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Research article

Extensions of Two Bivariate Strict Archimedean Copulas

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Abstract: The copula approach provides an option for capturing the structure of dependence between two quantitative variables. This approach is based on special bivariate functions called copulas. In brief, a copula can be presented as a "functional print" of a structure of dependence. Among the numerous types of copulas, the strict Archimedean copulas are the most famous. However, some of them appear to be unfairly neglected and understudied. We attempt to rehabilitate two of them, namely the Nelsen strict Archimedean copulas numbers 10 and 17 by extending their mathematical functionality. For each of them, the extension is made by the thorough addition of a parameter, which reveals to have a significant impact on several important aspects. As the main results, we determine the range of admissible values for the involved parameters. Then, the basic properties of the proposed extended copulas are studied, including symmetry, tail dependence, bounds, and correlation. A focus is put on the tau of Kendall as the main correlation measure. For one of the extended copulas, it is shown that there are some connections with the Frank copula and that the associated tau of Kendall almost reached the perfect interval of $[-1, 1]$, demonstrating the significance of our findings. Graphics, short statistical works, and numerical tables are given to support the theoretical results.

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1. Introduction

The involvement of two related variables is typical in both natural and man-made processes. To correctly capture the structure of dependence among these variables, an appropriate bivariate model must be found. To this end, various approaches are possible. When the variables are quantitative, the copula approach can be used. The minimal background on this approach is now recalled. To begin, the pioneer researcher was Sklar, who proposed the mathematical idea of a copula in 1959 (see [1]). He produced the central theorem on the subject, which today bears his name. Basically, in the bivariate case, this theorem states that a copula is a bivariate function that joins a bivariate distribution function (DF) to its marginal DFs. Since then, the theory and practice of copulas have been growing and are used in many scientific fields (finance, engineering, environmental science, biology, etc.). See [2], [3], [4] and [5] for a contemporary overview.

There are numerous types of bivariate copulas. We may mention the Gaussian copulas, Student copulas, extreme-value copulas, Farlie-Gumbel-Morgenstern (FGM) copulas, and Archimedean copulas. In addition, among the recent refrences, we may mention the new bivariate FGM copulas studied in [6] and [7], the bivariate copulas constructed via a unit Weibull distribution distortion in [8], the bivariate transformation of copulas proposed in [9], the bivariate copulas constructed via the Rüschendorf method in [10], the bivariate copulas based on the counter-monotonic shock method created in [11], the bivariate trigonometric- and hyperbolic-FGM-type copulas developed in [12], the bivariare exponential-type copulas developed in [13], the bivariate circular-linear copulas proposed in [14] and [15], and the bivariate ratio-type copulas elaborated in [16] and [17]. Recent applications of bivariate copulas in applied sectors include [18], [19] and [20].

Archimedean copulas remain the most used since they are simple to handle and implemented in most of the mathematical software (R, Python, SAS, Matlab, Mathematica, etc.). Furthermore, they are capable of modeling a wide variety of structures of dependence and have participated in an extensive range of applications. Surprisingly, despite the diversity of the Archimedean copulas, a few of them are used in nearly all of the existing works. They are the Clayton, Gumbel-Hougaard, Frank, and Joe copulas. See, for instance, [21], [22], [23] and [24]. Many of the other Archimedean copulas have received little attention to date but may be appropriate for various uses and data sets. Among the ”strict” Archimedean copulas, the most famous ones are the 13 presented in [3, Chapter 4]. It is true that the 13 are difficult to consider simultaneously, and it can be challenging to identify the copula that best describes the structure of dependence between two quantitative variables. So only those with the more flexible functionalities are selected, and the others are left behind. However, a closer look at some of these neglected copulas opens a window to increasing their functionalities, making them more attractive for dependence modeling.

This article contributes in this sense by considering two under-explored strict Archimedean copulas presented in [3, Chapter 4, Table 4.1], namely the Nelsen strict Archimedean copula number 10 (NSA10) and the Nelsen strict Archimedean copula number 17 (NSA17). Before going further, an overview of these two copulas is given below.

NSA10 copula: The NSA10 copula is defined by

$$C(x,y) = \left\{ \frac{2}{(2x^{-a}-1)(2y^{-a}-1)+1} \right\}^{1/a}, \quad (x,y) \in [0,1]^2,$$

with $a \in (0,1]$. We may eventually put $C(x,y) = xy$ for $a = 0$. In the Archimedean sense, the NSA10 copula is generated by the following function: $\varphi(u) = \log(2u^{-a}-1)$, $u \in [0,1]$. Clearly, it has not received much attention in the literature. Among the few related references, there is [25] which investigates it from theoretical and practical viewpoints. In [25, Table 2 N 10], it is indicated that the tau of Kendall (to be defined later) can be expressed as

$$\tau = 1 - \frac{2}{a} \int_0^1 u(2-u^a)\log(2u^{-a}-1)du,$$

and, for the admissible values of a, we have $\tau \in [-0.182, 0]$. Furthermore, the corresponding DF of Kendall (to be defined later) is indicated as

$$K(u) = u + \frac{1}{2a}u(2 - u^a)\log(2u^{-a} - 1), \quad u \in [0, 1].$$

It is also known that the NSA10 copula has no (lower and upper) tail dependence. Finally, in [25, Table 5], we see that the NSA10 copula can have quite acceptable fitting performance; this aspect is illustrated for truncated fire insurance claims data.

