Modeling Sigmoidal Growth Curves to Study the Confirmed Cases of COVID-19 in Egypt

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Abstract

Sigmoid growth models play an important role in describing many natural events that have a *sigmoidal curve* (S-shaped). In this paper, the two sigmoid growth models based on Burr Type XII distribution called the Burr 1 Type XII and Burr 2 Type XII sigmoid growth models are proposed to be able to describe various situations with accuracy. The methods of estimation of the non-linear least squares and maximum likelihood are used to estimate the parameters of the proposed models. The performance of the new proposed models is investigated and compared with the classical sigmoid growth, Brody and Weibull models in describing the growth of confirmed new cases of COVID-19 in Egypt. The results showed that the new proposed model, Burr 1 Type XII sigmoidal growth is superior over the other models with respect to the coefficient of determination R^2, mean squared error, root mean squared error, model efficiency, and the Akaike information corrected criterion especially when NLS estimation is used.

Keywords: *Non-linear regression, Sigmoid growth model, Weibull model, Non-linear least squares, Maximum likelihood, COVID-19.*

1 Introduction

The growth curves are found in a wide range of disciplines such as biology, chemistry, economy, fishery, demography, and medical science. These curves are used to model natural events that involve investigation changes of response in time. The growth rate curve is often bell-shaped, in which the rate of growth increases sharply at the initial stage, reaching a peak, and then decreases towards zero at an upper asymptote.

Many mathematical growth functions have been proposed by some others such as Brody (1945), von Bertalanffy (1957), and Richards (1959) to capture the growth trajectory with accuracy. These functions can be classified into three categories: the first with a diminishing returns behavior (Brody model), the second with sigmoidal (S- shaped) curve, and a fixed inflection point which is the point at when the rate of growth gets maximum value (Gompertz, Logistic, and von Bertalanffy models) and the third with a flexible inflection point (Richards model).

Sigmoid growth models are nonlinear regression models that have been applied in various fields with many different notations and parameterization [Tjørve and Tjørve(2017)]. The most in use are three-parameter growth functions as Logistic, Gompertz, and von Bertalanffy and four-parameter growth functions as Richards [Ratkowsky (1983), France *et al.* (1996), Maruyama *et al.*(2001), and Narushin and Takma (2003)]. General discussions of sigmoid growth models were presented in Malott (1990), Tsoularis and Wallace (2002), Seber and Wild (2003), Goshu and Koya (2013), Mahanta and Borah (2014), Archontoulis and Miguez (2015), Vrána *et al.* (2019), Omori *et al.* (2020), Shen (2020), and Utsunomiya *et al.* (2020).

The mathematical forms of some classical growth models are as follows:

Brody model: $$y_i = \alpha + (\beta - \alpha)e^{-k\,x_i} + \varepsilon_i\,, \qquad (1)$$

Logistic model: $y_i = \frac{\alpha}{1+\beta e^{-k x_i}} + \varepsilon_i$, (2)

Gompertz model: $y_i = \alpha\, e^{-\beta\, e^{-k x_i}} + \varepsilon_i$, (3)

Weibull model: $y_i = \alpha \left(1 - e^{-(k x_i)^c}\right) + \varepsilon_i$, (4)

where y_i ; $i = 1, \dots, n$ is the response variable, x_i is the independent variable, α, β, k, and c are parameters to be estimated which are defined as: α is the maximum value of the response variable in the data , $\alpha > 0$, $\beta$is the minimum value of the response variable in the data, k is the parameter governing the rate at which the response variable approaches its potential maximum, $k > 0$, and c is the allometric constant, and ε_i is a random error term which assumes that it is *independent and identically distributed* $(i.i.d.)$ with $N(0, \sigma^2)$.

Many studies used and proposed growth models in analyzing various growth phenomena such as Fernandes *et al*. (2017) used the Logistic and Gompertz growth models for analyzing the growth pattern of coffee berries, Souza *et al*. (2017) studied the growth models: Brody, Gompertz, Logistic, and von Bertalanffy models in analyzing the cross-section data of the live weight of the Mangalarga Marchador horses, Amarti (2018) proposed the Logistic growth model with the Allee effect for describing the growth of the population number, Ghaderi-Zefrehel *et al*. (2018) studied some general non-linear growth models such as von Bertalanffy, Gompertz, Logistic, and Brody along with hierarchical modeling to investigate the phenotypic growth pattern of Iranian Lori-Bakhtiari sheep, Ribeiro *et al*. (2018) explained the growth and development of the Asian pear fruit on the grounds of length, diameter, and fresh weight determined over time using the Gompertz, and Logistic models, Cao *et al*. (2019) presented a new sigmoid growth model for describing the growth of animals and plants when the growth rate curve is asymmetric and Ukalska and Jastrzebowski (2019) studied the dynamics of the epicotyls emergence of oak using the Lgoistic, Gompertz, and Richards models.

The aim of this paper is to introduce new sigmoid growth models based on Burr Type XII distribution for analyzing various growth situations with accuracy, the first model is called Burr 1 Type XII sigmoid growth model and the second is called Burr 2 Type XII sigmoid growth model. The organization of this paper is as follows: the new proposed models of sigmoidal growth, Burr 1 Type XII, and Burr 2 Type XII are introduced in Section 2. Estimation of the parameters of the proposed models is performed using the *non-linear least squares* (NLS) and *maximum likelihood* (ML) estimation methods in Section 3. The performance of new sigmoid growth models are investigated using daily confirmed new COVID-19 cases in Egypt from March 15, 2020 to May 4, 2020 in Section 4. Some concluding remarks are provided in Section 5.

