F) $r = 0.2$, $a = 0.8$, $\epsilon = 0.1$

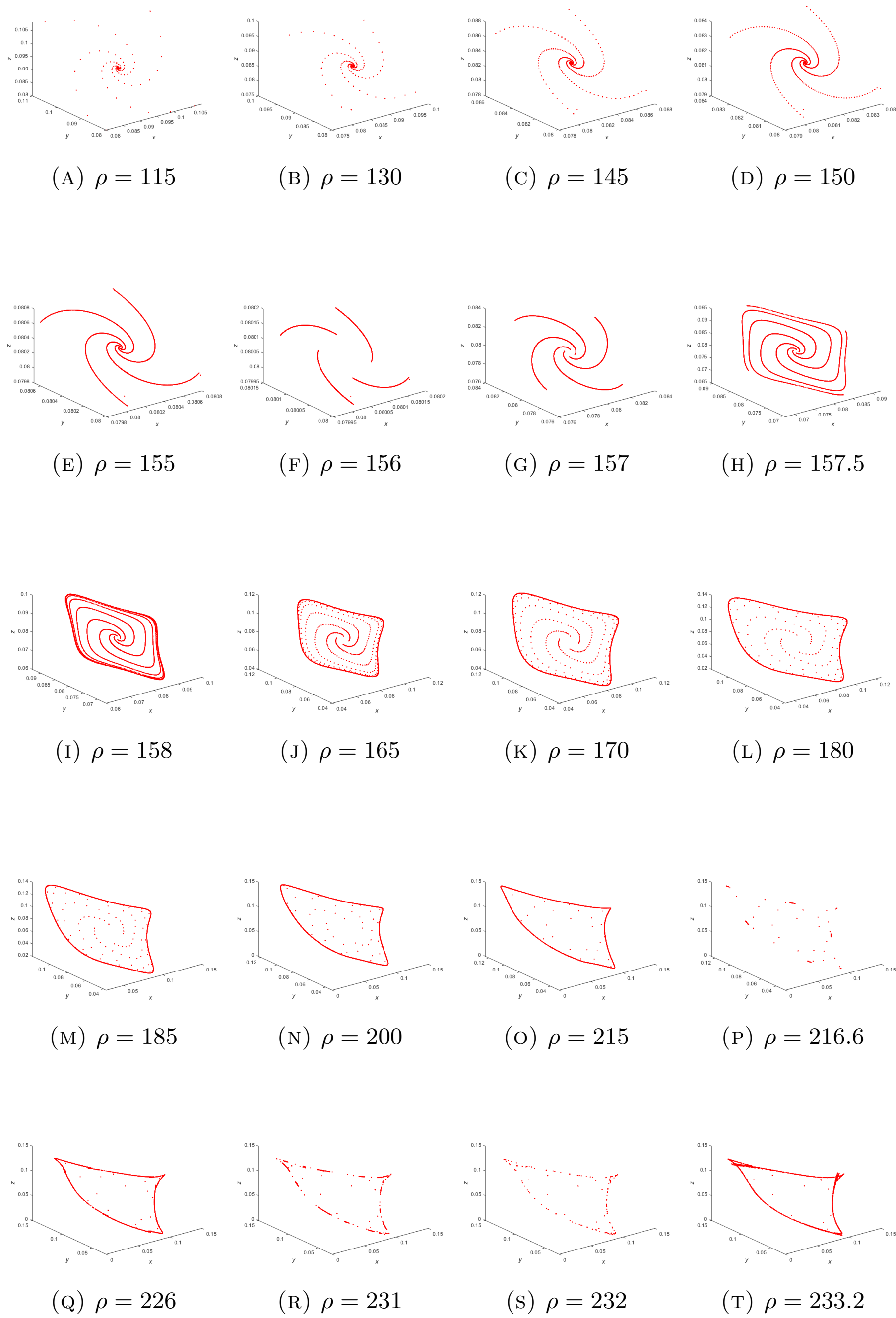


(G) $r = 0.1$, $a = 0.9$, $\epsilon = 0.1$ (H) $r = 0.1$, $a = 0.999$, $\epsilon = 0.001$