NSA17 copula: The NSA17 copula is defined by

$$C(x, y) = \left\{1 + \frac{[(1 + x)^{-a} - 1]\,[(1 + y)^{-a} - 1]}{2^{-a} - 1}\right\}^{-1/a} - 1, \quad (x, y) \in [0, 1]^2,$$

with $a \in \mathbb{R}/\{0\}$. We may eventually put $C(x, y) = xy$ for $a = -1$ and $C(x, y) = \min(x, y)$ when $a \to +\infty$. In the Archimedean sense, the NSA17 copula is generated by the following function: $\varphi(u) = -\log\{[(1 + u)^{-a} - 1]/(2^{-a} - 1)\}$, $u \in [0, 1]$. Like the NSA10 copula, it has not received much attention in the literature. Again, the reference [25] is one of the few that investigates it from both theoretical and practical viewpoints. In [25, Table 2 No 17], it is indicated that the tau of Kendall can be expressed as

$$\tau = 1 - \frac{4}{a}\int_0^1 (1 + u)[1 - (1 + u)^a]\log\left[\frac{(1 + u)^{-a} - 1}{2^{-a} - 1}\right]du,$$

and, for the admissible values of a, we have $\tau \in [-0.611, 1]$. Furthermore, the corresponding DF of Kendall is indicated as

$$K(u) = u + \frac{1}{a}(1 + u)[1 - (1 + u)^a]\log\left[\frac{(1 + u)^{-a} - 1}{2^{-a} - 1}\right], \quad u \in [0, 1].$$

Like the NSA10 copula, the NSA17 copula is known to have no tail dependence. Finally, in [25, Table 8], we see that the NSA17 copula is practicable; it is adapted to fit neighboring countries' data.

Hence, this article provides two-parameter extensions of the NSA10 and NSA17 copulas; an additional parameter, denoted by b, is thus put in each of them in such a way to improve their respective functionalities and dependence modeling capabilities. Also, the presence of b allows to significantly extend the admissible values for a, which may include negative values in some cases. After exhibiting wide ranges of values for a and b, we investigate some properties of the proposed extended copulas (symmetry, tail dependence, bounds, correlation, etc.). In particular, the Frank copula is obtained as a limit case of the extended version of the NSA17 copula. In the second part, we illustrate the findings graphically and numerically. Artificial data are simulated and the parameters are estimated to demonstrate the importance of b. Among other results, we show how the ranges of values of the tau of Kendall are expanded in comparison to those of the NSA10 and NSA17 copulas. Also, the DFs of Kendall are displayed. Hence, we provide the theoretical material for the use of new two-parameter extended strict Archimedean copulas in an applied setting.

The rest of the article is organized as follows: Section 2 presents the considered copulas, and our main theoretical results. The graphical and numerical work is developed in Section 3. A conclusion is given in Section 4.

2. Main Theoretical Results

2.1. Some Facts About the Archimedean Copulas

Archimedean copulas are a family of copulas that share certain fundamental characteristics. They offer a variety of options for modeling diverse structures of dependence. From an analytical viewpoint, the strict Archimedean copulas are the easiest to define in terms of support. Their basic definition is recalled below.

Definition 1. *Let us restrict our attention to the absolutely continuous bivariate case. A function $C(x, y)$, $(x, y) \in [0, 1]^2$ is said to be a strict Archimedean copula if and only if it can be expressed as*

$$C(x, y) = \varphi^{-1}\left[\varphi(x) + \varphi(y)\right], \quad (x, y) \in [0, 1]^2,$$

where $\varphi(u)$, $u \in [0, 1]$, denotes a twice differentiable function satisfying the following assumptions:

A1: $\varphi(1) = 0$,

A2: $\lim_{u \to 0} \varphi(u) = +\infty$,

A3: *for any* $u \in [0, 1]$, $\varphi'(u) < 0$,

A4: *for any* $u \in [0, 1]$, $\varphi''(u) \geq 0$,

and $\varphi^{-1}(v)$ with $v \geq 0$ denotes the inverse function of $\varphi(u)$, i.e., $\varphi^{-1}[\varphi(u)] = u$ for any $u \in [0, 1]$. Thus defined, $\varphi(u)$ is called a (strict) generator function (associated with $C(x, y)$).

Hereafter, we will omit the phrase "absolutely continuous" to lighten the writing.