2 The Proposed Models

There are different procedures for modeling the sigmoid functions to the sigmoid growth models, one important of these procedures formulas based on the *cumulative distribution function* (cdf) as proposed by Seber and Wild (2003). An obvious way for describing a sigmoid shape is to use the distribution function $F(x;\boldsymbol{\theta})$ of a continuous random variable with a unimodal distribution. The general formula of sigmoid model based on the distribution function can be defined as follows:

$$y_1 = \beta + (\alpha - \beta)F(k(x-\gamma);\, \boldsymbol{\theta}) + \varepsilon\,, \qquad (5)$$

where y_1 is the response variable in the general formula of sigmoid model, x is the independent variable, γ is the point of inflection, α is the maximum value of the dependent variable in the data, $\alpha > 0$, β is the minimum value of the response variable, k is as a scale parameter on x , $k > 0$, and ε is the random error.

Another formula of sigmoid model as special case when $\beta = 0$ in (5) as follows:

$$y_2 = \alpha\, F(k(x-\gamma);\, \boldsymbol{\theta}) + \varepsilon\,, \qquad (6)$$

where y_2 is the response variable in the special case of sigmoid model when $\beta = 0$ and ε is the random error. Also, when shifting the standard curve vertically at $\gamma = 0$ in (5), the special case of sigmoid model can be written as follows:

$$y_3 = \beta + (\alpha - \beta)F(kx; \boldsymbol{\theta}) + \varepsilon, \qquad (7)$$

where y_3 is the response variable in the special case of sigmoid model when $\gamma = 0$ and ε is the random error.

The Burr Type XII distribution was first introduced in the literature by Burr (1942) and has gained special attention due to its broad applications in different fields including the area of reliability, failure time modeling and acceptance sampling plan and so on. The Burr Type XII distribution is one of the most important distributions, since it is including several distributions as special cases such as Weibull, Pareto, Generalized Logistic, Logistic, Exponential, and Gompertz distributions [Kumar (2017)]. Thus, the Burr Type XII distribution will be using to construct different models of sigmoidal growth based on the cumulative distribution functions.

The *probability density function* (pdf) and the cdf of Burr Type XII (c, r) distribution are given respectively by:

$$f(x; c, r) = c\, r\, x^{c-1}(1 + x^c)^{-(r+1)}, \; X \geq 0, \; c > 0, r > 0 \qquad (8)$$

and

$$F(x; c, r) = 1 - (1 + x^c)^{-r}, \quad X \geq 0, c > 0, r > 0, \qquad (9)$$

where c and r are the shape and scale parameters, respectively.

From (5), the Burr Type XII sigmoid growth model is denoted $y_i(B)$ and is written in the following form:

$$y_i(B) = f_B(x_i, \boldsymbol{\theta}) + \varepsilon_i, \; \boldsymbol{\theta} = (\alpha, \beta, k, c, r)^T$$
$$= \beta + (\alpha - \beta)[1 - (1 + (kx_i)^c)^{-r}] + \varepsilon_i . \qquad (10)$$

When $r = 1$ in (10), the first new proposed model called the Burr 1 Type XII sigmoid growth model, denoted by $y_i(B1)$ and is defined as follows:

$$y_i(B1) = f_{B1}(x_i, \boldsymbol{\theta}) + \varepsilon_i\,, \qquad \boldsymbol{\theta} = (\alpha, \beta, k, c)^T$$
$$= \beta + (\alpha - \beta)[\,1 - (1 + (kx_i)^c)^{-1}] + \varepsilon_i\,. \qquad (11)$$

Also, when $c = 1$ and $r = 1$ in (10), the second new proposed model called the Burr 2 Type XII sigmoid growth model denoted by $y_i(B2)$ and is defined as follows:

$$y_i(B2) = f_{B2}(x_i, \boldsymbol{\theta}) + \varepsilon_i, \boldsymbol{\theta} = (\alpha, \beta, k)^T$$
$$= \beta + (\alpha - \beta)\left[1 - \left(1 + (kx_i)\right)^{-1}\right] + \varepsilon_i\,. \quad (12)$$

3 Estimation of the Parameters of the Proposed Models

In this section, the NLS and the ML estimation methods are used for estimating the parameters of the proposed sigmoid growth models.

3.1 Non-linear least squares estimation

First, we will discuss non-linear least squares estimation for the parameters of some classical growth models: Brody and Weibull models. It is required to minimize the following function:

$$S(\boldsymbol{\theta})_{NLS} = \sum_{i=1}^{n} \{y_i - f(x_i, \boldsymbol{\theta})\} \frac{\partial f(x_i, \boldsymbol{\theta})}{\partial \theta_j}\bigg|_{\theta_j = \hat{\theta}_j}, j = 1, 2, \dots, p. \qquad (13)$$

This provides a system of p non-linear equation with p unknown parameters. For Brody model as defined in (1),

$$f_{Brody}(x_i, \boldsymbol{\theta}) = \alpha + (\beta - \alpha)e^{-k\,x_i}, \ \boldsymbol{\theta} = (\alpha, k, \beta)^T. \qquad (14)$$

The NLS estimators can be obtained by minimizing

$$\sum_{i=1}^{n} \{y_i - [\alpha + (\beta - \alpha)e^{-k\,x_i}]\} \frac{\partial f_{Brody}(x_i, \boldsymbol{\theta})}{\partial \theta_j}\bigg|_{\theta_j = \hat{\theta}_j} = 0\,. \quad (15)$$

