Frontiers in Scientific Research and Technology

Online ISSN: 2682-2970
Print ISSN: 2682-2962
http://fsrt.journals.ekb.eg

FSRT J 6 (2023) 1 - 19 10.21608/FSRT.2023.193518.1084

Relative Efficiency of the Maximum Product Spacing Estimates of the Power Topp-Leone Distribution Parameters under Progressive Type-II Censoring Scheme

Mohamed A. Ahmed

Department of Statistics, Mathematics and Insurance, Al Madina Higher Institute of Management and Technology, Giza, Egypt.

ARTICLE INFO

Article history:
Received 13 February 2023
Received in revised form 16 March 2023
Accepted 20 March 2023
Available online 30 March 2023

Keywords
The power Topp-Leone distribution,
maximum likelihood estimation,
maximum product spacing estimation,
Bayesian approach,
progressive Type-II censoring scheme,
Markov Chain Monte Carlo technique.

ABSTRACT

Recently, the maximum product spacing method has become one of the most efficient parameters estimation methods because of its estimates efficiency which retains most of the maximum likelihood estimation method properties including the invariance property. In this paper, relative efficiency measures will be used to compare the efficiency of the maximum product spacing method with the maximum likelihood estimation method using classical and Bayesian approaches for the power Topp-Leone distribution. The relative efficiency of the maximum product spacing method will be investigated using progressive Type-II censoring scheme. A simulation study and a real data set are performed to compare the efficiency between estimation methods.

Mathematics Subject Classification: 60E05. 62E10. 62N05

1. Introduction

Cheng and Amin (1983) presented for the first time the maximum product spacing (MPS) method which becomes one of the most efficient parameters estimation methods because its estimators have most of the maximum likelihood estimation (MLE) properties especially the invariance property, the Geometric mean (GM) for a parameter θ, Singh et al.(2014) and Bhatti et al. (2021), is represented by

$$GM = \sqrt[n+1]{\prod_{i=1}^{n+1}\left[F(\theta;x_{(i)})-F(\theta;x_{(i-1)})\right]};\ i=1,2,...,n+1, \quad (1)$$

where, $F(x;\theta)$ is the cumulative distribution function (*CDF*) and $F(\theta;x_{(0)})=0$, $F(\theta;x_{(n+1)})=1$, suppose that $D(\theta;x)=\prod_{i=1}^{n+1}\left[F(\theta;x_{(i)})-F(\theta;x_{(i-1)})\right]$, then

$$D(\theta;x)=\prod_{i=2}^{n}\left[F(\theta;x_{(i)})-F(\theta;x_{(i-1)})\right]\times\left[F(\theta;x_{(1)})\right]\times\left[1-F(\theta;x_{(n)})\right]. \quad (2)$$

Singh et al. (2014), explained that product spacing (*D*) is an alternative to likelihood (*L*) for Bayesian inference and it retains most of the MLE method properties including the invariance property.

A progressive Type-II censoring scheme is described by Ng et al. (2004) where it is supposed that n units be participated in a lifetime test experiment and the number of failures decided beforehand to be r units. In the first failure, $R_{(1)}$ units are removed randomly from the remaining survived units $n-1$. In the second failure, $R_{(2)}$ units are removed randomly from the remaining survived units $n-R_{(1)}-1$. Finally, In the r-th failure, all the remaining survived units $R_{(r)}$, where $R_{(r)}=n-r-R_{(1)}-\cdots-R_{(r-1)}$, are removed from the experiment. Hence, the progressive Type-II censoring scheme containing r and $_{(1)}, \ldots, R_{(r)}$, gives the rule $R_{(1)}+\cdots+R_{(r)}=n-r$. One can see Balakrishnan and Aggrawalla (2000) and Balakrishnan and Saleh (2017) for more details.

Topp and Leone (1955) presented the bounded Topp-Leone (TL) distribution for empirical data with a J-shaped histogram as a powered band tool and automatically calculating machine failures. The Topp-Leone distribution is studied by many authors as Nadarajah and Kotz (2003), Ghitany et al. (2005), Zhou et al. (2006), van Dorp and Kotz (2006), Kotz and Seier (2007), Nadarajah (2009) and Genç (2012).

The CDF and the probability density function (PDF) of the classical TL distribution, Nadarajah and Kotz (2003), are

$$F_{TL}(x;\alpha)=\left[x\,(2-x)\right]^{\alpha};0<x<1;\alpha>0,$$

and

$$f_{TL}(x;\alpha)=2\alpha x^{\alpha-1}(2-x)^{\alpha-1}(1-x).$$

* Corresponding author at Al Madina Higher Institute of Management and Technology

E-mail addresses: mrmohamedali2005@yahoo.com (Mohamed A. Ahmed)

Ahmed (2021) presented for the first time the power Topp-Leone (PTL) distribution, Elgarhy et al. (2022), it has the following CDF and probability density function (PDF)

$$F_{PTL}(x;\alpha,\beta)=x^{\alpha\beta}\left(2-x^{\beta}\right)^{\alpha};0<x<1;\alpha>0,\beta>0;\alpha\beta\neq 1,$$

and

$$f_{PTL}(x;\alpha,\beta)=2\alpha\beta x^{\alpha\beta-1}\left(1-x^{\beta}\right)\left(2-x^{\beta}\right)^{\alpha-1},$$

when *β=1,* the PTL distribution reduces to TL distribution, Topp and Leone (1955), some shapes of the density function for the PTL distribution are illustrated in Figure (1), Ahmed (2021).

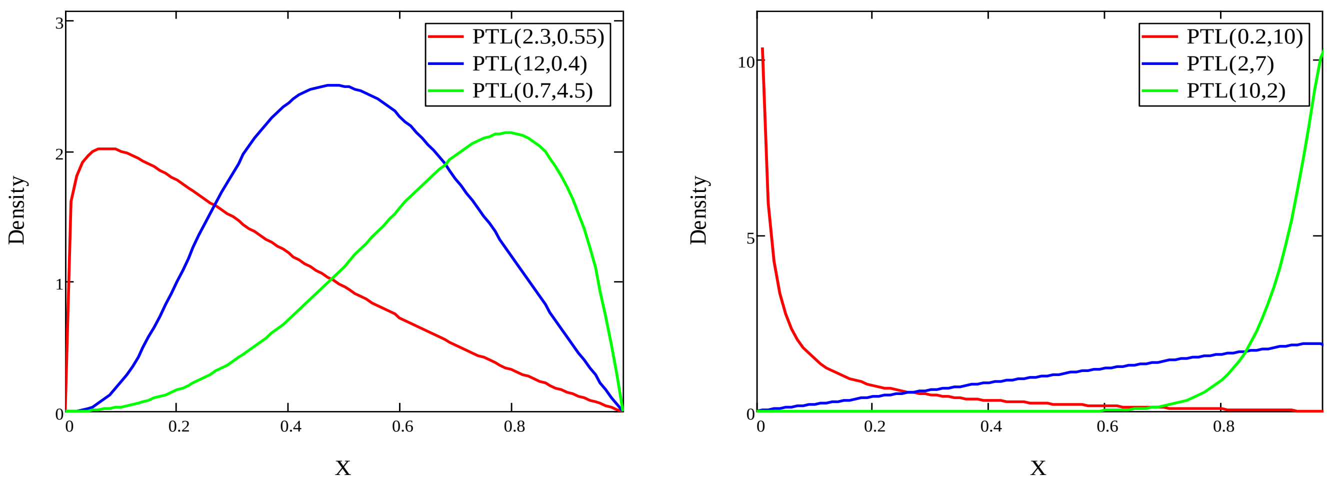

Figure (1): The PTL density functions

One can see, in Figure (1), the density function has unimodal with right-skewed, left-skewed and symmetric curves. Also, the density is suitable for the lifetime data having J, L and increasing linear shapes.

The main object of this manuscript is to investigate the efficiency of the MPS method using progressive Type-II censoring scheme for the PTL distribution via classical and Bayesian approaches using Markov Chain Monte Carlo (MCMC) technique.

The rest of this paper is organized as follows: In Section (2), a useful transformation for the PTL distribution is presented. In Section (3), the MLE estimation method under the progressive Type-II censoring scheme is investigated. In Section (4), the MPS estimation method under the progressive Type-II censoring scheme is used. In Section (5), the relative efficiency measures of MPS estimators are discussed. In Section (6), a simulation study is investigated between MPS and MLE methods of estimation for the PTL distribution in different cases. In Section (7), a real data set is applied. Finally, in Section (8), a conclusion about the study is provided.

2. A Useful Transformation for the PTL Distribution

The quantile function (QF) of the PTL distribution gives some problems in mathematical properties and generating random numbers because it has an implicit form, a simple transformation will be used to solve this problem and give an explicit form for the QF.

Completing the square will be used for CDF of the PTL distribution as follows:
Since,

$$F_{PTL}(x;\alpha,\beta)=\left[2x^{\beta}-x^{2\beta}\right]^{\alpha},$$

then, adding ±1 gives

$$F_{PTL}(x;\alpha,\beta)=\left[1-1+2x^{\beta}-x^{2\beta}\right]^{\alpha},$$

hence,

$$F_{PTL}(x;\alpha,\beta)=\left[1-\left(1-x^{\beta}\right)^{2}\right]^{\alpha};0<x<1;\alpha,\beta>0, \quad (3)$$

differentiating the last equation with respect to *x* yields

$$f_{PTL}(x;\alpha,\beta)=2\alpha\beta x^{\beta-1}\left(1-x^{\beta}\right)\left\{1-\left(1-x^{\beta}\right)^{2}\right\}^{\alpha-1}, \quad (4)$$

the QF of the PTL distribution is given by

$$x_q=\left\{1-\left[1-q^{\frac{1}{\alpha}}\right]^{\frac{1}{2}}\right\}^{\frac{1}{\beta}}. \quad (5)$$

One can see, easily, the simple and explicit forms of the CDF, PDF and QF of the PTL distribution in (3), (4) and (5) which give flexibility for using different methods of parameter estimation.