In a general copula setting, the tau of Kendall is defined by

$$\tau = 4E[C(X, Y)] - 1,$$

where (X, Y) is a random vector with a bivariate DF $C(x, y)$ and E is the mathematical expectation operator. The tau of Kendall is a famous correlation measure between X and Y. If $C(x, y)$ is a strict Archimedean copula with a generator function $\varphi(u)$, it is reduced to the following integral formula:

$$\tau = 1 + 4 \int_0^1 \frac{\varphi(u)}{\varphi'(u)} du. \tag{2.1}$$

On the other hand, the DF of Kendall is useful to describe the dependence of a copula. It is defined by

$$K(u) = Pr[C(X, Y) \leq u], \quad u \in [0, 1],$$

where (X, Y) is a random vector with a bivariate DF $C(x, y)$ and Pr is the probability operator. If $C(x, y)$ is a strict Archimedean copula with a generator function $\varphi(u)$, it is expressed as

$$K(u) = u - \frac{\varphi(u)}{\varphi'(u)}, \quad u \in [0, 1]. \tag{2.2}$$

In addition, the DF of Kendall can be used for data generation and is implied in efficient statistical procedures to identify the right Archimedean copula to choose based on data. It is also related to the tau of Kendall with the following formula: $\tau = 3 - 4 \int_0^1 K(u) du$. All the details about Archimedean copulas, tau of Kendall and DF of Kendall can be found in [3].

2.2. *Extended NSA10 Copula*

2.2.1. Presentation

The next result presents an extended version of the NSA10 copula with the addition of a new parameter b and determines the admissible ranges of values for the two involved parameters.

Proposition 2.1. *Let us consider the following bivariate function:*

$$C(x,y) = \left\{ \frac{b+1}{[(b+1)x^{-a} - b][(b+1)y^{-a} - b] + b} \right\}^{1/a}, \quad (x,y) \in [0,1]^2. \tag{2.3}$$

Then, under one of the following conditions:

C1: $b \in (-1, 0]$ *and* $a > 0$,

C2: $b > 0$ *and* $ab \in (0, 1]$,

the function $C(x,y)$ *is a strict Archimedean copula.*

Proof. Let us set

$$\varphi(u) = \log\left[(b+1)u^{-a} - b\right], \quad u \in [0,1],$$

which can be viewed as a two-parameter version of the generator function of the NSA10 copula. It is clear that, under **C1** or **C2**, $\varphi(u)$ is well-defined (except at $u = 0$, but this case will be discussed later). After some algebraic manipulations, the inverse function of $\varphi(u)$ is given by

$$\varphi^{-1}(v) = \left(\frac{b+1}{e^v + b} \right)^{1/a}, \quad v \geq 0.$$

Thanks to $\varphi(u)$ and $\varphi^{-1}(v)$, after some developments, we can write $C(x,y)$ as

$$C(x,y) = \varphi^{-1}\left[\varphi(x) + \varphi(y)\right], \quad (x,y) \in [0,1]^2.$$

Therefore, to demonstrate that $C(x,y)$ is a strict Archimedean copula, it is enough to prove that $\varphi(u)$ satisfies the assumptions **A1**, **A2**, **A3** and **A4** of Definition 1. Let us investigate them, in turns.

For A1: We have

$$\varphi(1) = \log\left[(b+1) \times 1^{-a} - b\right] = \log(1) = 0.$$

For A2: Since $a > 0$ under the two conditions, we have

$$\lim_{u \to 0} \varphi(u) = \lim_{u \to 0} \log\left[(b+1) \times u^{-a} - b\right] = \lim_{u \to 0} -a\log(u) = +\infty.$$

For A3: For any $u \in [0,1]$, we have

$$\varphi'(u) = -\frac{a(b+1)}{u[1 + b(1 - u^a)]}.$$

For any $b > -1$ and $a > 0$, which are implied by **C1** or **C2**, we have $b + 1 > 0$ and $1 - u^a \geq 0$, so $\varphi'(u) < 0$.

For A4: For any $u \in [0,1]$, we have

$$\varphi''(u) = a(b+1)\frac{b+1-(a+1)bu^a}{u^2[1+b(1-u^a)]^2}.$$

Under **C1** and **C2**, it is clear that the denominator term is positive. In order to conclude on the sign of $\varphi''(u)$, let us now investigate the numerator term by distinguishing **C1** and **C2**.

Under C1: Since $b \in (-1,0]$ and $a > 0$, we have $b+1 \geq 0$ and $-(a+1)bu^a \geq 0$. This implies that $\varphi''(u) \geq 0$.

Under C2: Since $b > 0$ and $ab \in (0,1]$ (so $a > 0$), we have $-(a+1)bu^a \geq -(a+1)b$ and

$$a(b+1)[b+1-(a+1)bu^a] \geq a(b+1)(1-ab) \geq 0.$$

Hence, we have $\varphi''(u) \geq 0$.

Hence, for **C1** and **C2**, we have $\varphi''(u) \geq 0$, demonstrating **A4**.

This ends the proof. □

The copula described in Equation (2.6) is called the extended NSA10 (ENSA10) copula. Clearly, by taking $b = 0$, it is reduced to the independence copula, and by taking $b = 1$, it is reduced to the NSA10 copula. To the best of our knowledge, in its general form, the ENSA10 copula is completely new in the literature.

From Proposition 2.1, we see that the inclusion of b has a strong influence on the admissible values for a. Also, the parameter b can take negative values. This suggests improved behavior in terms of dependence as well as more adaptable extended copula functions. These claims will be illustrated in Section 3 with the help of graphics and numerical tables.

2.2.2. Properties

Some immediate functions and properties are described below. To begin, it is clear that the ENSA10 copula is diagonally symmetric: for any $(x,y) \in [0,1]^2$, we have $C(x,y) = C(y,x)$.