(I) $r = 0.1$, $a = 0.9$, $\epsilon = 0.1$ (J) $r = 0.1$, $a = 0.999$, $\epsilon = 0.001$

FIGURE 1. Bifurcation diagrams of a system (2.11) and its accompanying maximum Lyapunov exponent



(A) $\rho = 115$ (B) $\rho = 130$ (C) $\rho = 145$ (D) $\rho = 150$

(E) $\rho = 155$ (F) $\rho = 156$ (G) $\rho = 157$ (H) $\rho = 157.5$

(I) $\rho = 158$ (J) $\rho = 165$ (K) $\rho = 170$ (L) $\rho = 180$

(M) $\rho = 185$ (N) $\rho = 200$ (O) $\rho = 215$ (P) $\rho = 216.6$

(Q) $\rho = 226$ (R) $\rho = 231$ (S) $\rho = 232$ (T) $\rho = 233.2$

Figure 2. Phase diagrams of system (2.11) with varying values of ρ

3. Conclusion

In this study, a Riccati differential equation with a delayed perturbation was studied. We looked at both the solution's existence and its continuous dependence on the initial conditions. Analyses of Hopf bifurcations and fixed points' local stability were presented. We adopted a discretization procedure to solve the delayed differential equation with piecewise constant arguments. When looking at the discrete system, we ran an investigation of its local stability. We validated our results using numerical simulations that generated bifurcation diagrams, Lyapunov exponents, and phase diagrams to better understand the underlying complicated dynamics. We contrasted the results of theoretical studies of the delayed Riccati differential equation (1.1) and its perturbed equation (1.3). We found that the dynamical system is sensitive to shifts in r, a and that even a little perturbation may cause a significant shift in the system's chaotic behavior. Moreover, when $a \to 1$ and $\epsilon \to 0$ the Riccati equation with perturbed delay (1.3) is equivalent to the Riccati differential equation (1.1) with the same dynamical properties.

References

[1] P. N. V. Tu, Dynamical Systems, An Introduction with Applications in Economics and Biology, Springer Science & Business Media, (2012).

[2] R. A. Frantz, J. C. Loiseau, J. C. Robinet, Krylov methods for large-scale dynamical systems: Application in fluid dynamics. Applied Mechanics Reviews, (2023).

[3] S. H. Strogatz, Nonlinear dynamics and chaos with applications to physics, Biology, chemistry, and engineering, CRC press, (2018).

[4] C. Zhu, A novel image encryption scheme based on improved hyperchaotic sequences, Optics communications, 285 (2012), 29–37.

[5] A. M. A. El-Sayed, A. Elsaid, H. M. Nour, A. Elsonbaty, Dynamical behavior, chaos control and synchronization of a memristor-based ADVP circuit, Communications in Nonlinear Science and Numerical Simulation, (2012).

[6] S. Vaidyanathan, K. Rajagopal, LabVIEW implementation of chaotic masking with adaptively synchronised forced Van der Pol oscillators and its application in real-time image encryption, International Journal of Simulation and Process Modelling, 12(2) (2017), 165-78.

[7] L. Kocarev, S. Lian, Chaos-based cryptography, Springer, (2011).

[8] N. Yadav, S. Ravela, A. R. Ganguly, Machine learning for robust identification of complex nonlinear dynamical systems: applications to earth systems modeling. arXiv preprint arXiv:2008.05590, (2020).

[9] K. Gajamannage, D. I. Jayathilake, Y. Park, E. M. Bollt, Recurrent neural networks for dynamical systems: Applications to ordinary differential equations, collective motion, and hydrological modeling. Chaos: An Interdisciplinary Journal of Nonlinear Science, 33(1) (2023).

[10] W. T. Reid, Riccati Differential Equations. Mathematics in Science and Engineering, Academic Press, vol. 86 (1972).

[11] B. D. Anderson, J. B. Moore, Optimal Control-linear Quadratic Methods, Courier Corporation, (2007).

