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HARMONIC UNIVALENT FUNCTIONS WITH FIXED FINITELY MANY COEFFICIENTS DEFINED BY $q-$ CALCULUS

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Abstract. In the present work by applying $q-$ calculus we investigate a new subclass of harmonic univalent functions with fixed finitely many coefficients. We obtain coefficient conditions, distortion bounds, extreme points, convolution conditions, and convex combinations for this class. Finally, we discuss an integral operator and a $q-$ Jackson type integral operator.

1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a simply-connected domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply-connected domain, we can write $f = h + \overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$, see [5].

Let us express the class $\mathcal{S}_H$ of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in \mathcal{S}_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \; g(z) = \sum_{k=1}^{\infty} b_k z^k, \; |b_1| < 1. \tag{1.1}$$

It is worthy noting that the class $\mathcal{S}_H$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the co-analytic part of its member is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

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Recently, Porwal [15] introduced the subclass $\mathcal{V}_H^n$ of $\mathcal{S}_H$ consisting of functions of form $f = h + \overline{g_n}$, where

$$h(z) = z + \sum_{k=2}^{\infty} |a_k| z^k,\ \ g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k,\ \ |b_1| < 1. \tag{1.3}$$

Recently, by using quantum calculus several subclasses of analytic and harmonic univalent functions were introduced and studied by various researchers. Noteworthy contributions in this direction may be found in [2, 4, 11, 15, 16, 18], see also [1, 19]. First, we recall the definition of q-calculus which was first introduced by Jackson [9, 10]. For $k \in \mathbb{N}$, the $q-$ number is defined as:

$$[k]_q = \frac{1-q^k}{1-q},\ \ 0 < q < 1. \tag{1.4}$$

Hence, $[k]_q = \sum_{i=0}^{k-1} q^i$, when $k \to \infty$ the series converges to $\frac{1}{1-q}$. Also

$$\lim_{q \to 1} [k]_q = k.$$

Jackson [9] (see also [1, 10]) defined $q-$ derivative of a function f in the following way

$$D_q(f(z)) = \frac{f(qz) - f(z)}{(q-1)z},\ \ q \neq 1,\ \ z \neq 0,$$

and $D_q(f(0)) = f'(0)$ provided $f'(0)$ exists.
It is easy to see that for a function $p(z) = z^k$, we observe that

$$D_q(p(z)) = D_q(z^k) = \frac{1-q^k}{1-q} z^{k-1} = [k]_q z^{k-1}.$$

Then

$$\lim_{q \to 1} D_q(p(z)) = p'(z)$$

where p' denotes the ordinary derivative.

Jackson [9] (see also [1, 10]) defined $q-$ Jackson definite integral of the function f as

$$\int_0^z f(t) d_q t = (1-q) z \sum_{n=0}^{\infty} f(zq^n) q^n,\ \ z \in \mathbb{C}.$$

Recently, Porwal [15] introduced the subclass $\mathcal{R}_H(n, q, \beta, \lambda)$ consisting of functions $f = h + \overline{g}$ of the form (1.1) which satisfy the condition

$$\Re \left\{ \frac{\Omega^n \left(D_q(h(z))\right) + (-1)^n \overline{\Omega^n \left(D_q(g(z))\right)}}{z} \right\} < \beta, \tag{1.5}$$

for some $\beta (1 < \beta \leq 2)$, $0 < q < 1$, $\lambda (0 \leq \lambda \leq 1)$, $n \in \mathbb{N}$, $z \in \mathbb{U}$ and the operator Ω^n was introduced by Dixit and Porwal [7].