By applying classical differentiation rules, the ENSA10 copula density is given by

$$c(x,y) = \frac{\partial}{\partial x \partial y}C(x,y) = \frac{1}{xy[1+b(1-x^a)(1-y^a)]^2}\left[\frac{x^a y^a}{1+b(1-x^a)(1-y^a)}\right]^{1/a} \times$$
$$\{[1+b(1-x^a)][1+b(1-y^a)] - abx^a y^a\}, \quad (x,y) \in [0,1]^2.$$

It is defined such that $C(x,y) = \int_0^x \int_0^y c(u,v)dudv$. The more the copula density has diverse shapes, the more the corresponding copula is able to capture versatile dependence.

On the other hand, the survival ENSA10 copula is obtained as

$$\begin{aligned}\hat{C}(x,y) &= x+y-1+C(1-x,1-y)\\ &= x+y-1+\left\{\frac{b+1}{[(b+1)(1-x)^{-a}-b][(b+1)(1-y)^{-a}-b]+b}\right\}^{1/a},\end{aligned}$$

$(x, y) \in [0, 1]^2$.

It is also a new two-parameter copula to add to the literature. Since there exists (x, y) such that $\hat{C}(x, y) \neq C(x, y)$, the ENSA10 copula is not radially symmetric.

The Fréchet-Hoeffding bounds can be applied. Hence, for any $(x, y) \in [0, 1]^2$, the ENSA10 copula satisfies the min-max-inequalities: $\max(x + y - 1, 0) \leq C(x, y) \leq \min(x, y)$, i.e.,

$$\max(x + y - 1, 0) \leq \left\{ \frac{b + 1}{[(b + 1)x^{-a} - b][(b + 1)y^{-a} - b] + b} \right\}^{1/a} \\ \leq \min(x, y).$$

These bivariate inequalities can be thought of as standalone multivariate analysis tools that are not limited to the copula topic.

Using standard limit techniques, the tail dependence parameters of the ENSA10 copula satisfy

$$\lambda_L = \lim_{x \to 0} \frac{C(x, x)}{x} = 0, \quad \lambda_U = \lim_{x \to 1} \frac{1 - 2x + C(x, x)}{1 - x} = 0.$$

Hence, like the NSA10 copula, the ENSA10 copula has no tail dependence.

The medial correlation of the ENSA10 copula is

$$\begin{aligned} M &= 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1 \\ &= 4\left\{ \frac{b + 1}{[(b + 1)2^a - b][(b + 1)2^a - b] + b} \right\}^{1/a} - 1. \end{aligned}$$

Despite the relative complexity of the roles of a and b, it is quite manageable from a computational perspective.

Based on Equation (2.1), the tau of Kendall of the ENSA10 copula is expressed as

$$\tau = 1 - \frac{4}{a(b + 1)} \int_0^1 u[1 + b(1 - u^a)] \log\left[(b + 1)u^{-a} - b\right] du. \tag{2.4}$$

There is no closed-form expression for this measure except for a few selected values of a and b. For example, by fixing $b = -1/2$, and taking $a = 1$, $a = 2$ and $a = 3$, we get

$$\tau = 1 - \frac{2}{3}[2 - \log(2)], \quad \tau = 1 - \frac{1}{2}[1 + \log(2)], \quad \tau = 1 - \frac{2}{45}[9 + 3\sqrt{3}\pi - \log(512)],$$

respectively.

Based on Equation (2.2), the DF of Kendall associated with the ENSA10 copula is given by

$$K(u) = u + \frac{1}{a(b + 1)} u[1 + b(1 - u^a)] \log\left[(b + 1)u^{-a} - b\right], \quad u \in [0, 1]. \tag{2.5}$$

The ENSA10 copula can serve as a generator of bivariate distributions. Indeed, by considering two univariate DFs, say $F(x)$ and $G(x)$, we define a new bivariate DF by

$$H(x, y) = C[F(x), G(y)]$$

$$= \left\{ \frac{b+1}{[(b+1)F(x)^{-a} - b][(b+1)G(y)^{-a} - b] + b} \right\}^{1/a}, \quad (x, y) \in \mathbb{R}^2.$$

Hence, new bivariate modeling perspectives are given for various purposes (bivariate data fitting, regression, clustering, etc.). We may refer to the survey in [26] for acceptable choices of univariate DFs in a lifetime modeling scenario.

In Subsection 3.1, a focus will be put on the ENSA10 copula density, the possible values of the tau of Kendall and the shape of the DF of Kendall depending on the parameter values.

2.3. Extended NSA17 Copula

2.3.1. Presentation

The next result offers an extended version of the NSA17 copula with the inclusion of a new parameter b and establishes the acceptable ranges of values for the two involved parameters.