The derivatives of $f_{Brody}(x_i, \boldsymbol{\theta})$ with respect to the parameters are given by

$$\frac{\partial f_{Brody}(x_i, \boldsymbol{\theta})}{\partial \alpha} = 1 - e^{-k\,x_i}, \qquad (16)$$

$$\frac{\partial f_{Brody}(x_i,\boldsymbol{\theta})}{\partial \beta} = e^{-k\,x_i}, \tag{17}$$

and

$$\frac{\partial f_{Brody}(x_i,\boldsymbol{\theta})}{\partial k} = (\alpha - \beta)\, x_i e^{-k\,x_i}\,. \tag{18}$$

Since the NLS method requires an iterative method, one of iterative methods can be used to solve these equations numerically, such as the Gauss-Newton method, the gradient descent method, and the Levenberg-Marquardt method [Fekedulegn *et al*. (1999), Seber and Wild (2003), and Gavin (2017)].

For the Weibull model as defined in (4),

$$f_{Weibull}(x_i, \boldsymbol{\theta}) = \alpha\left(1 - e^{-(k\,x_i)^c}\right)\,,\;\; \boldsymbol{\theta} = (\alpha, k, c)^T. \tag{19}$$

The NLS estimators can be obtained by minimizing

$$\sum_{i=1}^{n}\left\{y_i - \left[\alpha\left(1 - e^{-(k\,x_i)^c}\right)\right]\right\}\frac{\partial f_{Weibull}(x_i,\boldsymbol{\theta})}{\partial \theta_j}\bigg|_{\theta_j=\hat{\theta}_j} = 0\,. \tag{20}$$

Then,

$$\frac{\partial f_{Weibull}(x_i,\boldsymbol{\theta})}{\partial \alpha} = \left(1 - e^{-(k\,x_i)^c}\right), \tag{21}$$

$$\frac{\partial f_{Weibull}(x_i,\boldsymbol{\theta})}{\partial k} = \alpha\, c\;\, x_i^{\,c} k^{c-1} e^{-(k\,x_i)^c}\,, \tag{22}$$

and

$$\frac{\partial f_{Weibull}(x_i,\boldsymbol{\theta})}{\partial c} = \alpha\;\, e^{-(k\,x_i)^c}(k\,x_i)^c\, ln(k\,x_i). \tag{23}$$

The iterative methods can be used to get the solution of these equations numerically.

For the first new proposed model, the Burr 1 Type XII sigmoid growth model as defined in (11), the NLS estimators can be obtained by minimizing

$$\sum_{i=1}^{n}\left\{y_i - \left[\beta + (\alpha - \beta)[\,1 - (1 + (kx_i)^c)^{-1}]\right]\right\}\frac{f_{B1}(x_i,\boldsymbol{\theta})}{\partial \theta_j}\bigg|_{\theta_j=\hat{\theta}_j} = 0\,. \tag{24}$$

Then,

$$\frac{\partial f_{B1}(x_i,\boldsymbol{\theta})}{\partial \alpha} = 1 - (1 + (kx_i)^c)^{-1}, \tag{25}$$

$$\frac{\partial f_{B1}(x_i,\boldsymbol{\theta})}{\partial \beta} = (1 + (kx_i)^c)^{-1}, \tag{26}$$

$$\frac{\partial f_{B1}(x_i,\boldsymbol{\theta})}{\partial k} = c\, x_i{}^c k^{c-1}(\alpha - \beta)(1 + (kx_i)^c)^{-2}, \tag{27}$$

and

$$\frac{\partial f_{B1}(x_i,\boldsymbol{\theta})}{\partial c} = (\alpha - \beta)(1 + (kx_i)^c)^{-2}(kx_i)^c \ \ln(kx_i). \tag{28}$$

An iterative method can be used to get the solution of these equations numerically.

For the second new proposed model, the Burr 2 Type XII sigmoid growth model as defined in (12), the NLS estimators can be obtained by minimizing

$$\sum_{i=1}^{n}\left\{y_i - \left[\beta + (\alpha - \beta)\left[1 - \left(1 + (kx_i)\right)^{-1}\right]\right]\right\}.\frac{\partial f_{B2}(x_i,\boldsymbol{\theta})}{\partial \theta_j}\bigg|_{\theta_j = \hat{\theta}_j} = 0. \tag{29}$$

Then,

$$\frac{\partial f_{B2}(x_i,\boldsymbol{\theta})}{\partial \alpha} = 1 - \left(1 + (kx_i)\right)^{-1}, \tag{30}$$

$$\frac{\partial f_{B2}(x_i,\boldsymbol{\theta})}{\partial \beta} = (1 + (kx_i))^{-1}, \tag{31}$$

and

$$\frac{\partial f_{B2}(x_i,\boldsymbol{\theta})}{\partial k} = (\alpha - \beta)x_i(1 + (kx_i))^{-2}. \tag{32}$$

An iterative method can be used to get the solution of these equations numerically.