Further, we let

$$\overline{\mathcal{R}_H}(n, q, \beta, \lambda) \equiv \mathcal{R}_H(n, q, \beta, \lambda) \cap \mathcal{V}_H^n.$$

The study of analytic univalent functions with fixed finitely many coefficients is an interesting topic of research in geometric function theory. Dixit and Mishra [6], Dixit and Verma [8], Kwon [12], Owa and Srivastava [13], Thirupathi [20], Verma and Rosy [21] studied various subclasses of analytic univalent functions

with fixed finitely coefficients and obtained several interesting results. Motivating by the above-mentioned work Ahuja and Jahangiri [3] studied harmonic univalent functions with fixed second coefficients and opened up a new direction of research in the theory of harmonic univalent functions. After the appearance of this article, Pathak et al. [14], Porwal et al. [17] investigate some new subclasses of harmonic univalent functions with fixed finitely many coefficients. In the present paper, by using $q-$ calculus we introduce a new subclass of harmonic univalent function with fixed finitely many coefficients in the following way:
Let $\overline{\mathcal{R}_H}(n, q, \beta, \lambda, c_i)$ denotes the subclass of $\overline{\mathcal{R}}_H(n, q, \beta, \lambda)$ consisting of functions $f_n(z)$ of the form

$$f_n(z) = h(z) + \overline{g_n(z)}, \tag{1.6}$$

where $h(z)$ and $g_n(z)$ are given by

$$h(z) = z + \sum_{i=2}^{l} \frac{c_i(\beta - 1)}{[\phi(i, \lambda)]^n [i]_q} z^i + \sum_{k=l+1}^{\infty} |a_k| z^k, \;\; g_n(z) = (-1)^n \sum_{k=1}^{\infty} |b_k| z^k,$$

here

$$\phi(i, \lambda) = \frac{\Gamma(i+1)\Gamma(1-\lambda)}{\Gamma(i-\lambda)}$$

and $0 \leq c_i \leq 1$, $0 \leq \sum_{i=2}^{l} c_i \leq 1$.
In the present paper, we obtain the coefficient condition, distortion bounds, extreme points, convolution condition, convex combinations, an integral operator, and a $q-$ Jackson type integral operator.

2. Main Results

To prove our main results we shall require the following lemma.

Lemma 2.1. ([15]) Let $f_n = h + \overline{g_n}$ be given by (1.3). Then $f_n \in \overline{\mathcal{R}_H}(n, q, \beta, \lambda)$if and only if

$$\sum_{k=2}^{\infty} [\phi(k, \lambda)]^n [k]_q |a_k| + \sum_{k=1}^{\infty} [\phi(k, \lambda)]^n [k]_q |b_k| \leq \beta - 1. \tag{2.1}$$

In our first theorem, we obtain a necessary and sufficient coefficient bound for harmonic functions in $\overline{\mathcal{R}_H}(n, q, \beta, \lambda, c_i)$.

Theorem 2.1. Let f_n be given by (1.6). Then $f_n \in \overline{\mathcal{R}_H}(n, q, \beta, \lambda, c_i)$, if and only if

$$\sum_{k=l+1}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k, \lambda)]^n [k]_q}{\beta - 1} |b_k| \leq 1 - \sum_{i=2}^{l} c_i, \tag{2.2}$$

where $1 < \beta \leq 2$, $0 < q < 1$, $0 \leq \lambda \leq 1$, $0 \leq c_i \leq 1$, $0 \leq \sum_{i=2}^{l} c_i \leq 1$, $n \in \mathbb{N}$.
The result is sharp.

Proof. Putting

$$|a_i| = \frac{c_i(\beta - 1)}{[\phi(i, \lambda)]^n [i]_q}, \;\; (i = 1, 2, \ldots, l).$$

From Lemma 2.1, we have

$$\sum_{i=2}^{l} c_i + \sum_{k=l+1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1} |b_k| \leq 1,$$

which proves (2.2).
The result is sharp for the function $f_n(z)$ of the form

$$f_n(z) = z + \sum_{i=2}^{l} \frac{c_i(\beta-1)}{[\phi(i,\lambda)]^n [i]_q} z^i + \sum_{k=l+1}^{\infty} \frac{\beta-1}{[\phi(k,\lambda)]^n [k]_q} |x_k| z^k + (-1)^n \sum_{k=1}^{\infty} \frac{\beta-1}{[\phi(k,\lambda)]^n [k]_q} |y_k| \overline{z^k}, \tag{2.3}$$

where $\sum_{k=l+1}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| \leq 1 - \sum_{i=2}^{l} c_i$.