Proposition 2.2. *Let us consider the following bivariate function:*

$$C(x, y) = \frac{1}{b} \left[\left\{ 1 + \frac{[(1 + bx)^{-a} - 1]\,[(1 + by)^{-a} - 1]}{(1 + b)^{-a} - 1} \right\}^{-1/a} - 1 \right], \quad (x, y) \in [0, 1]^2. \tag{2.6}$$

Then, under one of the following conditions:

C1: $b > 0$ *and* $a \in \mathbb{R}/\{0\}$*,*

C2: $a \in (-1, +\infty)/\{0\}$ *and* $b \in \left[(a + 1)^{-1/a} - 1, 0\right)$*,*

the function $C(x, y)$ *is a strict Archimedean copula.*

Proof. Let us set

$$\varphi(u) = -\log\left[\frac{(1 + bu)^{-a} - 1}{(1 + b)^{-a} - 1}\right], \quad u \in [0, 1],$$

which can be viewed as a two-parameter version of the generator function of the NSA17 copula. It is clear that, under **C1** or **C2**, $\varphi(u)$ is well-defined (except at $u = 0$, but this case will be discussed later). After some mathematical manipulations, the inverse function of $\varphi(u)$ is given by

$$\varphi^{-1}(v) = \frac{1}{b}\left[\{1 + [(1 + b)^{-a} - 1]e^{-v}\}^{-1/a} - 1\right], \quad v \geq 0.$$

Thanks to $\varphi(u)$ and $\varphi^{-1}(v)$, after some algebraic developments, we can write $C(x, y)$ as

$$C(x, y) = \varphi^{-1}\left[\varphi(x) + \varphi(y)\right], \quad (x, y) \in [0, 1]^2.$$

Therefore, to demonstrate that $C(x, y)$ is a strict Archimedean copula, it is enough to prove that $\varphi(u)$ satisfies the assumptions **A1**, **A2**, **A3** and **A4** of Definition 1. Let us investigate them, in turns.

For A1: We have

$$\varphi(1) = -\log\left[\frac{(1 + b)^{-a} - 1}{(1 + b)^{-a} - 1}\right] = -\log(1) = 0.$$

For A2: We have

$$\lim_{u\to 0}\varphi(u) = \lim_{u\to 0} -\log\left[\frac{(1+bu)^{-a}-1}{(1+b)^{-a}-1}\right] = \lim_{x\to 0} -\log(x) = +\infty.$$

For A3: For any $u \in [0,1]$, we have

$$\varphi'(u) = -\frac{ab}{(1+bu)[(1+bu)^a - 1]}.$$

Under **C1** and **C2**, since $b > -1$, it is clear that $1 + bu > 0$. In order to conclude on the sign of $\varphi'(u)$, let us now study the other main terms by distinguishing **C1** and **C2**.

Under C1: Under this condition, we have $b > 0$. Let us now distinguish the cases $a > 0$ and $a < 0$.

Case $a > 0$: We have $-ab < 0$ and $(1+bu)^a \geq 1$, implying that $\varphi'(u) < 0$.

Case $a < 0$: We have $-ab > 0$ and $(1+bu)^a \leq 1$, implying that $\varphi'(u) < 0$.

Under C2: Under this condition, we have $b \in \left[(a+1)^{-1/a} - 1, 0\right)$ (so $b < 0$). Let us now distinguish the cases $a > 0$ and $a \in (-1, 0)$.

Case $a > 0$: We have $-ab > 0$, $(1+bu)^a \leq 1$, implying that $\varphi'(u) < 0$.

Case $a \in (-1,0)$: We have $-ab < 0$, $(1+bu)^a \geq 1$, implying that $\varphi'(u) < 0$.

Hence, for **C1** and **C2**, we have $\varphi'(u) < 0$, demonstrating **A3**.

For A4: For any $u \in [0,1]$, we have

$$\varphi''(u) = ab^2 \frac{(a+1)(1+bu)^a - 1}{(1+bu)^2[(1+bu)^a - 1]^2}.$$

Under **C1** and **C2**, it is clear that the denominator term is positive. In order to conclude on the sign of $\varphi''(u)$, let us now investigate the numerator term by distinguishing **C1** and **C2**.

Under C1: Under this condition, we have $b > 0$. Let us now distinguish the cases $a > 0$ and $a < 0$.

Case $a > 0$: To begin, we can write $(a+1)(1+bu)^a - 1 = a(1+bu)^a + (1+bu)^a - 1$. We have $ab^2 > 0$, $a(1+bu)^a > 0$ and $(1+bu)^a \geq 1$, implying that $\varphi''(u) \geq 0$.

Case $a < 0$: Again, we can write $(a+1)(1+bu)^a - 1 = a(1+bu)^a + (1+bu)^a - 1$. We have $ab^2 < 0$, $a(1+bu)^a < 0$ and $(1+bu)^a \leq 1$, implying that $\varphi''(u) \geq 0$.

Under C2: Under this condition, we have $b \in \left[(a+1)^{-1/a} - 1, 0\right)$ (so $b < 0$). Let us now distinguish the cases $a > 0$ and $a \in (-1, 0)$.

Case $a > 0$: We have $(1+bu)^a \geq (1+b)^a \geq (a+1)^{-1}$, which implies that

$$ab^2[(a+1)(1+bu)^a - 1] \geq ab^2[(a+1)(1+b)^a - 1] \geq 0.$$

It follows that $\varphi''(u) \geq 0$.

Case $a \in (-1,0)$: We have $(1+bu)^a \leq (1+b)^a \leq (a+1)^{-1}$, and we still have

$$ab^2[(a+1)(1+bu)^a - 1] \geq ab^2[(a+1)(1+b)^a - 1] \geq 0.$$

Hence $\varphi''(u) \geq 0$.