3.2 Maximum likelihood estimation

In this subsection, the ML estimation method is used for estimating the parameters of the proposed models. First, for the Brody growth model as in (1), suppose that $\boldsymbol{y} = (y_1, \dots, y_n)^{\boldsymbol{T}}$ be n independent random variables with pdf, $f(y_i|\boldsymbol{\theta}, \sigma_\varepsilon^2)$ depending on a vector-valued parameter $\boldsymbol{\theta}$ and the variance of error, σ_ε^2. Also, the $\varepsilon_i's$ are assumed to be independent and $i.i.d$ with $N(0,\sigma^2)$, then the likelihood function is:

$$L = f(\boldsymbol{y}|\boldsymbol{\theta},\sigma_\varepsilon^2)$$

$$= (2\pi\sigma_\varepsilon^2)^{-n/2}\, exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i-[\alpha+(\beta-\alpha)e^{-k\,x_i}]\right)^2}{\sigma_\varepsilon^2}\right)\right]. \quad (33)$$

The logarithm of the likelihood function denoted by $l(\boldsymbol{\theta},\sigma_\varepsilon^2;\,\boldsymbol{y})$ and is given as follows:

$$l(\boldsymbol{\theta},\sigma_\varepsilon^2;\,\boldsymbol{y}) = \log(L) \propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i-[\alpha+(\beta-\alpha)e^{-k\,x_i}]\right)^2}{\sigma_\varepsilon^2}\right). \quad (34)$$

Then, the ML estimator $\widehat{\boldsymbol{\theta}}$ can be obtained by solving the following equation:

$$\left.\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0\,,\quad \boldsymbol{\theta} = (\alpha,k,\beta)^T\,, \quad (35)$$

where

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\alpha} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\left[y_i-[\alpha+(\beta-\alpha)e^{-k\,x_i}]\right](1-e^{-k\,x_i})\right), \quad (36)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\beta} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\left[y_i-[\alpha+(\beta-\alpha)e^{-k\,x_i}]\right](e^{-k\,x_i})\right), \quad (37)$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial k} = \frac{(\alpha-\beta)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\left[y_i-[\alpha+(\beta-\alpha)e^{-k\,x_i}]\right]x_i e^{-k\,x_i}\right), \quad (38)$$

and

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{n}(y_i-[\alpha+(\beta-\alpha)e^{-k\,x_i}])^2. \quad (39)$$

An iterative method can be used to get the solution of these equations numerically.

For the Weibull growth model as in (4), suppose that the $\varepsilon_i's$ are $i.i.d.N(0,\sigma^2)$, then the likelihood function becomes:

$$L = f(\boldsymbol{y}|\boldsymbol{\theta},\sigma_\varepsilon^2) = (2\pi\sigma_\varepsilon^2)^{-n/2}\, exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i-\left[\alpha\,(1-e^{-(k\,x_i)^c})\right]\right)^2}{\sigma_\varepsilon^2}\right)\right]. \quad (40)$$

And the logarithm of the likelihood function is

$$l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y}) = \log(L) \propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - \left[\alpha\,(1 - e^{-(k\,x_i)^c})\right]\right)^2}{\sigma_\varepsilon^2}\right). \quad (41)$$

The ML estimator $\widehat{\boldsymbol{\theta}}$ can be obtained by solving the following equation:

$$\left.\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}} = 0\,, \quad \boldsymbol{\theta} = (\alpha, k, c)^T\,, \quad (42)$$

where

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial \alpha} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left[y_i - \alpha\left[1 - e^{-(k\,x_i)^c}\right]\right]^2\left[1 - e^{-(k\,x_i)^c}\right], \quad (43)$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial k} = \frac{\alpha\,c\,k^{c-1}}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\left[y_i - \left(1 - e^{-(k\,x_i)^c}\right)\right]\left(e^{-(k\,x_i)^c}\right)x_i{}^c\right), \quad (44)$$

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial c} = \frac{\alpha}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\left[\left[y_i - \left(1 - e^{-(k\,x_i)^c}\right)\right]\left(e^{-(k\,x_i)^c}(k\,x_i)^c\ln(k\,x_i)\right)\right]\right), \quad (45)$$

and

$$\frac{\partial l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y})}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{n}\left(y_i - \left[\alpha\,(1 - e^{-(k\,x_i)^c})\right]\right)^2. \quad (46)$$

An iterative method can be used to get the solution of these equations numerically.

For the new proposed model of sigmoidal growth, Burr 1 Type XII, suppose that the $\varepsilon_i's$ are $i.i.d.N(0, \sigma^2)$, then the likelihood function becomes:

$$L = f(\boldsymbol{y}|\boldsymbol{\theta}, \sigma_\varepsilon^2)$$

$$= (2\pi\sigma_\varepsilon^2)^{-n/2}\,exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - \left[\beta + (\alpha - \beta)\left[1 - (1 + (kx_i)^c)^{-1}\right]\right]\right)^2}{\sigma_\varepsilon^2}\right)\right]. \quad (47)$$

And the logarithm of the likelihood function is as follows:

$$l(\boldsymbol{\theta}, \sigma_\varepsilon^2;\ \boldsymbol{y}) = \log(L)$$

$$\propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i - \left[\beta + (\alpha - \beta)\left[1 - (1 + (kx_i)^c)^{-1}\right]\right]\right)^2}{\sigma_\varepsilon^2}\right). \quad (48)$$