Theorem 2.2. If $f_n \in \overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$, then for $|z| = r < 1$, we have

$$|f_n(z)| \leq (1+|b_1|)r + \sum_{k=2}^{l} |b_k| r^k + \sum_{k=2}^{l} \frac{c_k(\beta-1)}{[\phi(k,\lambda)]^n [k]_q} r^k + \frac{(\beta-1)r^{l+1}}{[\phi(l+1,\lambda)]^n [l+1]_q} \left\{ 1 - \sum_{k=2}^{l} c_k - \sum_{k=1}^{l} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1} |b_k| \right\}$$

and

$$|f_n(z)| \geq (1-|b_1|)r - \sum_{k=2}^{l} |b_k| r^k - \sum_{k=2}^{l} \frac{c_k(\beta-1)}{[\phi(k,\lambda)]^n [k]_q} r^k - \frac{(\beta-1)r^{l+1}}{[\phi(l+1,\lambda)]^n [l+1]_q} \left\{ 1 - \sum_{k=2}^{l} c_k - \sum_{k=1}^{l} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1} |b_k| \right\}.$$

Proof. Let $f_n \in \overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$. Taking the absolute value of f_n, we have

$$
\begin{aligned}
|f_n(z)| \leq & (1+|b_1|)r + \sum_{k=2}^{l} \frac{c_k\,(\beta-1)}{[\phi\,(k,\lambda)]^n\,[k]_q} r^k + \sum_{k=l+1}^{\infty} |a_k| r^k + \sum_{k=2}^{\infty} |b_k| r^k \\
& = (1+|b_1|)r + \sum_{k=2}^{l} |b_k| r^k + \sum_{k=2}^{l} \frac{c_k\,(\beta-1)}{[\phi\,(k,\lambda)]^n\,[k]_q} r^k + \sum_{k=l+1}^{\infty} (|a_k|+|b_k|)\, r^k \\
& = (1+|b_1|)r + \sum_{k=2}^{l} |b_k| r^k + \sum_{k=2}^{l} \frac{c_k\,(\beta-1)}{[\phi\,(k,\lambda)]^n\,[k]_q} r^k + \\
& \frac{(\beta-1) r^{l+1}}{[\phi\,(l+1,\lambda)]^n\,[l+1]_q} \sum_{k=l+1}^{\infty} \frac{[\phi\,(l+1,\lambda)]^n\,[l+1]_q}{\beta-1} (|a_k|+|b_k|)\, r^k \\
& \leq (1+|b_1|)r + \sum_{k=2}^{l} |b_k| r^k + \sum_{k=2}^{l} \frac{c_k\,(\beta-1)}{[\phi\,(k,\lambda)]^n\,[k]_q} r^k + \\
& \frac{(\beta-1) r^{l+1}}{[\phi\,(l+1,\lambda)]^n\,[l+1]_q} \sum_{k=l+1}^{\infty} \frac{[\phi\,(k,\lambda)]^n\,[k]_q}{\beta-1} (|a_k|+|b_k|)\, r^k \\
& \leq (1+|b_1|)r + \sum_{k=2}^{l} |b_k| r^k + \sum_{k=2}^{l} \frac{c_k\,(\beta-1)}{[\phi\,(k,\lambda)]^n\,[k]_q} r^k + \\
& \frac{(\beta-1) r^{l+1}}{[\phi\,(l+1,\lambda)]^n\,[l+1]_q} \left\{ 1 - \sum_{k=2}^{l} c_k - \sum_{k=1}^{l} \frac{[\phi\,(k,\lambda)]^n\,[k]_q}{\beta-1} |b_k| \right\}.
\end{aligned}
$$