Hence, for **C1** and **C2**, we have $\varphi''(u) \geq 0$, demonstrating **A4**.

This ends the proof. □

The copula described in Equation (2.6) is called the extended NSA17 (ENSA17) copula. Clearly, by taking $b = 1$, it is reduced to the NSA17 copula. To the best of our knowledge, in its general form, the ENSA17 copula does not appear in the literature.

From Proposition 2.2, we see that the inclusion of b has a strong influence on the admissible values for a. Furthermore, b can take negative values. This implies more functionalities for the related functions as well as enhanced behavior in terms of dependence. These claims will be illustrated in Section 3 with the help of graphics and numerical tables.

2.3.2. Properties

Some immediate functions and properties are described below. To begin, it is clear that the ENSA17 copula is diagonally symmetric: for any $(x, y) \in [0, 1]^2$, we have $C(x, y) = C(y, x)$.

By setting $a = \theta/b$, the generator function of the ENSA17 copula becomes

$$\varphi(u) = -\log\left[\frac{(1 + bu)^{-\theta/b} - 1}{(1 + b)^{-\theta/b} - 1}\right], \quad u \in [0, 1],$$

and it satisfies

$$\lim_{b\to 0} \varphi(u) = -\log\left(\frac{e^{-\theta u} - 1}{e^{-\theta} - 1}\right).$$

The limit function corresponds to the generator function of the Frank copula with parameter θ (see [3]). In this sense, under the considered configuration, the Frank copula can be viewed as a limit case of the ENSA17 copula.

The ENSA17 copula density can be calculated using some differentiation techniques by

$$\begin{aligned}
c(x, y) &= \frac{\partial}{\partial x \partial y} C(x, y) \\
&= \frac{1}{(1 + bx)(1 + by)\{(1 + bx)^a(1 + by)^a - (1 + b)^a(1 + bx)^a - (1 + b)^a(1 + by)^a + (1 + b)^a\}^2} \times \\
&\quad b(1 + b)^a \left\{1 + \frac{[(1 + bx)^{-a} - 1][(1 + by)^{-a} - 1]}{(1 + b)^{-a} - 1}\right\}^{-1/a} \times \\
&\quad \{(1 + b)^a[(1 + bx)^a - 1][(1 + by)^a - 1] + a[(1 + b)^a - 1](1 + bx)^a(1 + by)^a\}, \quad (x, y) \in [0, 1]^2.
\end{aligned}$$

We have $C(x, y) = \int_0^x \int_0^y c(u, v)dudv$. As usual, the panel of shapes of the copula density is determinant for understanding the modeling dependence capacity of the ENSA17 copula. This aspect will be investigated later.

The survival ENSA17 copula is obtained as

$$\begin{aligned}
\hat{C}(x, y) &= x + y - 1 + C(1 - x, 1 - y) \\
&= x + y - 1 + \frac{1}{b}\left[\left\{1 + \frac{\{[1 + b(1 - x)]^{-a} - 1\}\{[1 + b(1 - y)]^{-a} - 1\}}{(1 + b)^{-a} - 1}\right\}^{-1/a} - 1\right],
\end{aligned}$$

$(x, y) \in [0, 1]^2$.

Additionally, it adds a novel two-parameter copula to the body of knowledge. Since there exists (x, y) such that $\hat{C}(x, y) \neq C(x, y)$, the ENSA17 copula is not radially symmetric.

The Fréchet-Hoeffding bounds are satisfied, that is, for any $(x, y) \in [0, 1]^2$, we have $\max(x + y - 1, 0) \leq C(x, y) \leq \min(x, y)$, i.e.,

$$\begin{aligned}\max(x+y-1,0) &\leq \frac{1}{b}\left[\left\{1+\frac{[(1+bx)^{-a}-1]\,[(1+by)^{-a}-1]}{(1+b)^{-a}-1}\right\}^{-1/a}-1\right]\\ &\leq \min(x,y).\end{aligned}$$

These bivariate inequalities can be viewed as independent multivariate analysis tools.

Using standard limit techniques, the tail dependence parameters of the ENSA17 copula satisfy

$$\lambda_L = \lim_{x\to 0}\frac{C(x,x)}{x} = 0, \quad \lambda_U = \lim_{x\to 1}\frac{1-2x+C(x,x)}{1-x} = 0.$$

Hence, the ENSA17 copula has no tail dependence.

The medial correlation of the ENSA17 copula is

$$\begin{aligned}M &= 4C\left(\frac{1}{2},\frac{1}{2}\right)-1\\ &= \frac{4}{b}\left[\left\{1+\frac{\left[(1+2^{-1}b)^{-a}-1\right]\left[(1+2^{-1}b)^{-a}-1\right]}{(1+b)^{-a}-1}\right\}^{-1/a}-1\right]-1.\end{aligned}$$

Despite the relative complexity of the roles played by a and b, it is relatively manageable from a computational perspective.

Based on Equation (2.1), the tau of Kendall of the ENSA17 copula is expressed as

$$\tau = 1-\frac{4}{ab}\int_0^1 (1+bu)[1-(1+bu)^a]\log\left[\frac{(1+bu)^{-a}-1}{(1+b)^{-a}-1}\right]du. \tag{2.7}$$

There is no closed-form expression for the integral term, but it can be studied numerically.