3. The MLE Estimation Method under Censored Sample

In this section, the MLE estimation method will be investigated under a progressive Type-II censoring scheme via classical and Bayesian techniques.

3.1. The Censored MLE Method

Let $X_{(1)}$, $X_{(2)}$, …, $X_{(r)}$ be the ordered observed failures in a random sample from n components with $R_{(i)}$ removal for the *PTL(α,β;x)* distribution, the likelihood function for parameters α, β using the progressive Type-II (*PTII*) censoring Scheme $R_{(1)}, R_{(2)},\dots,R_{(r)}$, Balakrishnan and Aggrawalla (2000), Balakrishnan and Cramer (2014), is given by

$$L_{PTII}\left(\alpha,\beta;x\right)=A\prod_{i=1}^{r} f_{PTL}\left(\alpha,\beta;x_{(i)}\right)\prod_{i=1}^{r}\left[1-F_{PTL}\left(\alpha,\beta;x_{(i)}\right)\right]^{R_{(i)}}, \qquad (6)$$

where $A=n\left(n-R_{(1)}-1\right)\dots\left(n-\sum_{i=1}^{r-1}R_{(i)}-\left(r-1\right)\right)$ is a constant which does not depend on the parameters, substituting (3) and (4) into (6) gives

$$L_{PTII}\left(\alpha,\beta;x\right)=A\prod_{i=1}^{r}2\alpha\beta x_{(i)}^{\beta-1}\left(1-x_{(i)}^{\beta}\right)\left\{1-\left(1-x_{(i)}^{\beta}\right)^{2}\right\}^{\alpha-1}\prod_{i=1}^{r}\left[1-\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}\right]^{R_{(i)}}, \qquad (7)$$

the log likelihood function can be written as

$$\begin{aligned}\ell_{PTII}\left(\alpha,\beta;x\right)=&\log\left(A\right)+n\log\left(2\alpha\beta\right)+\left(\beta-1\right)\sum_{i=1}^{r}\log x_{(i)}+\sum_{i=1}^{n}\log\left(1-x_{(i)}^{\beta}\right)\\&+\left(\alpha-1\right)\sum_{i=1}^{r}\log\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]+\sum_{i=1}^{r}R_{(i)}\log\left\{1-\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}\right\}.\end{aligned}$$

The score functions for the parameters α and β are given by

$$\frac{\partial\ell_{PTII}\left(\alpha,\beta;x\right)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{r}\log\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]-\sum_{i=1}^{r}R_{(i)}\frac{\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}\log\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]}{1-\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}}, \qquad (8)$$

and

$$\begin{aligned}\frac{\partial\ell_{PTII}\left(\alpha,\beta;x\right)}{\partial\beta}=&\frac{n}{\beta}+\sum_{i=1}^{r}\log\left(x_{(i)}\right)-\sum_{i=1}^{r}\frac{x_{(i)}^{\beta}\log\left(x_{(i)}\right)}{1-x_{(i)}^{\beta}}+2\left(\alpha-1\right)\sum_{i=1}^{r}\frac{x_{(i)}^{\beta}\left[1-x_{(i)}^{\beta}\right]\log\left(x_{(i)}\right)}{1-\left(1-x_{(i)}^{\beta}\right)^{2}}\\&-\sum_{i=1}^{r}R_{(i)}\frac{2\alpha x_{(i)}^{\beta}\left(1-x_{(i)}^{\beta}\right)\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha-1}\log\left(x_{(i)}\right)}{1-\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}}.\end{aligned} \qquad (9)$$

The unknown parameters of the MLE are estimated numerically by solving the nonlinear Equations (8) and (9) using a suitable iterative technique such as the Newton–Raphson algorithm.

Obviously when $r=n$, the censored sample becomes a complete sample, Equation (6) can be reduced as follows

$$L\left(\alpha,\beta;x\right)=\prod_{i=1}^{n} f_{PTL}\left(\alpha,\beta;x\right),$$

estimating parameters via last equation gives estimates under a complete sample.

3.2. Bayesian Approach Based on the Censored MLE Method

In this subsection, the non-informative prior distributions for the parameters α and β respectively will be used, there is not enough information about the distribution of the parameters, Jeffreys (1998), Singh et al. (2013) and Chandra and Rathaur (2017), as follows

$$\pi(\alpha)=\frac{1}{\alpha};0<\alpha<a, \tag{10}$$

and

$$\pi(\beta)=\frac{1}{\beta};0<\beta<b, \tag{11}$$

In this section, the joint posterior distribution can be given by substituting (10) , (11) and (7) into the next equation

$$\pi(\alpha,\beta;x)=\frac{L_{PTII}\left(\alpha,\beta;x\right)\pi(\alpha)\pi(\beta)}{\int_0^b\int_0^a L_{PTII}\left(\alpha,\beta;x\right)\pi(\alpha)\pi(\beta)d\alpha d\beta};0<\alpha<a;0<\beta<b;\ 0<x<1.$$

The marginal posterior distribution of α and β can be given by, respectively,

$$\pi(\alpha;x)=\int_0^b\pi(\alpha,\beta;x)d\beta;0<\beta<b;0<x<1, \tag{12}$$

and

$$\pi(\beta;x)=\int_0^a\pi(\alpha,\beta;x)d\alpha;0<\alpha<a;0<x<1, \tag{13}$$

estimating α and β can be obtained using the SE loss function or LINEX loss function.

3.2.1. The SE Loss Function

In this subsection, estimation of the marginal posterior distributions will be performed using the quadratic loss function or the SE loss function which is a symmetric loss function given by substituting (12) and (13) into the following equations, Guure et al.(2012),

$$E_{SE}(\alpha;x)=\int_0^a\alpha\,\pi(\alpha;x)d\alpha;0<\alpha<a;0<x<1, \tag{14}$$

and

$$E_{SE}(\beta;x)=\int_0^b\beta\ \pi(\beta;x)d\beta;0<\beta<b;0<x<1. \tag{15}$$

The unknown parameters estimators of the Bayesian approach using integrations in (14) and (15) are not possible to be obtained numerically so the MCMC technique will be used.

3.2.2. The LINEX Loss Function

In this subsection, estimation of the marginal posterior distributions will be performed using the LINEX loss function which is an asymmetric loss function given by substituting (12) and (13) into the following equations, Guure *et al*.(2012),

$$E_{LINEX}(\alpha;x)=\frac{-1}{h}\ln\left[\int_0^a e^{-h\alpha}\ \pi(\alpha;x)d\alpha\right];0<\alpha<a;0<x<1, \tag{16}$$

and

$$E_{LINEX}(\beta;x)=\frac{-1}{h}\ln\left[\int_0^b e^{-h\beta}\ \pi(\beta;x)d\beta\right];0<\beta<b;\ 0<x<1. \tag{17}$$

The unknown parameters of the Bayesian technique via integrations in (16) and (17) cannot be estimated numerically therefore the MCMC technique will be used.

3.3. The MCMC Technique

In this subsection, the MCMC method will be discussed using the Gibbs sampling procedure in order to generate a sample from conditional posterior densities for the parameters α and β, for more details about the MCMC technique one can see Gelfand and Smith (1990) and Singh et al. (2013). The conditional posterior densities of the parameters α and β are given respectively by

$$\pi(\alpha|\beta;x) \propto \alpha^{r-1}\prod_{i=1}^{r}\left\{1-\left(1-x_{(i)}^{\beta}\right)^{2}\right\}^{\alpha-1}\prod_{i=1}^{r}\left[1-\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}\right]^{R_{(i)}}, \qquad (18)$$

and

$$\pi(\beta|\alpha;x) \propto \beta^{r-1}\prod_{i=1}^{r}x_{(i)}^{\beta-1}\left(1-x_{(i)}^{\beta}\right)\left\{1-\left(1-x_{(i)}^{\beta}\right)^{2}\right\}^{\alpha-1}\prod_{i=1}^{r}\left[1-\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}\right]^{R_{(i)}}. \qquad (19)$$

The Bayes estimates of the parameter α and β under SE loss function respectively are

$$E_{SE}\left(\pi(\alpha|\beta;x)\right)=\frac{1}{N}\sum_{j=1}^{N}\pi_{j}(\alpha|\beta;x), \qquad (20)$$

and

$$E_{SE}\left(\pi(\beta|\alpha;x)\right)=\frac{1}{N}\sum_{j=1}^{N}\pi_{j}(\beta|\alpha;x), \qquad (21)$$

where N is the number of iteration in the MCMC process, the Bayes estimates of the parameter α and β under LINEX loss function respectively are:

$$E_{LINEX}\left(\pi(\alpha|\beta;x)\right)=\frac{-1}{h}\ln\left(\frac{1}{N}\sum_{j=1}^{N}e^{-h\,\pi_{j}(\alpha|\beta;x)}\right), \qquad (22)$$

and

$$E_{LINEX}\left(\pi(\beta|\alpha;x)\right)=\frac{-1}{h}\ln\left(\frac{1}{N}\sum_{j=1}^{N}e^{-h\,\pi_{j}(\beta|\alpha;x)}\right), \qquad (23)$$

where N is the number of iteration in the MCMC process.

4. The MPS Estimation Method under Censored Sample

In this section, the MPS estimation method will be applied under a progressive Type-II censoring scheme via classical and Bayesian techniques.