The ML estimator $\widehat{\boldsymbol{\theta}}$ can be obtained by solving the following equation:

$$\left.\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0\,, \quad \boldsymbol{\theta} = (\alpha, k, \beta, c)^T\,, \tag{49}$$

where

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \alpha} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\begin{matrix} y_i - \left[\beta + (\alpha-\beta)\left[1-(1+(kx_i)^c)^{-1}\right]\right] \\ \left[1-(1+(kx_i)^c)^{-1}\right]\end{matrix}\right), \tag{50}$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \beta} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(y_i - \left[\begin{matrix}\beta + (\alpha-\beta)\left[1-(1+(kx_i)^c)^{-1}\right] \\ (1+(kx_i)^c)^{-1}\end{matrix}\right]\right), \tag{51}$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial k} = \frac{ck^{c-1}(\alpha-\beta)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\begin{matrix} y_i - \left[\beta + (\alpha-\beta)\left[1-(1+(kx_i)^c)^{-1}\right]\right] \\ (1+(kx_i)^c)^{-2}{x_i}^c\end{matrix}\right), \tag{52}$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial c} = \frac{(\alpha-\beta)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(\begin{matrix} y_i - \left[\beta + (\alpha-\beta)\left[1-(1+(kx_i)^c)^{-1}\right]\right] \\ (1+(kx_i)^c)^{-2}(kx_i)^c\ln(kx_i)\end{matrix}\right), \tag{53}$$

and

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial \sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{n}\left(y_i - \left[\begin{matrix}\beta + (\alpha-\beta) \\ \left[1-\left(1+\left(k(x_i-\gamma)\right)\right)^{-1}\right]\end{matrix}\right]\right)^2. \tag{54}$$

An iterative method can be used to get the solution of these equations numerically.

For the new proposed model of sigmoidal growth, Burr 2 Type XII, suppose that the $\varepsilon_i's$ are $i.i.d.N(0,\sigma^2)$, then the likelihood function is given as:

$$L = f(\boldsymbol{y}|\boldsymbol{\theta},\sigma_\varepsilon^2)$$

$$= (2\pi\sigma_\varepsilon^2)^{-n/2}\,exp\left[-\frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i-\left[\beta+(\alpha-\beta)\left[1-(1+(kx_i)^c)^{-1}\right]\right]\right)^2}{\sigma_\varepsilon^2}\right)\right]. \tag{55}$$

By taking the logarithm of likelihood function:

$$l(\boldsymbol{\theta},\sigma_\varepsilon^2;\ \boldsymbol{y}) = \log(L)$$

$$\propto -\frac{n}{2}\log(\sigma_\varepsilon^2) - \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\left(y_i-\left[\beta+(\alpha-\beta)\left[1-(1+(kx_i)^c)^{-1}\right]\right]\right)^2}{\sigma_\varepsilon^2}\right). \tag{56}$$

The ML estimator, $\widehat{\boldsymbol{\theta}}$ can be obtained by solving the following equation:

$$\left.\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\widehat{\boldsymbol{\theta}}} = 0\,,\quad \boldsymbol{\theta} = (\alpha,k,\beta)^T\,, \tag{57}$$

where

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\alpha} = \frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\begin{pmatrix} y_i - \left[\beta + (\alpha-\beta)\left[1-\left(1+(kx_i)\right)^{-1}\right]\right] \\ \left[1-\left(1+(kx_i)\right)^{-1}\right]\end{pmatrix}, \tag{58}$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\beta} = -\frac{1}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\left(y_i - \begin{bmatrix}\beta + (\alpha-\beta)\left[1-\left(1+(kx_i)\right)^{-1}\right] \\ [1-(1+(kx_i))^{-1}]\end{bmatrix}\right), \tag{59}$$

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial k} = \frac{(\alpha-\beta)}{\sigma_\varepsilon^2}\sum_{i=1}^{n}\begin{pmatrix} y_i - \left[\beta + (\alpha-\beta)\left[1-\left(1+(kx_i)\right)^{-1}\right]\right] \\ \left(1+(kx_i)\right)^{-2}x_i\end{pmatrix}, \tag{60}$$

and

$$\frac{\partial l(\boldsymbol{\theta},\sigma_\varepsilon^2;\boldsymbol{y})}{\partial\sigma_\varepsilon^2} = -\frac{n}{2\sigma_\varepsilon^2} + \frac{1}{2\sigma_\varepsilon^4}\sum_{i=1}^{n}\left(y_i - \begin{bmatrix}\beta + (\alpha-\beta) \\ [1-(1+(kx_i))^{-1}]\end{bmatrix}\right)^2. \tag{61}$$

After that, one of iterative methods can be used to get the solution of these equations numerically.

Determining the initial values of the parameters is needed to obtain the estimators when the iterative methods are used. Initial value specification is one of the most difficult problems encountered in estimating parameters of non-linear models [Fekedulegn *et al.* (1999)].

The starting value of $\boldsymbol{\alpha}$: The parameter α_0 is specified as the maximum value of the dependent variable in the data. Then, the new value of α is calculated for the different sigmoidal equations.

The starting value of $\boldsymbol{k}$: The parameter k is defined as the constant rate at which the response variable approaches its maximum possible value. Based on this definition, one can write

$$k = \frac{(y_n-y_1)}{\alpha_0(x_n-x_1)}\,, \tag{62}$$

where y_1 and y_n are the values of the response variable corresponding to the first x_1 and the last x_n observations, and α_0 is the initial value specified for the parameter α.

The starting value of $\boldsymbol{\gamma}$: The parameter γ is defined as the point of inflection value of the curve of the response variable, or, it can be assumed that γ is the value of the response variable corresponding to $\frac{\alpha_0}{2}$ value of the dependent variable.

The starting value of $\boldsymbol{\beta}$: The starting value for the constant β_0 is specified by evaluating the model at the start of the growth and the assumption that β is the minimum of the dependent variable in the data. Then, when the predictor variable is zero, the new value of β is considered for the different sigmoid equations.