Based on Equation (2.2), the DF of Kendall associated with the ENSA17 copula is given by

$$K(u) = u+\frac{1}{ab}(1+bu)[1-(1+bu)^a]\log\left[\frac{(1+bu)^{-a}-1}{(1+b)^{-a}-1}\right], \quad u\in[0,1]. \tag{2.8}$$

Like the ENSA10 copula, the ENSA17 copula can serve as a bivariate distribution generator. Indeed, by considering two univariate DFs, say $F(x)$ and $G(x)$, we define a new bivariate DF by

$$\begin{aligned}H(x,y) &= C[F(x),G(y)]\\ &= \frac{1}{b}\left[\left\{1+\frac{\{[1+bF(x)]^{-a}-1\}\{[1+bG(y)]^{-a}-1\}}{(1+b)^{-a}-1}\right\}^{-1/a}-1\right], \quad (x,y)\in\mathbb{R}^2.\end{aligned}$$

As a result, fresh viewpoints on bivariate modeling are provided for various statistical uses.

In Subsection 3.2, a focus will be put on the ENSA17 copula density, the possible values of the tau of Kendall and the forms of the DF of Kendall depending on the parameter values.

3. Graphical and Numerical Work

In this section, we provide additional contributions to the ENSA10 and ENSA17 copulas by plotting graphics illustrating the validity of Propositions 2.1 and 2.2, as well as the shapes of their respective copula densities and DFs of Kendall. We also perform short statistical works showing the importance of the newly added parameter b and numerical studies on the possible values of the tau of Kendall for the two copulas. This graphical and numerical work is made using the software R (see [27]).

3.1. On the ENSA10 Copula

Let us consider the ENSA10 copula as defined in Equation (2.3) and the involved parameter conditions **C1** and **C2** in Proposition 2.1. To begin, Figures 1 and 2 display the graphics of the ENSA10 copula for $b = -1/2$ and $a = 1$, satisfying **C1**, and for $b = 2$ and $a = 1/2$ satisfying **C2**, respectively.

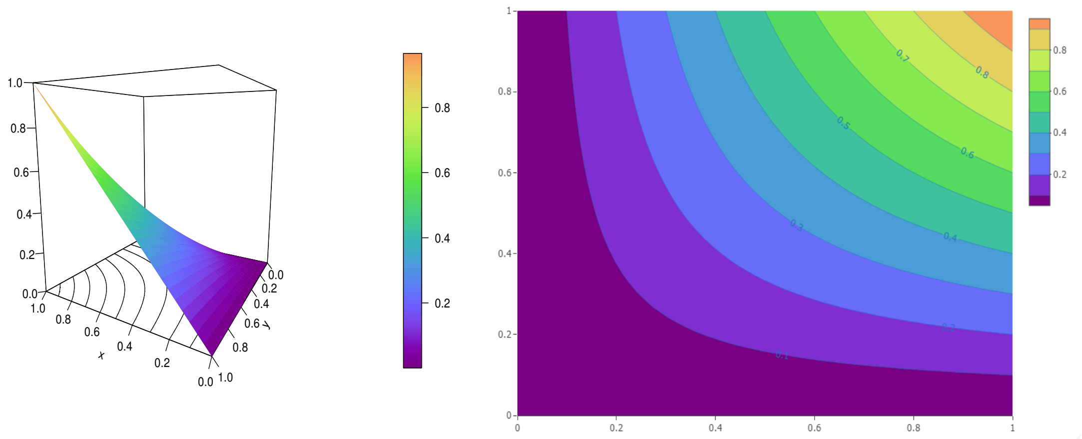

Figure 1. Graphics of the ENSA10 copula for $b = -1/2$ and $a = 1$: standard (left) and intensity/contour (right)

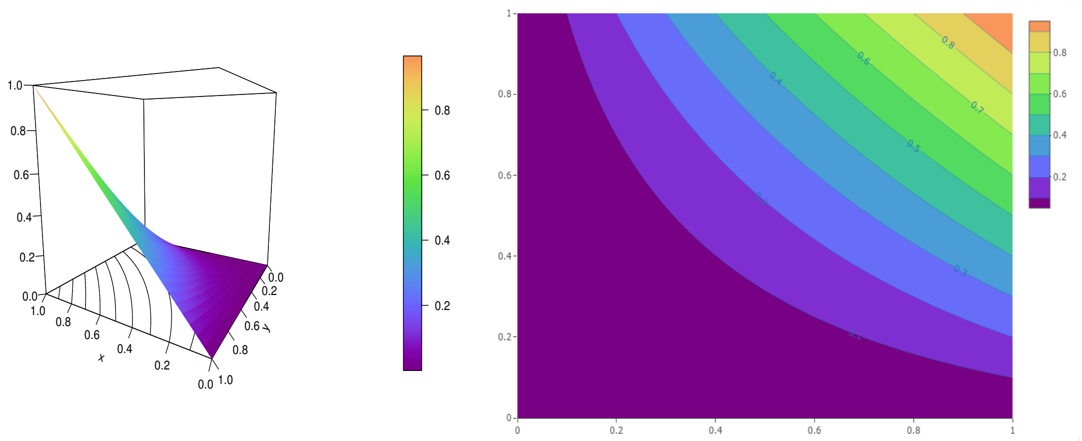

Figure 2. Graphics of the ENSA10 copula for $b = 2$ and $a = 1/2$: standard (left) and intensity/contour (right)

From these figures, the classical shape characteristics of a copula can be seen. This illustrates

Proposition 2.1 for the considered parameter values.