4.1 The Censored MPS Method

Let $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ be the ordered observed failures in a random sample from n components with $R_{(i)}$ removal for the *PTL(α,β;x)* distribution, the MPS for parameters α,β using the progressive Type-II censoring scheme $R_{(1)}, R_{(2)},\ldots,R_{(r)}$, Ng et al. (2012), Almetwally and Almongy (2019), is given by

$$D_{PTII}\left(\alpha,\beta;x\right)=A\prod_{i=1}^{r+1}\left[F_{PTL}\left(\alpha,\beta;x_{(i)}\right)-F_{PTL}\left(\alpha,\beta;x_{(i-1)}\right)\right]\prod_{i=1}^{r}\left[1-F_{PTL}\left(\alpha,\beta;x_{(i)}\right)\right]^{R_{(i)}}, \qquad (24)$$

where $A=n\left(n-R_{(1)}-1\right)\ldots\left(n-\sum_{i=1}^{r-1}R_{(i)}-(r-1)\right)$ is a constant which does not depend on the parameters, using (3), (2) and (24) give

$$D_{PTII}(\alpha,\beta;x)=A\prod_{i=2}^{r}\left\{\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}-\left[1-\left(1-x_{(i-1)}^{\beta}\right)^{2}\right]^{\alpha}\right\}\left[1-\left(1-x_{(1)}^{\beta}\right)^{2}\right]^{\alpha}$$
$$\left\{1-\left[1-\left(1-x_{(r)}^{\beta}\right)^{2}\right]^{\alpha}\right\}\prod_{i=1}^{r}\left[1-\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}\right]^{R_{(i)}}, \tag{25}$$

taking logarithm of (25) gives

$$\log\left[D_{PTII}(\alpha,\beta;x)\right]=\log A+\sum_{i=2}^{r}\log\left[\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}-\left[1-\left(1-x_{(i-1)}^{\beta}\right)^{2}\right]^{\alpha}\right]+\alpha\log\left[1-\left(1-x_{(1)}^{\beta}\right)^{2}\right]$$
$$+\log\left[1-\left[1-\left(1-x_{(r)}^{\beta}\right)^{2}\right]^{\alpha}\right]+\sum_{i=1}^{r}R_{(i)}\log\left\{1-\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}\right\}.$$

The score functions for the parameters α and β are given by

$$\frac{\partial\log\left[D_{PTII}(\alpha,\beta;x)\right]}{\partial\alpha}=\sum_{i=2}^{n}\left\{\frac{\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}\log\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]}{\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}-\left[1-\left(1-x_{(i-1)}^{\beta}\right)^{2}\right]^{\alpha}}\right.$$
$$\left.-\frac{\left[1-\left(1-x_{(i-1)}^{\beta}\right)^{2}\right]^{\alpha}\log\left[1-\left(1-x_{(i-1)}^{\beta}\right)^{2}\right]}{\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}-\left[1-\left(1-x_{(i-1)}^{\beta}\right)^{2}\right]^{\alpha}}\right\}+\log\left[1-\left(1-x_{(1)}^{\beta}\right)^{2}\right] \tag{26}$$
$$-\frac{\left[1-\left(1-x_{(r)}^{\beta}\right)^{2}\right]^{\alpha}\log\left[1-\left(1-x_{(r)}^{\beta}\right)^{2}\right]}{1-\left[1-\left(1-x_{(r)}^{\beta}\right)^{2}\right]^{\alpha}}-\sum_{i=1}^{r}R_{(i)}\frac{\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}\log\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]}{1-\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}},$$

and

$$\frac{\partial\log\left[D_{PTII}(\alpha,\beta;x)\right]}{\partial\beta}=2\alpha\sum_{i=2}^{r}\left\{\frac{x_{i}^{\beta}\left(1-x_{(i)}^{\beta}\right)\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha-1}\log(x_{(i)})}{\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}-\left[1-\left(1-x_{(i-1)}^{\beta}\right)^{2}\right]^{\alpha}}\right.$$
$$\left.-\frac{x_{(i-1)}^{\beta}\left(1-x_{(i-1)}^{\beta}\right)\left[1-\left(1-x_{(i-1)}^{\beta}\right)^{2}\right]^{\alpha-1}\log(x_{(i-1)})}{\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}-\left[1-\left(1-x_{(i-1)}^{\beta}\right)^{2}\right]^{\alpha}}\right\}+\frac{2\alpha x_{(1)}^{\beta}\left(1-x_{(1)}^{\beta}\right)\log(x_{(1)})}{1-\left(1-x_{(1)}^{\beta}\right)^{2}}$$
$$-\frac{2\alpha x_{(r)}^{\beta}\left(1-x_{(r)}^{\beta}\right)\left[1-\left(1-x_{(r)}^{\beta}\right)^{2}\right]^{\alpha-1}\log(x_{(r)})}{1-\left[1-\left(1-x_{(r)}^{\beta}\right)^{2}\right]^{\alpha}}$$
$$-\sum_{i=1}^{r}R_{(i)}\frac{2\alpha x_{(i)}^{\beta}\left(1-x_{(i)}^{\beta}\right)\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha-1}\log\left(x_{(i)}\right)}{1-\left[1-\left(1-x_{(i)}^{\beta}\right)^{2}\right]^{\alpha}}. \tag{27}$$

The unknown parameters of the MPS are estimated by solving the nonlinear Equations (26) and (27), numerically, using a suitable iterative technique.

Obviously when *r=n* , the censored sample becomes a complete sample, Equation (24) can be reduced as follows

$$D(\alpha,\beta;x)=\prod_{i=1}^{n+1}\left[F_{PTL}\left(\alpha,\beta;x_{(i)}\right)-F_{PTL}\left(\alpha,\beta;x_{(i-1)}\right)\right],$$

estimating parameters via last equation gives estimates under a complete sample.

4.2. Bayesian Approach Based on the Censored MPS Method

The joint posterior distribution can be given by substituting (10) , (11) and (25) into the next equation

$$\pi^*(\alpha,\beta;x)=\frac{D_{PTII}\left(\alpha,\beta;x\right)\pi(\alpha)\pi(\beta)}{\int_0^b\int_0^a D_{PTII}\left(\alpha,\beta;x\right)\pi(\alpha)\pi(\beta)d\alpha d\beta};0<\alpha<a;0<\beta<b;\ 0<x<1.$$

The marginal posterior distribution of α and β respectively can be given by

$$\pi^*(\alpha;x)=\int_0^b\pi^*(\alpha,\beta;x)d\beta;0<\beta<b;0<x<1, \tag{28}$$

and

$$\pi^*(\beta;x)=\int_0^a\pi^*(\alpha,\beta;x)d\alpha;0<\alpha<a;0<x<1, \tag{29}$$

estimating α and β can be performed using the SE loss function or LINEX loss function.

4.2.1. The SE Loss Function

Estimation of the marginal posterior distributions will be performed using the SE loss function, or the quadratic loss function given which is a symmetric loss function, substituting (28) and (29) into the following equations

$$E_{SE}(\alpha;x)=\int_0^a\alpha\,\pi^*(\alpha;x)d\alpha;0<\alpha<a;0<x<1, \tag{30}$$

and

$$E_{SE}(\beta;x)=\int_0^b\beta\ \pi^*(\beta;x)d\beta;0<\beta<b;0<x<1. \tag{31}$$

The unknown parameters estimators of the Bayesian technique using integrations in (30) and (31) cannot be obtained numerically so the MCMC technique will be used.

4.2.2. The LINEX Loss Function

Estimation of the marginal posterior distributions will be performed using the LINEX loss function given by substituting (28) and (29) into the following equations

$$E_{LINEX}(\alpha;x)=\frac{-1}{h}\ln\left[\int_0^a e^{-h\alpha}\ \pi^*(\alpha;x)d\alpha\right];0<\alpha<a;0<x<1, \tag{32}$$

and

$$E_{LINEX}(\beta;x)=\frac{-1}{h}\ln\left[\int_0^b e^{-h\beta}\ \pi^*(\beta;x)d\beta\right];0<\beta<b;\ 0<x<1. \tag{33}$$

The unknown parameters of the Bayesian approach via integrations in (32) and (33) cannot be estimated numerically therefore the MCMC technique will be used.

4.3. The MCMC Technique

In this subsection, the MCMC technique will be discussed using the Gibbs sampling procedure. The conditional posterior densities of the parameters α and β are given respectively by:

$$\pi^*(\alpha|\beta;x) \propto \frac{1}{\alpha}\prod_{i=2}^{r}\left\{\left[1-\left(1-x_{(i)}^{\beta}\right)^2\right]^{\alpha}-\left[1-\left(1-x_{(i-1)}^{\beta}\right)^2\right]^{\alpha}\right\}\left[1-\left(1-x_{(1)}^{\beta}\right)^2\right]^{\alpha}$$
$$\times\left\{1-\left[1-\left(1-x_{(r)}^{\beta}\right)^2\right]^{\alpha}\right\}\prod_{i=1}^{r}\left[1-\left[1-\left(1-x_{(i)}^{\beta}\right)^2\right]^{\alpha}\right]^{R(i)}, \tag{34}$$

and

$$\pi^*(\alpha|\beta;x) \propto \frac{1}{\beta}\prod_{i=2}^{r}\left\{\left[1-\left(1-x_{(i)}^{\beta}\right)^2\right]^{\alpha}-\left[1-\left(1-x_{(i-1)}^{\beta}\right)^2\right]^{\alpha}\right\}\left[1-\left(1-x_{(1)}^{\beta}\right)^2\right]^{\alpha}$$
$$\times\left\{1-\left[1-\left(1-x_{(r)}^{\beta}\right)^2\right]^{\alpha}\right\}\prod_{i=1}^{r}\left[1-\left[1-\left(1-x_{(i)}^{\beta}\right)^2\right]^{\alpha}\right]^{R_{(i)}}. \tag{35}$$

The Bayes estimates of the parameter α and β under SE loss function respectively are:

$$E_{SE}\left(\pi^*(\alpha|\beta;x)\right)=\frac{1}{N}\sum_{j=1}^{N}\pi_j^*(\alpha|\beta;x), \tag{36}$$

and

$$E_{SE}\left(\pi^*(\beta|\alpha;x)\right)=\frac{1}{N}\sum_{j=1}^{N}\pi_j^*(\beta|\alpha;x), \tag{37}$$

where *N* is the number of iterations in the MCMC process, the Bayes estimates of the parameter α and β under LINEX loss function respectively are:

$$E_{LINEX}\left(\pi^*(\alpha|\beta;x)\right)=\frac{-1}{h}\ln\left(\frac{1}{N}\sum_{j=1}^{N}e^{-h\,\pi_j^*(\alpha|\beta;x)}\right), \tag{38}$$

and

$$E_{LINEX}\left(\pi^*(\beta|\alpha;x)\right)=\frac{-1}{h}\ln\left(\frac{1}{N}\sum_{j=1}^{N}e^{-h\,\pi_j^*(\beta|\alpha;x)}\right), \tag{39}$$

where *N* is the number of iterations in the MCMC process.