Now, the inflection points for growth curves, namely, Brody, Weibull, Burr 1 Type XII, and Burr 2 Type XII functions are derived as follows:

Brody: From (1), consider $f_{Brody}(x_i, \boldsymbol{\theta}) = \alpha + (\beta - \alpha)e^{-k\,x_i}$,

$\boldsymbol{\theta} = (\alpha, k, \beta)^T$.Then, set $\beta = y_0$, $y_0 = f_{Brody}(x_i, \boldsymbol{\theta})$ at $x = 0$.

Then, according to the inflection point of Brody function, the first and second derivatives of $f_{Brody}(x_i, \boldsymbol{\theta})$ denoted as $f'_{Brody}(x_i, \boldsymbol{\theta})$ and $f''_{Brody}(x_i, \boldsymbol{\theta})$ are given respectively by:

$$f'_{Brody}(x_i, \boldsymbol{\theta}) = k(\alpha - \beta)e^{-kx_i},$$

and

$$f''_{Brody}(x_i, \boldsymbol{\theta}) = k^2(\alpha - \beta)e^{-kx_i},$$

where when $f''_{Brody}(x_i, \boldsymbol{\theta}) = 0$, the Brody growth function does not possess any point of inflection.

Weibull: From (4), consider $f_{Weibull}(x_i, \boldsymbol{\theta}) = \alpha\left(1 - e^{-(k\,x_i)^c}\right)$,

$\boldsymbol{\theta} = (\alpha, k, c)^T$. The first and second derivatives of $f_{Weibull}(x_i, \boldsymbol{\theta})$

denoted as $f'_{Weibull}(x_i, \boldsymbol{\theta})$ and $f''_{Weibull}(x_i, \boldsymbol{\theta})$ are given respectively by:

$$f'_{Weibull}(x_i, \boldsymbol{\theta}) = \alpha\, c\, k^c e^{(kx_i)^c} x_i^{\,c-1} \,,$$

and

$$f''_{Weibull}(x_i, \boldsymbol{\theta}) = c\, \alpha\, k^c e^{(kx_i)^c} x_i^{\,c-2}[(c-1) - kc] \,.$$

When $f''_{Weibull}(x_i, \boldsymbol{\theta}) = 0$, the Weibull growth function does not possess any point of inflection.

Burr 1 Type XII: From (11),

$$\begin{aligned} f_{B1}(x_i, \boldsymbol{\theta}) &= \beta + (\alpha - \beta)[\,1 - (1 + (kx_i)^c)^{-1}],\, \boldsymbol{\theta} = (\alpha, \beta, k, c)^T \\ &= \alpha - (\alpha - \beta)(1 + (kx_i)^c)^{-1} \,. \end{aligned} \qquad (63)$$

By solving (63) and setting $\beta = y_0$ where $y_0 = f_{B1}(x_i, \boldsymbol{\theta})$ at $x = 0$, then, according to the inflection point of Burr 1 Type XII, the first and second derivatives of $f_{B1}(x_i, \boldsymbol{\theta})$ denoted $f'_{B1}(x_i, \boldsymbol{\theta})$ and $f''_{B1}(x_i, \boldsymbol{\theta})$ are given respectively by:

$$f'_{B1}(x_i, \boldsymbol{\theta}) = ck^c(\alpha - \beta)x_i^{\,c-1}(1 + (kx_i)^c)^{-2} \,,$$

$$f''_{B1}(x_i, \boldsymbol{\theta}) = ck^c(\alpha - \beta)\begin{Bmatrix} x_i^{\,c-1}(-2ck^c x_i^{\,c-1})(1 + (kx_i)^c)^{-3} \\ +(1 + (kx_i)^c)^{-2}\,(c-1)x_i^{\,c-2}) \end{Bmatrix}.$$

When $f''_{B1}(x_i, \boldsymbol{\theta}) = 0$, then, $(1 + (kx_i)^c)^{-2} = 0$. Hence, $x_i = \frac{(-1)^{1/c}}{k}$, then, by substituting the new x_i in (63), the new value of $\alpha_{\max} = f_{B1}(x_i, \boldsymbol{\theta}) = \alpha$.

Burr 2 Type XII: From (12),

$$\begin{aligned} f_{B2}(x_i, \boldsymbol{\theta}) &= \beta + (\alpha - \beta)\left[1 - \left(1 + (kx_i)\right)^{-1}\right], \boldsymbol{\theta} = (\alpha, \beta, k, c)^T \\ &= \alpha - (\alpha - \beta)\left(1 + (kx_i)\right)^{-1}. \end{aligned} \qquad (64)$$

By solving (64) and setting $\beta = y_0$ where $y_0 = f_{B2}(x_i, \boldsymbol{\theta})$ at $x = 0$, the first and second derivatives of $f_{B2}(x_i, \boldsymbol{\theta})$ denoted as $f'_{B2}(x_i, \boldsymbol{\theta})$

and $f''_{B2}(x_i, \boldsymbol{\theta})$ are given respectively by:

$$f'_{B2}(x_i, \boldsymbol{\theta}) = k(\alpha - \beta)(1 + kx_i)^{-2} ,$$

and

$$f''_{B2}(x_i, \boldsymbol{\theta}) = -2\, k^2(\alpha - \beta)(1 + kx_i)^{-3}. \qquad (65)$$

When $f''_{B2}(x_i, \boldsymbol{\theta}) = 0$, then, $(1 + kx_i)^{-3} = 0$. Hence, $x_i = \frac{1}{k}$, then, the new value of α_{max} after substituting the new x_i in (65) is

$$\alpha_{max} = f_{B2}(x_i) = \alpha - (\alpha - \beta)(2)^{-2} .$$

4 Application

COVID-19 is an emerging pandemic of Corona virus disease 2019 caused by severe acute respiratory syndrome Coronavirus2 (SARS-Cov2). It was first detected in Wuhan, China in December, 2019. The epidemic was declared by the *World Health Organization* (WHO) as a public health emergency of international importance on January, 2020.