Next, we prove the left-hand inequality as

$$
\begin{aligned}
|f_n(z)| \geq & (1-|b_1|)r - \sum_{k=2}^{l} \frac{c_k\,(\beta-1)}{[\phi\,(k,\lambda)]^n\,[k]_q} r^k - \sum_{k=l+1}^{\infty} |a_k| r^k - \sum_{k=2}^{\infty} |b_k| r^k \\
& = (1-|b_1|)r - \sum_{k=2}^{l} |b_k| r^k - \sum_{k=2}^{l} \frac{c_k\,(\beta-1)}{[\phi\,(k,\lambda)]^n\,[k]_q} r^k - \sum_{k=l+1}^{\infty} (|a_k|+|b_k|)\, r^k \\
& = (1-|b_1|)r - \sum_{k=2}^{l} |b_k| r^k - \sum_{k=2}^{l} \frac{c_k\,(\beta-1)}{[\phi\,(k,\lambda)]^n\,[k]_q} r^k - \\
& \frac{(\beta-1) r^{l+1}}{[\phi\,(l+1,\lambda)]^n\,[l+1]_q} \sum_{k=l+1}^{\infty} \frac{[\phi\,(l+1,\lambda)]^n\,[l+1]_q}{\beta-1} (|a_k|+|b_k|)\, r^k \\
& \geq (1-|b_1|)r - \sum_{k=2}^{l} |b_k| r^k - \sum_{k=2}^{l} \frac{c_k\,(\beta-1)}{[\phi\,(k,\lambda)]^n\,[k]_q} r^k - \\
& \frac{(\beta-1) r^{l+1}}{[\phi\,(l+1,\lambda)]^n\,[l+1]_q} \sum_{k=l+1}^{\infty} \frac{[\phi\,(k,\lambda)]^n\,[k]_q}{\beta-1} (|a_k|+|b_k|)\, r^k \\
& \geq (1-|b_1|)r - \sum_{k=2}^{l} |b_k| r^k - \sum_{k=2}^{l} \frac{c_k\,(\beta-1)}{[\phi\,(k,\lambda)]^n\,[k]_q} r^k - \\
& \frac{(\beta-1) r^{l+1}}{[\phi\,(l+1,\lambda)]^n\,[l+1]_q} \left\{ 1 - \sum_{k=2}^{l} c_k - \sum_{k=1}^{l} \frac{[\phi\,(k,\lambda)]^n\,[k]_q}{\beta-1} |b_k| \right\}.
\end{aligned}
$$

In the following theorem, we obtain the extreme points of the closed convex hull of $\overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$ which is denoted by clco$\overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$.

Theorem 2.3. Let $f_n \in \text{clco}\overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$, if and only if

$$f_n(z)=\sum_{k=l}^{\infty} x_k h_k(z)+\sum_{k=1}^{\infty} y_k g_{n_k}(z), \tag{2.4}$$

where $h_l(z)=z+\sum_{i=2}^{l}\frac{c_i(\beta-1)}{[\phi(i,\lambda)]^n[i]_q}z^i$

$$h_k(z) \quad = z+\sum_{i=2}^{l}\frac{c_i(\beta-1)}{[\phi(i,\lambda)]^n[i]_q}z^i+\frac{\left(1-\sum_{i=2}^{l}c_i\right)(\beta-1)}{[\phi(k,\lambda)]^n[k]_q}z^k, \qquad (k=l+1,l+2,\dots)$$

$$g_{n_k}(z) \quad = z+\sum_{i=2}^{l}\frac{c_i(\beta-1)}{[\phi(i,\lambda)]^n[i]_q}z^i+(-1)^n\frac{\left(1-\sum_{i=2}^{l}c_i\right)(\beta-1)}{[\phi(k,\lambda)]^n[k]_q}z^k, \qquad (k=1,2,\dots)$$

where $x_k\ge 0, y_k\ge 0$ and

$$\sum_{k=l}^{\infty}x_k+\sum_{k=1}^{\infty}y_k=1.$$

In particular, the extreme points of $\overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. Suppose that

$$f_n(z)=\sum_{k=l}^{\infty} x_k h_k(z)+\sum_{k=1}^{\infty} y_k g_{n_k}(z),$$

$$f_n(z)=z+\sum_{i=2}^{l}\frac{c_i(\beta-1)}{[\phi(i,\lambda)]^n[i]_q}z^i+\sum_{k=l+1}^{\infty}\frac{\left(1-\sum_{i=2}^{l}c_i\right)(\beta-1)}{[\phi(k,\lambda)]^n[k]_q}x_kz^k+(-1)^n\sum_{k=1}^{\infty}\frac{\left(1-\sum_{i=2}^{l}c_i\right)(\beta-1)}{[\phi(k,\lambda)]^n[k]_q}y_k\overline{z}^k.$$