5. The Relative Efficiency of the MPS Estimators

In this section, some relative efficiency (RE) measures, Vasudeva (1991), will be given in order to discuss the efficiency of MPS compared to other methods. Determining the root of mean square error (RMSE) ratio of the classical MLE to the classical MPS, for any parameter θ, under a complete sample can be given by

$$RE_1=\frac{\text{RMSE of MLE (Complete) for }\theta}{\text{RMSE of MPS (Complete) for }\theta}, \tag{40}$$

one can see that if $RE_1 > 1$ it means that the MPS method gives more efficiency than the MLE method. The RMSE ratio of the classical MLE to the Bayesian MLE using the SE loss function, for any parameter θ, under the complete sample can be represented by

$$RE_2=\frac{\text{RMSE of MLE (Complete) for }\theta}{\text{RMSE of MLE (Complete) using Bayesian (SE) for }\theta}, \tag{41}$$

the RMSE ratio of the classical MLE to the Bayesian MPS using the SE loss function, for any parameter θ, under the complete sample can be represented by

$$RE_3=\frac{\text{RMSE of MLE (Complete) for }\theta}{\text{RMSE of MPS (Complete) using Bayesian (SE) for }\theta}, \tag{42}$$

The RMSE ratio of the classical MLE to the Bayesian MLE using the LINEX loss function, for any parameter θ, under the complete sample can be written as

$$RE_4=\frac{\text{RMSE of MLE (Complete) for }\theta}{\text{RMSE of MLE (Complete) using Bayesian (LINEX) for }\theta}, \tag{43}$$

the RMSE ratio of the classical MLE to the Bayesian MPS using the LINEX loss function, for any parameter θ, under the complete sample can be given by

$$RE_5 = \frac{\text{RMSE of MLE (Complete) for } \theta}{\text{RMSE of MPS (Complete) Bayesian (LINEX) for } \theta}, \tag{44}$$

the RMSE ratio of the classical MLE to the classical MPS, for any parameter θ, under a progressive Type-II censoring scheme can be written as

$$RE_6 = \frac{\text{RMSE of MLE (Censored) for } \theta}{\text{RMSE of MPS (Censored) for } \theta}, \tag{45}$$

the RMSE ratio of the classical MLE to the Bayesian MLE using SE loss function, for any parameter θ, under progressive Type-II censoring scheme can be represented by

$$RE_7 = \frac{\text{RMSE of MLE (Censored) for } \theta}{\text{RMSE of MLE (Censored) Bayesian (SE) for } \theta}, \tag{46}$$

the RMSE ratio of the classical MLE to the Bayesian MPS using SE loss function, for any parameter θ, under progressive Type-II censoring scheme can be written as

$$RE_8 = \frac{\text{RMSE of MLE (Censored) for } \theta}{\text{RMSE of MPS (Censored) Bayesian (SE) for } \theta}, \tag{47}$$

the RMSE ratio of the classical MLE to the Bayesian MLE using LINEX loss function, for any parameter θ, under progressive Type-II censoring scheme can be given by

$$RE_9 = \frac{\text{RMSE of MLE (Censored) for } \theta}{\text{RMSE of MLE (Censored) Bayesian (LINEX) for } \theta}, \tag{48}$$

the RMSE ratio of the classical MLE to the Bayesian MPS using LINEX loss function, for any parameter θ, under progressive Type-II censoring scheme can be represented by

$$RE_{10} = \frac{\text{RMSE of MLE (Censored) for } \theta}{\text{RMSE of MPS (Censored) Bayesian (LINEX) for } \theta}. \tag{49}$$

Equations (40) to (49) will be used in a simulation study, in the next section, to determine the relative efficiency of MPS compared to the MLE method.

6. A Simulation Study

In this study, the efficiency of maximum product spacing for estimators of the PTL distribution parameters will be illustrated relative to maximum likelihood estimators in Bayesian and non-Bayesian approaches under complete and progressive Type-II censoring samples. The simulation study will be performed using random numbers generated with fixed seeds. Bias and RMSE for a parameter θ will be given by: $\text{Bias}\left(\hat{\theta},\theta\right) = E\left(\hat{\theta}\right) - \theta,$

and $\text{RMSE}\left(\hat{\theta}\right) = \sqrt{\text{Var}\left(\hat{\theta}\right) + \left[\text{Bias}\left(\hat{\theta},\theta\right)\right]^2}$, total bias and total RMSE for parameters θ_1, θ_2 are given by: $\text{Total Bias} = \sqrt{\left[\text{Bias}\left(\hat{\theta}_1,\theta_1\right)\right]^2 + \left[\text{Bias}\left(\hat{\theta}_2,\theta_2\right)\right]^2},$ and $\text{Total RMSE} = \sqrt{\left[\text{RMSE}\left(\hat{\theta}_1\right)\right]^2 + \left[\text{RMSE}\left(\hat{\theta}_2\right)\right]^2}.$

6.1. Under Censored Sample (Non-Bayesian)

In this subsection, the algorithm for MLE and MPS methods under a progressive Type-II censoring scheme using a non-Bayesian approach will be illustrated in the following steps:

Step (1): Generating an ordered random sample $X_{(1)}, X_{(2)}, \ldots, X_{(r)}$ of sizes $r=(5,10,15,25,50,150)$ where r represents failures for $n=(10,20,30,50,100,300)$ respectively from the PTL distribution with fixed removals $R_{(1)}, R_{(2)}, \ldots, R_{(r-1)}$ and $R_{(r)} = n - r - R_{(1)} - \cdots - R_{(r-1)}$ using fixed seeds, one can see that if $r=n$ it is reduced to the complete sample case.

Step (2): Using a set of values of parameters as: $(\alpha=3, \beta=2)$.

Step (3): Solving normal equations of estimators for every method independently as follows:

In the MLE method under progressive Type-II censoring scheme: Solve (8) and (9), in the MPS method under progressive Type-II: Solve (26) and (27).

Step (4): Calculating biases, MLEs and RMSEs of the PTL distribution for every method independently.

Step (5): Calculating the relative efficiency measure from (45) in the censored sample case, the relative efficiency measure can be calculated from (40) in the complete sample case.

Step (6): Repeating Step (1) to Step (4) 10000 times.

The simulation results are indicated in appendix I and II, Tables: (3), (4), (9) and (10), Figures: (6), (9), (12) and (15), for MLE and MPS methods under progressive Type-II censoring scheme using the non-Bayesian approach. As sample size increases, biases and RMSEs decrease, moreover, when the sample size increases, the consistency of estimators' increases. On the other hand, as sample size increases, the relative efficiency measures of the MPS method decrease depending on the convergence of the estimators for both methods to the true value of parameters which makes RMSEs decrease.

One can see that the best efficient estimation method, according to biases and RMSEs, is the MPS method. On the other hand, relative efficiency measures of MPS estimators reflect that the MPS estimators' performance is better than the MLE estimators' performance through different sample sizes.

6.2. Under Censored Sample (Bayesian)

In this subsection, the algorithm for MLE and MPS methods under progressive Type-II censoring scheme using the Bayesian approach with MCMC technique, via the Gibbs sampling procedure, will be illustrated in the following steps:

Step (1): Generating an ordered random sample $X_{(1)},X_{(2)},\ldots,X_{(r)}$ of sizes $r=(5,10,15,25,50,150)$ where r represents failures for $n=(10,20,30,50,100,300)$ respectively from the PTL distribution with fixed removals $R_{(1)}, R_{(2)},\ldots,R_{(r-1)}$ and $R_{(r)} = n - r - R_{(1)} - \cdot\cdot\cdot - R_{(r-1)}$ using fixed seeds, one can see that if $r=n$ it is reduced to the complete sample case.

Step (2): Using a set values of parameters as: $(\alpha=3, \beta=2)$.

Step (3): Generating posterior for α and β for every method independently as follows:

In the MLE method using MCMC under progressive Type-II censoring scheme: Generate posterior for α and β from (18) and (19) respectively where the Bayes estimate of the parameters under SE loss function is given by (20) and (21), the Bayes estimate of the parameters under LINEX loss function is given by (22) and (23).

In the MPS method using MCMC with complete sample: Generate posterior for α and β from (34) and (35) respectively where the Bayes estimate of the parameters under SE loss function is given by (36) and (37), the Bayes estimate of the parameters under LINEX loss function is given by (38) and (39).

Step (4): Calculating biases, MLEs and RMSEs of the PTL distribution for every method independently.

Step (5): Calculating the relative efficiency measures from (46) to (49), the relative efficiency measures can be calculated from (41) to (44) in the complete sample case.

Step (6): Repeating Step (1) to Step (4) 10000 times.