To check the performance of the new proposed sigmoid growth models, the data set on the number of daily confirmed new COVID-19 cases in Egypt from March 15, 2020 to May 4, 2020, which is taken from ministry of health and population in Egypt (2020). The data was recorded every day for a period of 51 days. The explanatory variable considered in this study is days (x) and the number of confirmed new cases of COVID-19 (y) is considered as a response variable. Fig. 1 displays the relationship between the number of confirmed new cases of COVID-19 as response variable y, and the days as explanatory variable x after the data are refined by multiplying by inverted variance transformation.

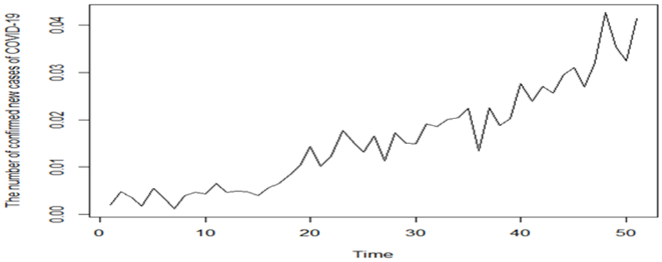

Fig. 1: Description of the number of confirmed new cases of COVID-19 over time.

The initial values are calculated as α_0 = 0.0426, k_0 = 0.0194, $\gamma \cong 15$, c = 3, and β_0 = 0.0011. Plots of growth curves, Burr 1 Type XII, Burr 2 Type XII, Brody, and Weibull using their inflection points are displayed in Fig. 2. Also, fitted growth curves of the Burr 1 Type XII, Burr 2 Type XII, Brody, and Weibull growth models for the data set are displayed in Fig. 3. Estimation of the model parameters are performed by NLS method using Levenberg-Marquardt iteration algorithm by nlsLM function of the minpack.lm package of R.3.6.3. In addition, the estimate parameters of these models by ML method are obtained by Newton-Raphson maximization using maxLik package of R.3.6.3. Table 1 shows the parameter estimates by NLS and ML estimation, *approximate standard error* (ASE) and asymptotic 95% confidence intervals for each parameter by these two methods. Also, for comparison between the models, the *Akaike Information corrected criterion* (AICc) and *Likelihood Ratio Test* (LRT) are used (Table 2) according to the following formulas:

$$\text{AICc} = -2\,l + 2\,p + \frac{2\,p(p+1)}{n-p-1}, \qquad (66)$$

where l is the logarithm of likelihood function for the model, and p represents the number of parameters in the model.

$$\text{LRT} = 2\,log\left(\frac{L_{full}}{L_{reduced}}\right) = 2\left(log(L_{full}) - log(L_{reduced})\right), \qquad (67)$$

where L_{full} and $L_{reduced}$ are the likelihood functions for the full and reduced models, respectively.

For evaluating the selection models to the data, the following criteria are used: the coefficient of determination, R^2, *mean squared error* (MSE), *root mean squared error* (RMSE) and *model efficiency* (ME) as shown in Table 3 according to the following formulas:

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + \sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2} , \qquad (68)$$

$$\text{MSE} = \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n} , \qquad (69)$$

$$\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{n}} , \qquad (70)$$

and

$$\text{ME} = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y})^2} , \qquad (71)$$

where n is the sample size, y_i , $\hat{y}_i$ are the observed and predicted values, respectively, $\bar{y}$ is the mean of observed values, and p is the number of parameters in the model.

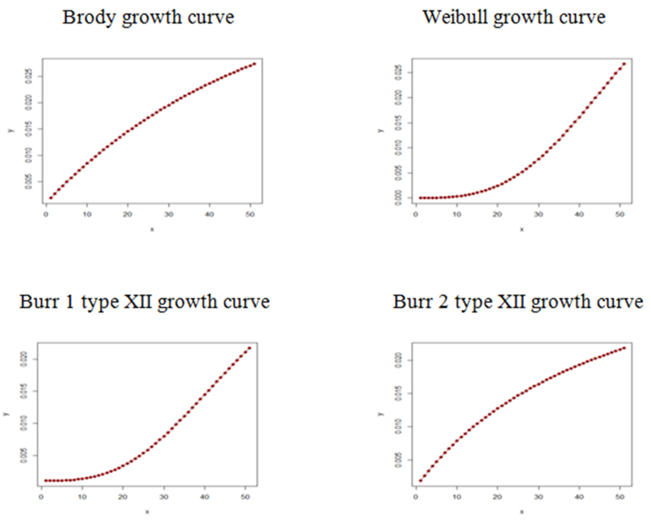

Fig.2. Plots of growth curves with their respective inflection points.