Therefore, $f_n\in \text{clco}\overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$, since

$$\sum_{i=2}^{l}c_i+\sum_{k=l+1}^{\infty}\frac{[\phi(k,\lambda)]^n[k]_q}{\beta-1}\left[\frac{\left(1-\sum_{i=2}^{l}c_i\right)(\beta-1)}{[\phi(k,\lambda)]^n[k]_q}\right]x_k+\sum_{k=1}^{\infty}\frac{[\phi(k,\lambda)]^n[k]_q}{\beta-1}\left[\frac{\left(1-\sum_{i=2}^{l}c_i\right)(\beta-1)}{[\phi(k,\lambda)]^n[k]_q}\right]y_k$$

$$=\sum_{i=2}^{l}c_i+\left(1-\sum_{i=2}^{l}c_i\right)\left(\sum_{k=l+1}^{\infty}x_k+\sum_{k=1}^{\infty}y_k\right)$$

$$=\sum_{i=2}^{l}c_i+\left(1-\sum_{i=2}^{l}c_i\right)(1-x_l)$$

$$=1-\left(1-\sum_{i=2}^{l}c_i\right)x_l$$

$$\le 1.$$

Conversely, suppose that $f_n=h+\overline{g_n}\in clco\overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$, where h and g_n are given by

$$h(z)=z+\sum_{i=2}^{l}\frac{c_i(\beta-1)}{[\phi(i,\lambda)]^n[i]_q}z^i+\sum_{k=l+1}^{\infty}|a_k|z^k$$

and

$$g_n(z)=(-1)^n\sum_{k=1}^{\infty}|b_k|\overline{z}^k.$$

Since

$$|a_k| \leq \frac{\left(1-\sum_{i=2}^{l} c_i\right)(\beta-1)}{[\phi(k,\lambda)]^n [k]_q},\ (k=l+1,l+2,\ldots)$$

and

$$|b_k| \leq \frac{\left(1-\sum_{i=2}^{l} c_i\right)(\beta-1)}{[\phi(k,\lambda)]^n [k]_q},\ (k=1,2,\ldots).$$

We may set

$$x_k = \frac{[\phi(k,\lambda)]^n [k]_q}{\left(1-\sum_{i=2}^{l} c_i\right)(\beta-1)} |a_k|,\ (k=l+1,l+2,\ldots),$$

$$y_k = \frac{[\phi(k,\lambda)]^n [k]_q}{\left(1-\sum_{i=2}^{l} c_i\right)(\beta-1)} |b_k|,\ (k=l+1,l+2,\ldots),$$

and

$$x_l = 1 - \sum_{k=l+1}^{\infty} x_k - \sum_{k=1}^{\infty} y_k.$$

Then the proof is completed by noting that

$$\begin{aligned}
f_n(z) =& z + \sum_{i=2}^{l} \frac{c_i(\beta-1)}{[\phi(i,\lambda)]^n [i]_q} z^i + \sum_{k=l+1}^{\infty} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k| \overline{z}^k \\
=& h_l(z) + \sum_{k=l+1}^{\infty} \frac{\left(1-\sum_{i=2}^{l} c_i\right)(\beta-1)}{[\phi(k,\lambda)]^n [k]_q} x_k z^k + (-1)^n \sum_{k=1}^{\infty} \frac{\left(1-\sum_{i=2}^{l} c_i\right)(\beta-1)}{[\phi(k,\lambda)]^n [k]_q} y_k \overline{z}^k \\
=& h_l(z) + \sum_{k=l+1}^{\infty} x_k (h_k(z) - h_l(z)) + \sum_{k=1}^{\infty} y_k (g_k(z) - h_l(z)) \\
=& x_l h_l(z) + \sum_{k=l+1}^{\infty} x_k h_k(z) + \sum_{k=1}^{\infty} y_k g_k(z) \\
=& \sum_{k=l}^{\infty} x_k h_k(z) + \sum_{k=1}^{\infty} y_k g_k(z).
\end{aligned}$$

For our next result, we define the convolution of two harmonic functions. For harmonic functions $f_n(z)$ of the form (1.3) and

$$F_n(z) = z + \sum_{k=2}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| \overline{z}^k$$

we define their convolution

$$(f_n * F_n)(z) = f_n(z) * F_n(z) = z + \sum_{k=2}^{\infty} |a_k A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k B_k| \overline{z}^k, \qquad (2.5)$$

using this definition, we obtain the convolution properties of the class $\overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$.