The study results are indicated in appendix I and II, Tables: from (5) to (8) and from (11) to (14), Figures: (7), (8), (10), (11), (13), (14), (16) and (17) for the MLE and MPS methods under progressive Type-II censoring scheme using Bayesian approach with MCMC technique. As sample size increases, biases and RMSEs decrease. Moreover, when sample size increases, the consistency of estimators increases. On the other hand, as sample size increases, the relative efficiency measures decrease depending on the convergence of the estimators for both methods to the true value of parameters which makes RMSEs decrease.

One can see that, the Bayesian approach gives the MPS and MLE estimation methods under progressive Type-II censoring scheme more efficiency than classical methods, according to biases and RMSEs, especially the MPS method. On the other hand, relative efficiency measures reflect that the Bayesian MPS estimators' performance is better than the Bayesian MLE estimators' performance through different sample sizes.

Moreover, it is clear that, using the Bayesian approach in estimation methods under the progressive Type-II censoring scheme with LINEX loss function gives, according to biases and RMSEs, more efficient estimators than the SE loss function estimators.

7. Application

In this section, the MPS efficiency will be investigated practically using a real data set, where Ahmed (2021) provided a real dataset following the PTL (*2.286, 4.328*) distribution, the dataset represents the lifetime (Hours) of classical lamps for 50 devices as follows: 0.913, 0.786, 0.860, 0.904, 0.971, 0.616, 0.961, 0.789, 0.817, 0.722, 0.956, 0.835, 0.853, 0.692, 0.850, 0.677, 0.898, 0.965, 0.820, 0.964, 0.865, 0.947, 0.798, 0.746, 0.926, 0.709, 0.615, 0.747, 0.931, 0.913, 0.895, 0.745, 0.839, 0.766, 0.690, 0.531, 0.838, 0.846, 0.876, 0.817, 0.719, 0.907, 0.915, 0.879, 0.890, 0.865, 0.869, 0.772, 0.933, 0.875.

The estimators, standard error (ST) and Kolmogorov–Smirnov (KS) test statistic of MLE and MPS parameters estimation methods will be calculated under complete and censored sample cases.

7.1. Under Complete Sample

In this example, MLE and MPS methods are performed under the complete sample, via Mathematica package version 10, it provides powerful data visualization tools. All results are included in the Table (1), the graph of the probability density functions (PDFs) for different parameters estimation methods is indicated in the Figure (2) and the graph of the empirical CDF compared to the cumulative distribution functions (CDFs) of the PTL distribution using different parameters estimation methods is indicated in the Figure (3).

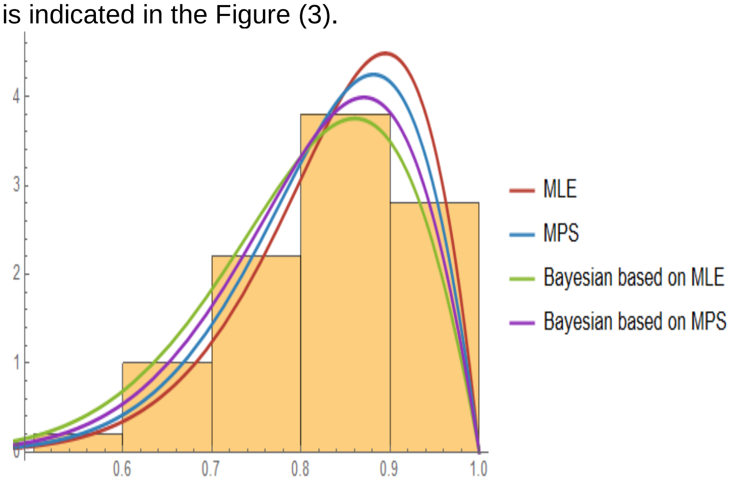

Figure (2): Probability density functions for different parameters estimation methods.

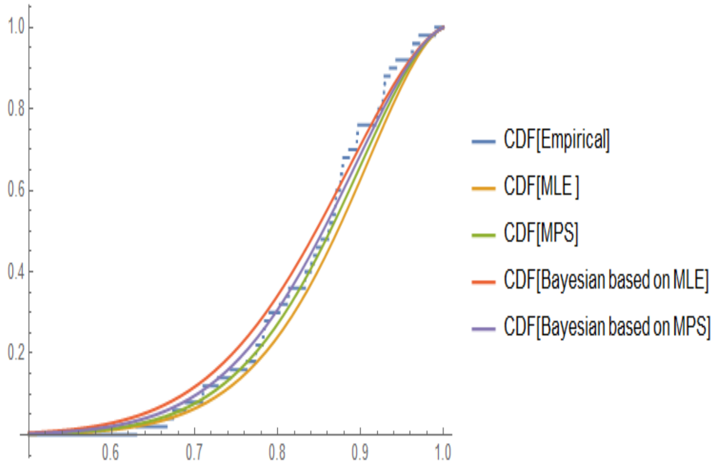

Figure (3): The empirical CDF compared to some CDFs of the PTL distribution using different parameters estimation methods.

Table (1): Estimators, ST, KS and P-value for MLE and MPS methods.

Method	parameters	Estimators	Standard Error	*KS*	p-value
MLE	α β	2.286 4.328	1.054 0.008	0.110	0.537
MPS	α β	3.251 3.694	0.954 0.005	0.075	0.683
Bayesian based on MLE	α β	3.877 3	0.748 0.003	0.054	0.816
Bayesian based on MPS	α β	3.914 3	0.519 0.002	0.041	0.928

It is clear that, from the Table (1), the MPS method using the Bayesian approach with the MCMC technique can be considered the most efficient estimation method in this example because it has the smallest ST and KS values and the largest P-value.

7.2. Under Censored Sample

In this example, MLE and MPS methods are performed under the progressive Type-II censoring scheme where the failed items are considered 45 items and the removed items are considered 5 items, via Mathematica package version 10. All results are included in the Table (2), the graph of PDFs of the PTL distribution for different parameters estimation methods is indicated in the Figure (4) and the graph of the empirical CDF compared to CDFs of the PTL distribution using different parameters estimation methods is indicated in the Figure (5).

It is clear that, from Table (2), the MPS method using the Bayesian approach with the MCMC technique, via the Gibbs sampling procedure, under the progressive Type-II censoring scheme can be considered the most efficient estimation method in this example because it has the smallest ST and KS values and the largest P-value.

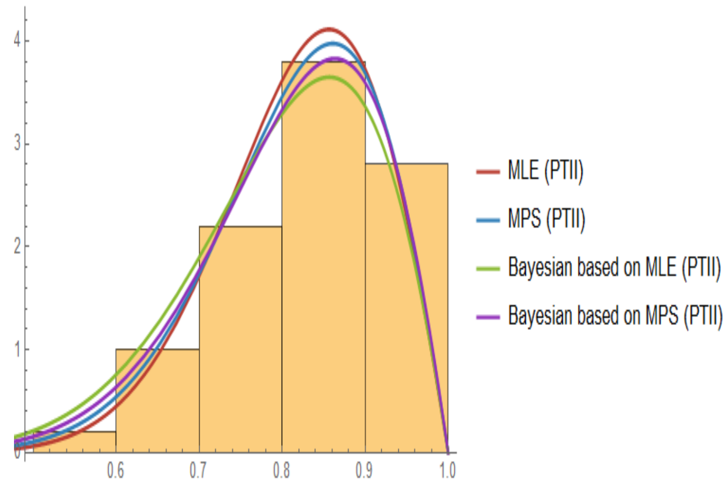

Figure (4): Probability density functions for different parameters estimation methods in the censored sample.

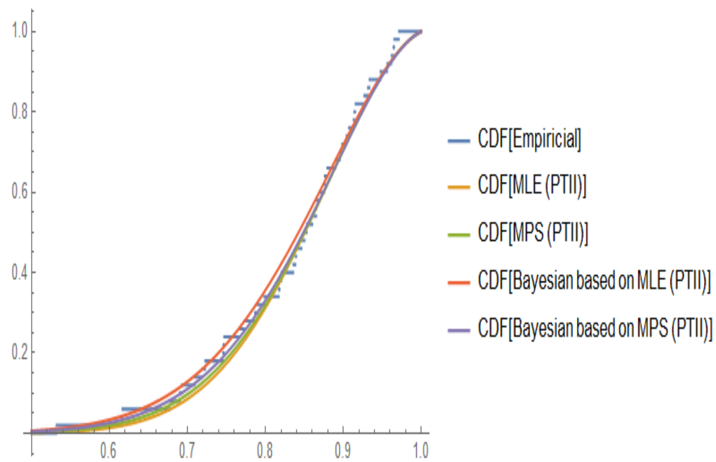

Figure (5): The empirical CDF compared to some CDFs of the PTL distribution using different parameters estimation methods in the censored sample.

Table (2): Estimators, ST, KS and P-value for MLE and MPS methods.

Method	parameters	Estimators	Standard Error	KS	P-value
MLE (PTII)	α β	11.24 1.494	3.172 1.518	0.213	0.174
MPS (PTII)	α β	5.724 2.178	1.825 0.672	0.180	0.218
Bayesian based on MLE (PTII)	α β	3.366 2.779	1.024 0.429	0.169	0.301
Bayesian based on MPS (PTII)	α β	3.614 2.806	0.792 0.157	0.147	0.508

8. Conclusion

The transformation of completing the square gives the PTL distribution more flexibility in mathematical properties and generating random numbers which helps easily to use MLE and MPS methods. The MPS method is an efficient estimation method having good performance with small biases and small RMSEs compared to the MLE method in Bayesian and non-Bayesian approaches under a complete sample and progressive Type-II censoring scheme. The Bayesian estimation approach with the MCMC technique has a better performance with the smallest biases and the smallest RMSEs especially when the LINEX loss function is used. The author encourages studying more about the MPS estimation method with the Bayesian approach in other censoring schemes.