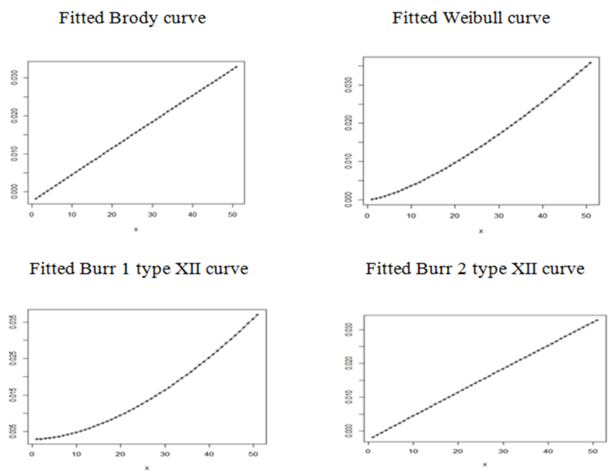

Fig. 3. Plots of the fitted growth curves.

Table1. Parameter estimates, approximate standard errors and confidence intervals of parameters for each model.

Model	Estimation method	parameter	Estimate	ASE	Approximate 95% confidence limits: Lower bound	Upper bound
Burr 1 Type XII	NLS	α	1.3977	30.6130	0	62.9834
		β	0.0029	0.0013	0.0001	0.0057
		k	0.0024	0.0329	0	0.0688
		c	1.7780	0.6564	0.4574	3.0986
	ML	α	0.5316	52.8030	0	104.0200
		β	0.0029	0.3825	-0.7467	0.7525
		k	0.0044	0.3792	0	0.7478
		c	1.8200	71.3600	0	141.7000
Burr 2 Type XII	NLS	α	1.8481	18.2570	0	38.55
		β	-0.0024	0.0015	-0.0006	0.0006
		k	0.0004	0.0038	0	0.0081
	ML	α	3.2040	700.0200	0	1375.2000
		β	-0.0024	0.2858	-0.0563	0.5570
		k	0.0002	0.0480	0	0.0944
Weibull	NLS	α	0.8545	16.1270	0	33.2800
		c	1.4145	0.3257	0.7596	2.0695
		k	0.0021	0.0298	0	0.0621
	ML	α	0.52397	48.8900	0	96.3620
		c	0.0030	0.2450	0	0.4849
		k	1.4243	30.9700	0	62.1200
Brody	NLS	α	1.4497	22.2220	0	46.1310
		β	-0.0024	0.0016	-0.0056	0.0007
		k	0.0005	0.0075	0	0.0156
	ML	α	1.5018	150.9200	0	297.3000
		β	-0.0025	0.2860	-0.5631	0.5581
		k	0.0005	0.0472	0	0.0931

From Table 1, since the estimate of the parameter α indicates the number of confirmed new cases when the maximum rate of growth is reached in the respective stages, the highest value of the upper asymptote α was obtained for the Burr 2 Type XII growth model and the smallest for the Weibull model. Also, the k parameter which indicates the growth rate of confirmed new cases is similar for the Weibull and Burr 1Type XII growth curves by NLS estimation. The largest value of this parameter was obtained for the Weibull model by ML estimation and the smallest for the Burr 2 Type XII model by NLS estimation. On the other hand, the c parameter, which is presented as an adjustment factor, shifts a sigmoidal curve parallel to the time axis; as the value of the parameter is smaller, the curve shifts to the more left side, and vice versa. That is, with a smaller value of c, the model describes a growth curve with a shorter lag period as achieved by Weibull model.

Table 2. Evaluation of AICc and *p-values* of LRT test for Burr 1 Type XII, Burr 2 Type XII, Brody, and Weibull growth models

Model	AICc	*p-value*
Burr 1 Type XII	-442.4	0.0019
Burr 2 Type XII	-423.6	0.0114
Brody	-423.9	0.0114
Weibull	-438.5	0.0058

Table 3. The R^2, MSE, RMSE, and ME for Burr 1 Type XII, Burr 2 Type XII, Brody, and Weibull growth models

Model	**Method**	R^2	**MSE**	**RMSE**	**ME**
Burr 1Type XII	NLS	0.9310	8.96×10^{-6}	0.0029	0.9310
	ML	0.9210	9.62×10^{-6}	0.0031	0.9260
Burr 2Type XII	NLS	0.8950	1.32×10^{-5}	0.0036	0.8950
	ML	0.8820	1.40×10^{-5}	0.0037	0.8900
Brody	NLS	0.8950	1.31×10^{-5}	0.0036	0.8950
	ML	0.8920	1.33×10^{-5}	0.0036	0.8950
Weibull	NLS	0.9280	9.91×10^{-6}	0.0031	0.9220
	ML	0.9230	1.11×10^{-5}	0.0033	0.9190

As observed from Table 2, the LRT is significant ($p - value < 0.05$) in all models, and the model, Burr 1 Type XII with four parameters is the most suitable to describe the growth of confirmed new cases of COVID-19 in Egypt over time since it has the lowest AICc.

Moreover, as observed from Fig. 3 and Table 3, all evaluated models fitted well the investigated curves of confirmed new cases of COVID-19 in Egypt with R^2 and ME values and the Burr 1 Type XII sigmoid growth model is the best since it has the largest value of R^2 and ME and the lowest value of MSE and RMSE specially when NLS estimation is used.

5. Conclusions

In this paper, two sigmoid growth models have been proposed to be able to describe the most diverse situations of growth data. The new proposed models based on the Burr Type XII distribution with two formulas of cdf. Estimating the parameters of the new proposed models were provided by NLS and ML estimation methods. Moreover, the performance of new sigmoid growth models was investigated using daily confirmed new COVID-19 cases in Egypt from March 15, 2020 to May 4, 2020. The results showed that the new proposed model, Burr 1 Type XII sigmoidal growth is superior over the other models with respect to R^2, MSE, RMSE, ME, and AICc especially when NLS estimation is used.

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