Theorem 2.4. Let $f_n(z)$ given by (1.6) and $F_n(z)$ of the form

$$F_n(z) = z + \sum_{i=2}^{l} \frac{c_i(\beta-1)}{[\phi(i,\lambda)]^n [i]_q} z^i + \sum_{k=l+1}^{\infty} |A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |B_k| \overline{z}^k, \qquad (2.6)$$

be the members of $\overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$, then $(f_n * F_n)(z) \in \overline{\mathcal{R}_H}(n,q,\beta,\lambda,d_i)$, where $d_i = \frac{c_i^2(\beta-1)}{[\phi(i,\lambda)]^n[i]_q}$.

Proof. The convolution of $f_n(z)$ and $F_n(z)$ defined by (1.6) and (2.6), respectively, is given by

$$(f_n * F_n)(z) = z + \sum_{i=2}^{l} \left(\frac{c_i\,(\beta-1)}{[\phi\,(i,\lambda)]^n\,[i]_q} \right)^2 z^i + \sum_{k=l+1}^{\infty} |a_k A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k B_k| \overline{z}^k.$$

This can also be rewritten as

$$(f_n * F_n)(z) = z + \sum_{i=2}^{l} \frac{d_i\,(\beta-1)}{[\phi\,(i,\lambda)]^n\,[i]_q} z^i + \sum_{k=l+1}^{\infty} |a_k A_k| z^k + (-1)^n \sum_{k=1}^{\infty} |b_k B_k| \overline{z}^k,$$

where $d_i = \frac{c_i^2(\beta-1)}{[\phi(i,\lambda)]^n[i]_q}$. Since $0 \le c_i \le 1$, $0 \le \sum_{i=2}^{l} c_i \le 1$, this suggests that $0 \le d_i \le 1$, $0 \le \sum_{i=2}^{l} d_i \le 1$. To show that $(f_n * F_n)(z) \in \overline{\mathcal{R}_H}(n,q,\beta,\lambda,d_i)$ it is sufficient to prove that the coefficients of $(f_n * F_n)(z)$ satisfy the required condition given in Theorem 2.1. For $F_n(z) \in \overline{\mathcal{R}_H}(n,q,\alpha,\lambda,c_i)$, we note that $|A_k| \le 1$ and $|B_K| \le 1$. Now, for the convolution function $(f_n * F_n)(z)$ we have

$$\begin{aligned}
&\sum_{k=l+1}^{\infty} \frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1} |a_k A_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1} |b_k B_k| \\
&\le \sum_{k=l+1}^{\infty} \frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1} |a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1} |b_k| \\
&\le 1 - \sum_{i=2}^{l} c_i, \ \text{(since } f_n \in \overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)), \\
&\le 1 - \sum_{i=2}^{l} d_i, \ \text{(since } 0 \le d_i \le c_i \le 1).
\end{aligned}$$

Therefore $(f_n * F_n)(z) \in \overline{\mathcal{R}_H}(n,q,\beta,\lambda,d_i)$.

Next, we shall discuss the convex combination of the class $\overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$.

Theorem 2.5. The family $\overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$ is closed under convex combination.