References

[1] Ahmed, M. A. (2021). On the Identified Power Topp-Leone Distribution: Properties, Simulation and Applications. *Thailand Statistician*, 19(4), 838-854.

[2] Almetwally, E.M. and Almongy, H.M. (2019). Maximum Product Spacing and Bayesian Method for Parameter Estimation for Generalized Power Weibull Distribution under Censoring Scheme. *Journal of Data Science,*17(2), 407-444.

[3] Balakrishnan, N. and Aggarwala, R. (2000). *Progressive censoring: theory, methods, and applications*. Springer Science & Business Media.

[4] Balakrishnan, N. and Cramer, E. (2014). *The Art of Progressive Censoring: Statistics for Industry and Technology*. Springer.

[5] Balakrishnan, N. and Saleh, H.M. (2017). Recurrence relations for single and product moments of progressively Type-II censored order statistics from generalized logistic distribution with applications to inference. *Communications in Statistics-Simulation and Computation*, 46(6), 4559-4577.

[6] Bhatti, F.A., Hamedani, G.G., Korkmaz, M.Ç., Sheng, W. and Ali, A. (2021). On the Burr XII-moment exponential distribution. *Plos one*, 16(2), e0246935.

[7] Chandra, N. and Rathaur, V.K. (2017). Bayes Estimation of Augmenting Gamma Strength Reliability of a System under Non-informative Prior Distributions. *Calcutta Statistical Association Bulletin,* 69(1), 87-102.

[8] Cheng, R.C.H. and Amin, N.A.K. (1983). Estimating parameters in continuous univariate distributions with a shifted origin. *Journal of the Royal Statistical Society: Series B*, 45(3), 394-403.

[9] Elgarhy, M., Hassan, A.S. and Nagy, H.F. (2022). Parameter Estimation Methods and Applications of the Power Topp-Leone Distribution. *Gazi Univeristy Jouranl of Science,* 35(2), 731-746.

[10] Gelfand, A.E., and Smith, A.F. (1990). Sampling-based approaches to calculating marginal densities. *Journal of the American statistical association*, 85(410), 398-409.

[11] Genç, A.I. (2012). Moments of order statistics of Topp–Leone distribution. *Statistical Papers*, 53(1), 117-131.

[12] Ghitany, M.E., Kotz, S. and Xie, M. (2005). On some reliability measures and their stochastic orderings for the Topp–Leone distribution. *Journal of Applied Statistics*, 32(7), 715-722.

[13] Guure, C.B., Ibrahim, N.A. and Ahmed, A.O.M. (2012). Bayesian estimation of two-parameter Weibull distribution using extension of Jeffreys' prior information with three loss functions. *Mathematical Problems in Engineering*.

[14] Jeffreys, H. (1998). *The theory of probability*. OUP Oxford.

[15] Kotz, S. and Seier, E. (2007). Kurtosis of the Topp-Leone distributions. *Interstat*, 1, 1-15.

[16] Nadarajah, S. and Kotz, S. (2003). Moments of some J-shaped distributions. *Journal of Applied Statistics*, 30(3), 311-317.

[17] Nadarajah, S. (2009). Bathtub-shaped failure rate functions. *Quality & Quantity*, 43(5), 855-863.

[18] Ng, H.K.T., Chan, P.S. and Balakrishnan, N. (2004). Optimal progressive censoring plans for
the Weibull distribution. *Technometrics*, 46(4), 470-481.

[19] Ng, H.K.T., Luo, L., Hu, Y. and Duan, F. (2012). Parameter estimation of three-parameter Weibull distribution based on progressively Type-*II* censored samples. *Journal of Statistical Computation and Simulation*, 82(11), 1661-1678.

[20] Singh, S.K., Singh, U., and Kumar, D. (2013). Bayes estimators of the reliability function and parameter of inverted exponential distribution using informative and non-informative priors. *Journal of Statistical computation and simulation*, 83(12), 2258-2269.

[21] Singh, U., Singh, S.K. and Singh, R.K. (2014). A comparative study of traditional estimation methods and maximum product spacings method in generalized inverted exponential distribution. *Journal of Statistics Applications & Probability*, 3(2), 153.

[22] Topp, C.W. and Leone, F.C. (1955). A family of J-shaped frequency functions. *Journal of the American Statistical Association*, 50(269), 209-219

[23] Van Dorp, J.R. and Kotz, S. (2006).Modeling income distributions using elevated distributions on a bounded domain. In *Distribution models theory*, 1-25.

[24] Vasudeva Rao, A., Dattatreya Rao, A.V. and Narasimham, V.L. (1991). Asymptotic Relative Efficiencies of the ML Estimates of the Weibull Parameters in Grouped Type-*I* Right Censored Samples. *Reports of Statistical Application Research, Union of Japanese Scientists and Engineers*, 4, 1-12.

[25] Zhou, M., Yang, D.W., Wang, Y. and Nadarajah, S. (2006). Some J-shaped distributions: Sums, products and ratios. *In: Proceedings of the Annual Reliability and Maintainability Symposium*, 175-181.

Appendix I

Estimation methods tables

Table (3): MLE Method (Complete)

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE
10	$\alpha=3$	266.034	263.034	263.052	634.357	634.418
	$\beta=2$	5.074	3.074		8.78	
20	$\alpha=3$	140.205	137.205	137.211	428.885	428.916
	$\beta=2$	3.374	1.374		5.136	
30	$\alpha=3$	87.271	84.271	84.275	319.847	319.866
	$\beta=2$	2.784	0.784		3.464	
50	$\alpha=3$	38.851	35.851	35.854	179.422	179.438
	$\beta=2$	2.452	0.452		2.437	
100	$\alpha=3$	10.17	7.17	7.172	70.798	70.809
	$\beta=2$	2.172	0.172		1.248	
300	$\alpha=3$	3.519	0.519	0.524	2.272	2.36
	$\beta=2$	2.069	0.069		0.639	

Table (4): MPS Method (Complete)

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE	RE_1
10	$\alpha=3$	22.17	19.17	21.375	108.6	109.603	5.79
	$\beta=2$	11.455	9.455		14.795		
20	$\alpha=3$	11.355	8.355	9.74	49.125	49.992	8.58
	$\beta=2$	7.006	5.006		9.272		
30	$\alpha=3$	7.94	4.94	5.829	28.681	29.353	10.90
	$\beta=2$	5.094	3.094		6.247		
50	$\alpha=3$	5.716	2.716	3.195	17.675	18.054	9.94
	$\beta=2$	3.682	1.682		3.679		
100	$\alpha=3$	3.749	0.749	1.078	9.463	9.611	7.37
	$\beta=2$	2.775	0.775		1.682		
300	$\alpha=3$	2.904	-0.096	0.315	1.647	1.806	1.31
	$\beta=2$	2.3	0.3		0.742		

Table (5): Bayesian approach (SE loss function) based on MLE method (Complete)

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE	RE_2
10	$\alpha=3$	2.066	-0.934	0.934	1.07	1.08	587.42
	$\beta=2$	1.985	-0.015		0.15		
20	$\alpha=3$	2.46	-0.54	0.54	0.733	0.733	585.15
	$\beta=2$	2	-2.496×10^{-6}		4×10^{-6}		
30	$\alpha=3$	2.62	-0.38	0.38	0.584	0.584	547.72
	$\beta=2$	2	-2.104×10^{-6}		4×10^{-6}		
50	$\alpha=3$	2.749	-0.251	0.251	0.454	0.454	395.24
	$\beta=2$	2	-1.824×10^{-6}		4×10^{-6}		
100	$\alpha=3$	2.86	-0.14	0.14	0.318	0.318	222.67
	$\beta=2$	2	-1.312×10^{-6}		4×10^{-6}		
300	$\alpha=3$	3.068	0.068	0.068	0.261	0.261	9.04
	$\beta=2$	2	1.6×10^{-7}		4×10^{-6}		

Table (6): Bayesian approach (SE loss function) based on MPS method (Complete)

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE	RE_3
10	$\alpha=3$	2.345	-0.655	0.673	0.897	0.952	666.41
	$\beta=2$	1.846	-0.154		0.317		
20	$\alpha=3$	2.66	-0.34	0.34	0.641	0.641	669.14
	$\beta=2$	2	-1.872×10^{-6}		4×10^{-6}		
30	$\alpha=3$	2.776	-0.224	0.224	0.523	0.523	611.60
	$\beta=2$	2	-1.56×10^{-6}		4×10^{-6}		
50	$\alpha=3$	2.86	-0.14	0.14	0.418	0.418	429.28
	$\beta=2$	2	-1.232×10^{-6}		4×10^{-6}		
100	$\alpha=3$	2.927	-0.073	0.073	0.301	0.301	235.25
	$\beta=2$	2	-8.32×10^{-7}		4×10^{-6}		
300	$\alpha=3$	2.977	-0.023	0.023	0.169	0.169	13.96
	$\beta=2$	2	-4.8×10^{-7}		4×10^{-6}		

Table (7): Bayesian approach (LINEX Loss Function at h= -1) based on MLE method (Complete)

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE	RE_4
10	$\alpha=3$	2.221	-0.779	0.779	0.95	0.962	659.48
	$\beta=2$	1.992	-7.802×10^{-3}		0.15		
20	$\alpha=3$	2.599	-0.401	0.401	0.653	0.653	656.84
	$\beta=2$	2	-2.496×10^{-6}		4×10^{-6}		
30	$\alpha=3$	2.727	-0.273	0.273	0.532	0.532	601.25
	$\beta=2$	2	-2.104×10^{-6}		4×10^{-6}		
50	$\alpha=3$	2.827	-0.173	0.173	0.424	0.424	423.20
	$\beta=2$	2	-1.824×10^{-6}		4×10^{-6}		
100	$\alpha=3$	2.903	-0.097	0.097	0.305	0.305	232.16
	$\beta=2$	2	-1.312×10^{-6}		4×10^{-6}		
300	$\alpha=3$	3.036	0.036	0.036	0.252	0.252	9.37
	$\beta=2$	2	1.6×10^{-7}		4×10^{-6}		