[12] G. Männel, M. Siebert, C. Brendle, P. Rostalski, Robust model predictive control of an anaesthesia workstation ventilation unit. at-Automatisierungstechnik, 68(11) (2020).

[13] P. Acquistapace, and F. Bucci, Uniqueness for Riccati equations with application to the optimal boundary control of composite systems of evolutionary partial differential equations. Annali di Matematica Pura ed Applicata, (2023).

[14] J. F. Carinena, G. Marmo, A. M. Perelomov, M. F. Z. Ranada, Related operators and exact solutions of Schrödinger equations, International Journal of Modern Physics A, 13(28) (1998), 4913–4929.

[15] A. M. A. El-Sayed, S. M. Salman, Dynamic behavior and chaos control in a complex Riccati-type map, Quaestions Mathematicae, 39(5) (2016), 665–681.

[16] R. D. Driver, Ordinary and delay differential equations, Springer Science & Business Media, Vol. 20, (2012).
[17] A. A. Elsadany, S. M. Salman, On the bifurcation of Marotto's map and its application in image encryption, Journal of Computational and Applied Mathematics, 328 (2018), 177-196.
[18] S. M. Salman, A. M. Yousef, A. A. Elsadany, Stability, bifurcation analysis and chaos control of a discrete predator-prey system with square root functional response Chaos, Solitons and Fractals, 93 (2016), 20-31.
[19] M. C. Mackey, L. Glass, Oscillations and Chaos in Physiological Control Systems, Science, 197 (1997), 287-289.
[20] A. M. A. EL-Sayed, S. M. Salman, A. M. A. Abo-Bakr, On the Dynamics of the Logistic Delay Differential Equation with Two Different Delays. Journal of Applied and Computational Mechanics, 7(2) (2021).
[21] N. Guglielmi, E. Iacomini, A. Viguerie, Delay differential equations for the spatially resolved simulation of epidemics with specific application to COVID-19. Mathematical Methods in the Applied Sciences, 45(8) (2022).
[22] C. Zhang, Q. Zhu, Exponential stability of random perturbation nonlinear delay systems with intermittent stochastic noise. Journal of the Franklin Institute, 360(2) (2023).
[23] A. M. A. EL-Sayed, S. M. Salman, A. M. A. Abo-Bakr, On the dynamics of a class of difference equations with continuous arguments and its singular perturbation. Alexandria Engineering Journal, 66 (2023).
[24] N. MacDonald, Two delays may not destabilize although either can delay, Mathematical Biosciences, 82(2) (2006), 127-140.
[25] J. K. Hale, E. F. Infante, F. S. Tsen, Stability in linear delay equations, Journal of Mathematical Analysis and Applications, 105(2) (1985), 533-555.
[26] A. M. A. El-Sayed, S. M. Salman, Chaos and bifurcation of discontinuous dynamical systems with piecewise constant arguments, Malaya Journal of Matematik, 1(1) (2012), 14-18.
[27] A. M. A. El-Sayed, S. M. Salman, Chaos and bifurcation of the Logistic discontinuous dynamical systems with piecewise constant arguments, Malaya Journal of Matematik, 3(1) (2013), 14-20.
[28] M. U. Akhmet, D. Altntana, T. Ergenc, Chaos of the logistic equation with piecewise constant arguments, arXiv preprint arXiv:1006.4753, (2010).
[29] A. M. A. El-Sayed, S. M. Salman, On a discretization process of fractional-order Riccati differential equation, J. Fract. Calc. Appl, 4(2) (2013), 251-259.
[30] K. Ogata, Discrete-time control systems, Prentice-Hall, Inc., (1995).

Ahmed M. A. El-Sayed
Faculty of Science, Alexandria University, Alexandria, Egypt
Email address: amasayed@alexu.edu.eg

Sanaa M. Salman
Faculty of Education, Alexandria University, Alexandria, Egypt
Email address: samastars9@alexu.edu.eg

AbdAllah A. F. AbdElfattah
Faculty of Science, Alexandria University, Alexandria, Egypt
Email address: abdallah.awad_pg@alexu.edu.eg