Proof. For $j = 1,2,3...$ let $f_{n_j}(z) \in \overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$ where $f_{n_j}(z)$ is given by

$$f_{n_j}(z) = z + \sum_{i=2}^{l} \frac{c_i\,(\beta-1)}{[\phi\,(i,\lambda)]^n\,[i]_q} z^i + \sum_{k=l+1}^{\infty} |a_{k_j}| z^k + (-1)^n \sum_{k=1}^{\infty} |b_{k_j}| \overline{z}^k.$$

Then by Theorem 2.1, we have

$$\sum_{k=l+1}^{\infty} \frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1} |a_{k_j}| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n\,[k]_q}{\beta-1} |b_{k_j}| \le 1 - \sum_{i=2}^{l} c_i.$$

For $\sum_{j=1}^{\infty} t_j = 1$, $0 \le t_j \le 1$, the convex combination of $f_{n_j}(z)$ may be written as

$$\sum_{j=1}^{\infty} t_j f_j(z) = z + \sum_{i=2}^{l} \frac{c_i\,(\beta-1)}{[\phi\,(i,\lambda)]^n\,[i]_q} z^i + \sum_{k=l+1}^{\infty} \left(\sum_{j=1}^{\infty} t_j |a_{k_j}| \right) z^k + (-1)^n \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} t_j |b_{k_j}| \right) \overline{z}^k.$$

Then by Theorem 2.1, we have

$$\begin{aligned}
&\sum_{i=2}^{l} c_i + \sum_{k=l+1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1} \left(\sum_{j=1}^{\infty} t_j |a_{k_j}|\right) + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1} \left(\sum_{j=1}^{\infty} t_j |b_{k_j}|\right) \\
&= \sum_{i=2}^{l} c_i + \sum_{j=1}^{\infty} t_j \left(\sum_{k=l+1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1} |a_{k_j}| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1} |b_{k_j}|\right) \\
&\leq \sum_{i=2}^{l} c_i + \sum_{j=1}^{\infty} t_j \left(1 - \sum_{i=2}^{l} c_i\right) \\
&= 1.
\end{aligned}$$

Therefore

$$\sum_{j=1}^{\infty} t_j f_{n_j}(z) \in \overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i).$$

3. An Integral Operator

Let $f_n(z) = h(z) + \overline{g_n(z)} \in \mathcal{S}_H$ be given by (1.3) then $F_n(z)$ defined by relation

$$F_n(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t) dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g_n(t) dt}, \ (c > -1). \tag{3.1}$$

Theorem 3.1. Let $f_n(z) = h(z) + \overline{g_n(z)} \in \mathcal{S}_H$ be given by (1.6) and $f_n(z) \in \overline{\mathcal{R}_H}(n,q,\beta,\lambda,c_i)$ then $F_n(z)$ be defined by (3.1) also belong to $\overline{\mathcal{R}_H}(n,q,\beta,\lambda,\rho_i)$, where $\rho_i = \frac{c+1}{c+i} c_i$.

Proof.
From representation of $F_n(z)$ we have

$$F_n(z) = z + \sum_{i=2}^{l} \frac{c+1}{c+i} \frac{c_i(\beta-1)}{[\phi(i,\lambda)]^n [i]_q} z^i + \sum_{k=l+1}^{\infty} \frac{c+1}{c+k} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k| \overline{z}^k.$$

The function $F_n(z)$ can also be written as

$$F_n(z) = z + \sum_{i=2}^{l} \frac{\rho_i(\beta-1)}{[\phi(i,\lambda)]^n [i]_q} z^i + \sum_{k=l+1}^{\infty} \frac{c+1}{c+k} |a_k| z^k + (-1)^n \sum_{k=1}^{\infty} \frac{c+1}{c+k} |b_k| \overline{z}^k,$$

where $\rho_i = \frac{c+1}{c+i} c_i$. Since $0 \le c_i \le 1$, $0 \le \sum_{i=2}^{l} c_i \le 1$, this suggests that $0 \le \rho_i \le 1$, $0 \le \sum_{i=2}^{l} \rho_i \le 1$. Now

$$\begin{aligned}
&\sum_{k=l+1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}\left(\frac{c+1}{c+k}|a_k|\right) + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}\left(\frac{c+1}{c+k}|b_k|\right)\\
\le &\sum_{k=l+1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|b_k|\\
\le &1 - \sum_{i=2}^{l} c_i, \text{ (Using inequality (2.2) of Theorem 2.1, we have}\\
\le &1 - \sum_{i=2}^{l} \rho_i, \text{ (since } 0 \le \rho_i \le c_i \le 1).
\end{aligned}$$

Thus $F_n(z) \in \overline{\mathcal{R}_H}(n, q, \beta, \lambda, \rho_i)$.