Table (8): Bayesian approach (LINEX Loss Function at h= -1) based on MPS method (Complete)

Sample Size	Parameters	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE	RE_5
10	$\alpha=3$	2.569	-0.431	0.446	0.782	0.839	756.16
	$\beta=2$	1.884	-0.116		0.302		
20	$\alpha=3$	2.83	-0.17	0.17	0.594	0.594	722.08
	$\beta=2$	2	-1.872×10^{-6}		4×10^{-6}		
30	$\alpha=3$	2.899	-0.101	0.101	0.498	0.498	642.30
	$\beta=2$	2	-1.56×10^{-6}		4×10^{-6}		
50	$\alpha=3$	2.944	-0.056	0.056	0.407	0.407	440.88
	$\beta=2$	2	-1.232×10^{-6}		4×10^{-6}		
100	$\alpha=3$	2.972	-0.028	0.028	0.297	0.297	238.41
	$\beta=2$	2	-8.32×10^{-7}		4×10^{-6}		
300	$\alpha=3$	2.992	-8.422×10^{-3}	8.422×10^{-3}	0.168	0.168	14.05
	$\beta=2$	2	-4.8×10^{-7}		4×10^{-6}		

Table (9): Progressive Type-*II* Censoring Scheme (*r = 0.5 n*) *MLE* Method

Sample Size	Parameters	*R*	Mean of Estimators	Biases	Total Bias	*RMSEs*	Total *RMSE*
10	$\alpha=3$	5	490.206	487.206	487.374	1.028×10^3	1.0283×10^3
	$\beta=2$		14.799	12.799		24.858	
20	$\alpha=3$	10	158.715	155.715	156.266	509.633	510.173
	$\beta=2$		15.108	13.108		23.466	
30	$\alpha=3$	15	69.883	66.883	68.092	261.408	262.365
	$\beta=2$		14.775	12.775		22.394	
50	$\alpha=3$	25	15.754	12.754	17.554	55.862	59.659
	$\beta=2$		14.061	12.061		20.944	
100	$\alpha=3$	50	4.905	1.905	10.579	14.368	22.645
	$\beta=2$		12.406	10.406		17.503	
300	$\alpha=3$	150	3.15	0.15	0.879	0.699	1.546
	$\beta=2$		2.866	0.866		1.379	

Table (10): Progressive Type-*II* Censoring Scheme (*r = 0.5 n*) via MPS Method

Sample Size	Parameters	R	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE	RE_6
10	$\alpha=3$	5	82.727	79.727	83.635	270.025	272.275	3.78
	$\beta=2$		27.267	25.267		34.93		
20	$\alpha=3$	10	28.147	25.147	34.378	133.21	137.105	3.72
	$\beta=2$		25.441	23.441		32.448		
30	$\alpha=3$	15	12.968	9.968	23.406	58.554	65.622	4.00
	$\beta=2$		23.177	21.177		29.626		
50	$\alpha=3$	25	4.928	1.928	17.848	21.361	33.139	1.80
	$\beta=2$		19.744	17.744		25.335		
100	$\alpha=3$	50	2.493	-0.507	14.083	6.821	21.734	1.04
	$\beta=2$		16.074	14.074		20.636		
300	$\alpha=3$	150	3.018	0.018	0.155	0.217	1.005	1.54
	$\beta=2$		2.154	0.154		0.982		

Table (11): Bayesian approach (SE loss function at h= -1) based on progressive type-II censoring scheme (*r = 0.5 n*) via MLE method

Sample Size	Parameters	R	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE	RE_7
10	$\alpha=3$	5	7.477	4.477	4.762	4.501	4.787	214.81
	$\beta=2$		3.622	1.622		1.629		
20	$\alpha=3$	10	4.516	1.516	1.606	1.97	2.092	243.87
	$\beta=2$		2.53	0.53		0.703		
30	$\alpha=3$	15	4.65	1.65	1.778	1.868	2.007	130.72
	$\beta=2$		2.66	0.66		0.735		
50	$\alpha=3$	25	4.648	1.648	1.791	1.772	1.921	31.06
	$\beta=2$		2.702	0.702		0.743		
100	$\alpha=3$	50	4.62	1.62	1.776	1.664	1.821	12.44
	$\beta=2$		2.727	0.727		0.741		
300	$\alpha=3$	150	3.658	0.658	0.712	1.3	1.423	1.09
	$\beta=2$		2.271	0.271		0.578		

Table (12): Bayesian approach (SE loss function at h= -1) based on progressive type-II censoring scheme (*r = 0.5 n*) via MPS method

Sample Size	Parameters	R	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE	RE_8
10	*α=3*	5	4.565	1.565	1.714	1.768	1.892	543.50
	β=2		2.7	0.7		0.674		
20	*α=3*	10	4.546	1.546	1.685	1.703	1.837	277.72
	β=2		2.67	0.67		0.687		
30	*α=3*	15	4.486	1.486	1.607	1.67	1.816	144.47
	β=2		2.611	0.611		0.713		
50	*α=3*	25	4.427	1.427	1.528	1.691	1.805	33.05
	β=2		2.546	0.546		0.632		
100	*α=3*	50	4.246	1.246	1.326	1.631	1.783	12.70
	β=2		2.453	0.453		0.723		
300	*α=3*	150	3.49	0.49	0.528	1.144	1.257	1.23
	β=2		2.197	0.197		0.522		

Table (13): Bayesian approach (LINEX loss function at h= -1) based on progressive type-II censoring scheme (*r = 0.5 n*) via MLE method

Sample Size	Parameters	R	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE	RE_9
10	*α=3*	5	4.654	1.654	1.763	2	2.131	482.54
	β=2		2.612	0.612		0.736		
20	*α=3*	10	4.434	1.434	1.583	1.56	1.716	297.30
	β=2		2.671	0.671		0.714		
30	*α=3*	15	4.267	1.267	1.405	1.491	1.641	159.88
	β=2		2.607	0.607		0.685		
50	*α=3*	25	4.055	1.055	1.178	1.421	1.567	38.07
	β=2		2.525	0.525		0.66		
100	*α=3*	50	3.782	0.782	0.887	1.288	1.426	15.88
	β=2		2.418	0.418		0.612		
300	*α=3*	150	3.066	0.066	0.146	0.955	1.081	1.43
	β=2		2.13	0.13		0.507		

Table (14): Bayesian approach (LINEX loss function at h= -1) based on progressive type-II censoring scheme (*r = 0.5 n*) via MPS method

Sample Size	Parameters	R	Mean of Estimators	Biases	Total Bias	RMSEs	Total RMSE	RE_{10}
10	*α=3*	5	4.457	1.457	1.61	1.524	1.68	612.08
	β=2		2.685	0.685		0.707		
20	*α=3*	10	4.342	1.342	1.487	1.47	1.621	314.73
	β=2		2.641	0.641		0.684		
30	*α=3*	15	4.137	1.137	1.267	1.365	1.508	173.98
	β=2		2.56	0.56		0.641		
50	*α=3*	25	3.902	0.902	1.015	1.276	1.412	42.25
	β=2		2.466	0.466		0.605		
100	*α=3*	50	3.626	0.626	0.718	1.146	1.273	17.79
	β=2		2.352	0.352		0.554		
300	*α=3*	150	2.97	-0.03	0.075	0.894	1.011	1.53
	β=2		2.069	0.069		0.471		

Appendix II

Estimation methods graphs: *RMSE*s

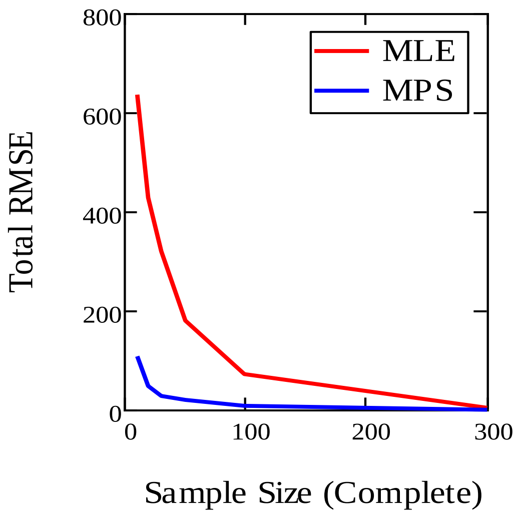

Figure (6): The total RMSEs of MLE and MPS methods for different complete sample sizes

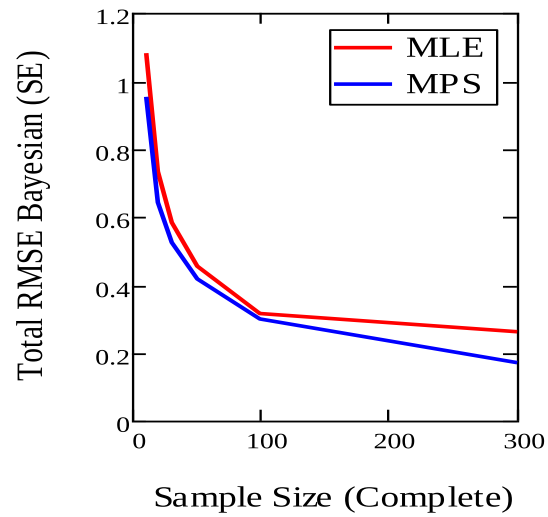

Figure (7): The total RMSEs of Bayesian MLE and MPS methods for different complete sample sizes