Definition 3.2. Let $f_n = h + \overline{g_n}$ be defined by (1.3). Then, the q-Jackson integral operator F_{n_q} is defined by the relation

$$F_{n_q}(z) = \frac{[c]_q}{z^{c+1}} \int_0^z t^c h(t) d_q t + \overline{\frac{[c]_q}{z^{c+1}} \int_0^z t^c g(t) d_q t}, \tag{3.2}$$

where $[c]_q$ is the q-number defined by (1.4).

Theorem 3.3. Let $f_n(z) = h(z) + \overline{g_n(z)}$ be given by (1.6) and $f_n(z) \in \overline{\mathcal{R}_H}(n, q, \beta, \lambda, c_i)$. Then $F_{n_q}(z)$ defined by (3.2) is also in the class $\overline{\mathcal{R}_H}(n, q, \beta, \lambda, \epsilon_i)$.

Proof. Let $f_n(z)$ be defined by (1.6) belongs to the class $\overline{\mathcal{R}_H}(n, q, \beta, \lambda, c_i)$. Then by Theorem 2.1, the condition (2.2) is satisfied.

From the representation (3.2) of F_{n_q}, it follows that,

$$F_{n_q}(z) = z + \sum_{i=2}^{l} \frac{[c]_q}{[c+i+1]_q} \frac{c_i(\beta-1)}{[\phi(i,\lambda)]^n [i]_q} z^i + \sum_{k=l+1}^{\infty} \frac{[c]_q}{[k+c+1]_q}|a_k| z^k + (-1)^n \sum_{k=1}^{\infty} \frac{[c]_q}{[k+c+1]_q}|b_k| \overline{z}^k.$$

The function $F_n(z)$ can also be written as

$$F_{n_q}(z) = z + \sum_{i=2}^{l} \frac{\epsilon_i(\beta-1)}{[\phi(i,\lambda)]^n [i]_q} z^i + \sum_{k=l+1}^{\infty} \frac{[c]_q}{[k+c+1]_q}|a_k| z^k + (-1)^n \sum_{k=1}^{\infty} \frac{[c]_q}{[k+c+1]_q}|b_k| \overline{z}^k,$$

where $\epsilon_i = \frac{[c]_q}{[c+i+1]_q} c_i$.

Since

$$[k+c+1]_q - [c]_q = \sum_{i=0}^{k+c} q^i - \sum_{i=0}^{c-1} q^i = \sum_{i=c}^{k+c} q^i > 0$$

$$[k+c+1]_q > [c]_q$$

$$\text{or } \frac{[c]_q}{[k+c+1]_q} < 1.$$

Using the above result, we observe that $0 \le \epsilon_i \le 1$, $0 \le \sum_{i=2}^{l} \epsilon_i \le 1$, since $0 \le c_i \le 1$, $0 \le \sum_{i=2}^{l} c_i \le 1$. Now

$$\sum_{k=l+1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1} \frac{[c]_q}{[k+c+1]_q}|a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1} \frac{[c]_q}{[k+c+1]_q}|b_k|$$

$$\sum_{k=l+1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|a_k| + \sum_{k=1}^{\infty} \frac{[\phi(k,\lambda)]^n [k]_q}{\beta-1}|b_k|$$

$$\leq 1 - \sum_{i=2}^{l} c_i, \text{ (Using inequality (2.2) of Theorem 2.1, we have}$$

$$\leq 1 - \sum_{i=2}^{l} \epsilon_i, \text{ (since } 0 \leq \epsilon_i \leq c_i \leq 1).$$

Thus the proof of Theorem 3.3 is established. □

4. Concluding Remark

In this paper, authors obtain results regarding coefficient inequality, extreme points, and bounds for harmonic functions belonging to the class defined in this article. We also obtain convolution and convex combinations for this class. Finally, we also discuss a class preserving integral operator and $q-$ Jackson type integral operator. We conclude that these results play an important role in the theory of harmonic univalent functions, especially for functions with fixed second coefficients or fixed finitely many coefficients associated with $q-$ calculus.

Competing interests

The authors declare that they have no competing interests.

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