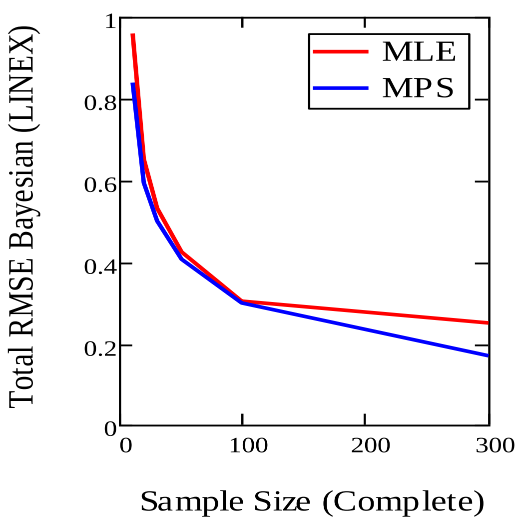

Figure (8): The total RMSEs of Bayesian MLE and MPS methods for different complete sample sizes

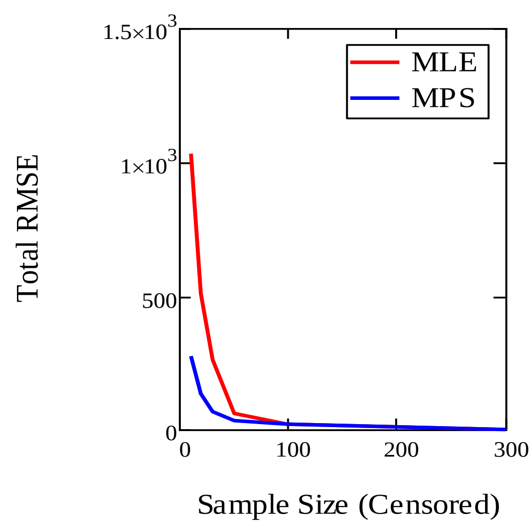

Figure (9): The total RMSEs of MLE and MPS methods for different censored sample sizes

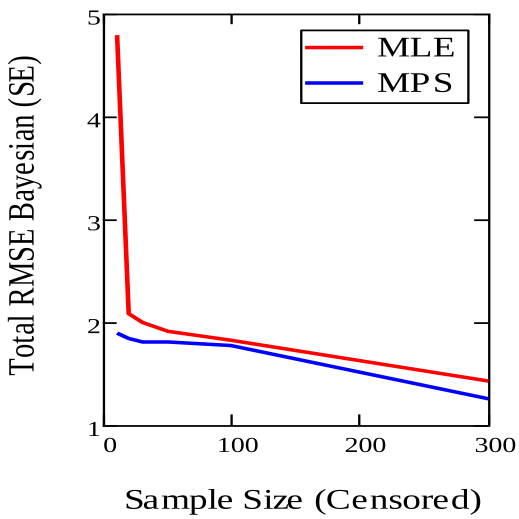

Figure (10): The total RMSEs of Bayesian MLE and MPS methods for different censored sample sizes

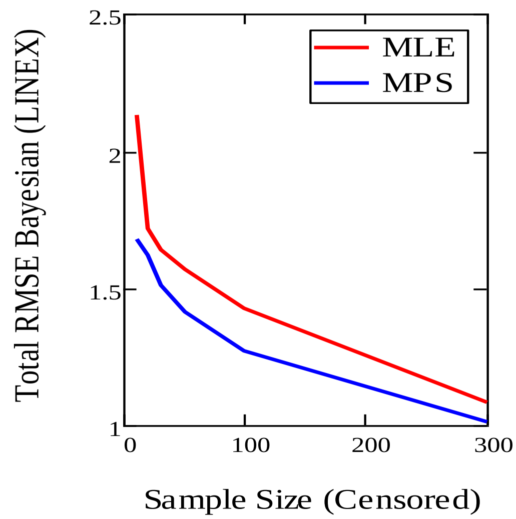

Figure (11): The total RMSEs of Bayesian MLE and MPS methods for different censored sample sizes

Estimation methods graphs: *Biases*

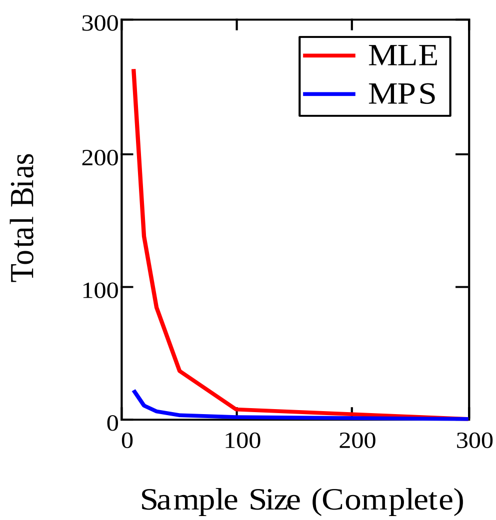

Figure (12): The total Biases of MLE and MPS methods for different complete sample sizes

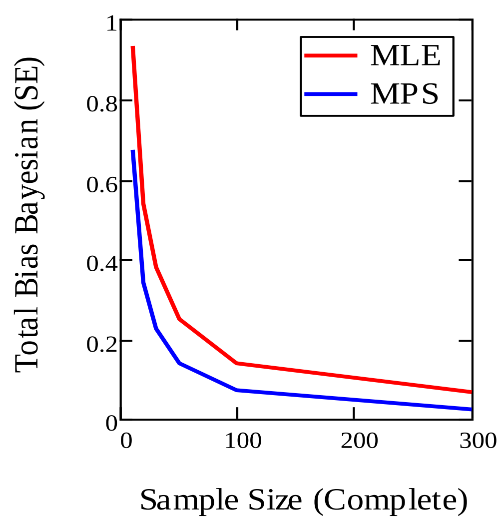

Figure (13): The total Biases of Bayesian MLE and MPS methods for different complete sample sizes

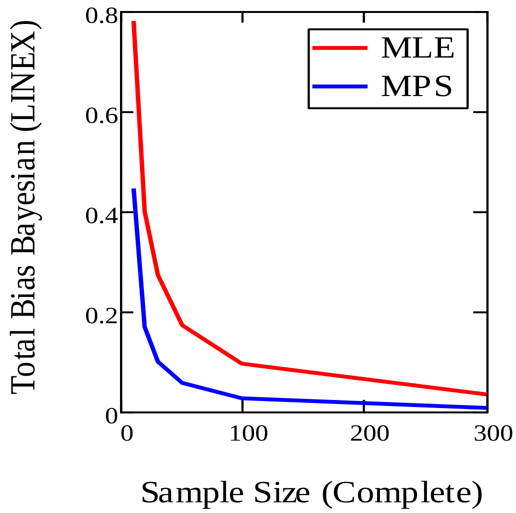

Figure (14): The total Biases of Bayesian *MLE* and *MPS* methods for different complete sample sizes

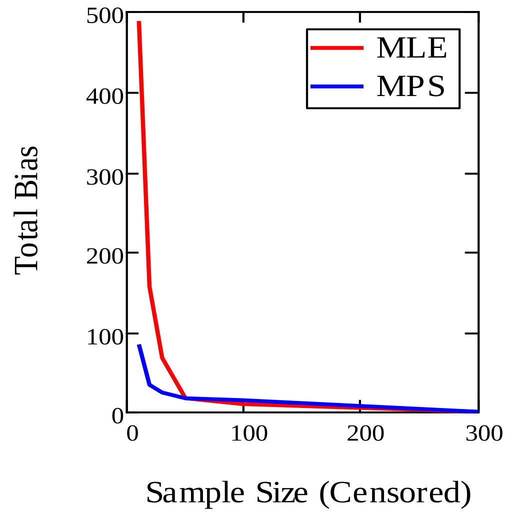

Figure (15): The total Biases of MLE and MPS methods for different censored sample sizes

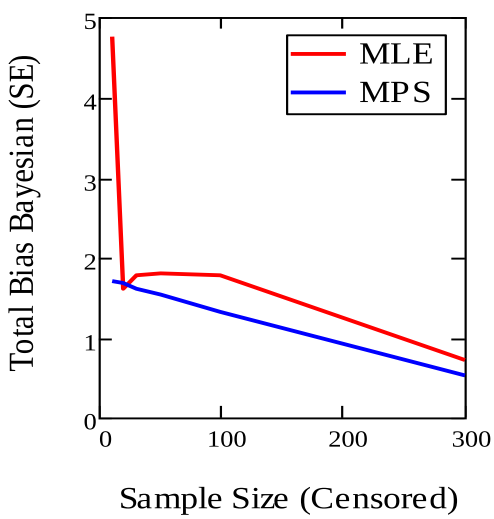

Figure (16): The total Biases of Bayesian *MLE* and *MPS* methods for different censored sample sizes

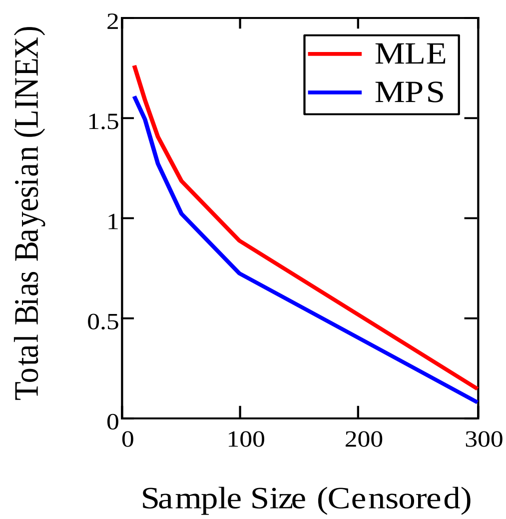

Figure (17): The total Biases of Bayesian MLE and MPS methods for different censored sample sizes