Figures 3 and 4 display the graphics of the ENSA10 copula density under the same parameter configurations as above, i.e., for $b = -1/2$ and $a = 1$, satisfying **C1**, and for $b = 2$ and $a = 1/2$ satisfying **C2**, respectively.

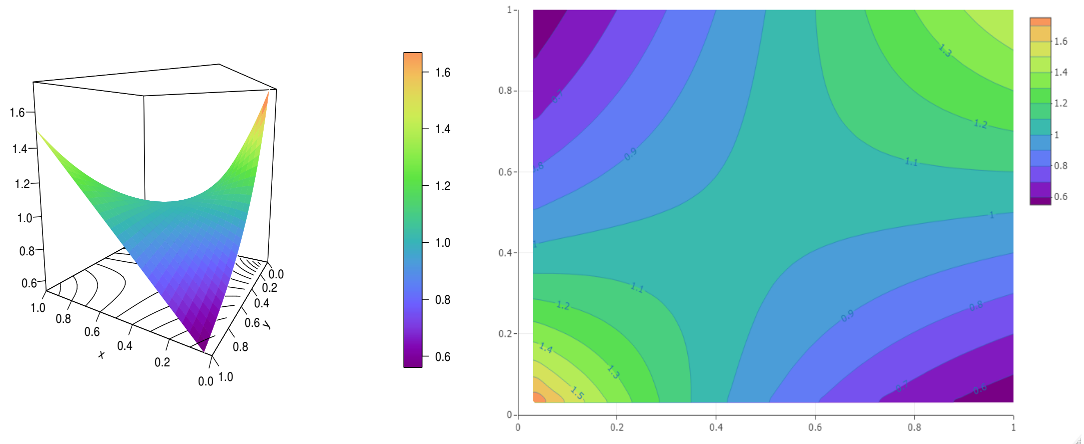

Figure 3. Graphics of the ENSA10 copula density for $b = -1/2$ and $a = 1$: standard (left) and intensity/contour (right)

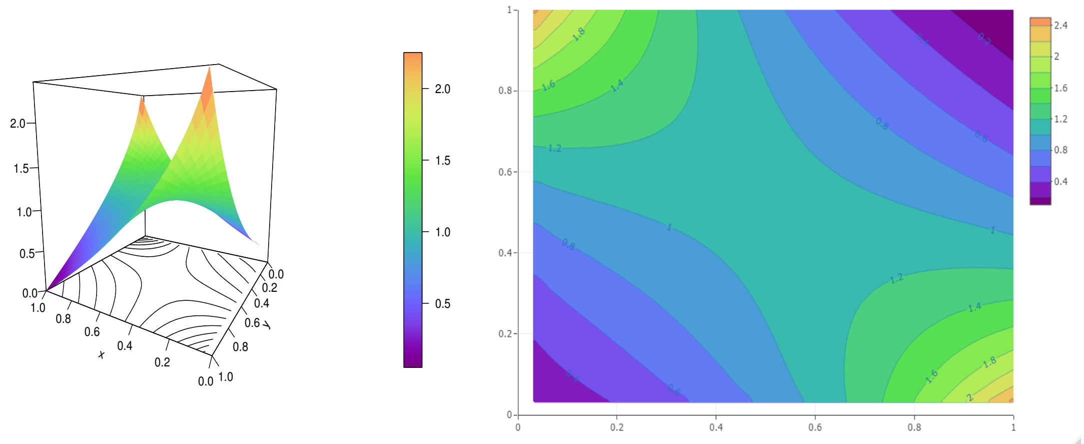

Figure 4. Graphics of the ENSA10 copula density for $b = 2$ and $a = 1/2$: standard (left) and intensity/contour (right)

We see a wide range of shapes that validate the interest of the ENSA10 copula for dependence modeling purposes.

A quick statistical and graphical analysis is now complete to highlight the significance of the findings and the relevance of the parameter b. We generate two pairs of $n = 1000$ related artificial data following the two different schemes:

Artificial dataset 1: $(u_1, \cos^2(\pi u_1) + v_1), \ldots, (u_n, \cos^2(\pi u_n) + v_n)$,

Artificial dataset 2: $(u_1, \exp[\cos(\pi u_1)] + v_1), \ldots, (u_n, \exp[\cos(\pi u_n)] + v_n)$, where, for any $i = 1, \ldots, n$, u_i and v_i are the observations of two independent random variables U and V that follow the standard normal distribution. For each data set, we estimate the parameters a and b by the omnibus method (or pseudo-likelihood method) (see [28]), and show how the shape of the ENSA10 copula density defined with these estimated parameters behaves.

Figures 5 and 6 display the scatterplots of the data and the shape of the estimated ENSA10 copula density for Artificial datasets 1 and 2, respectively.

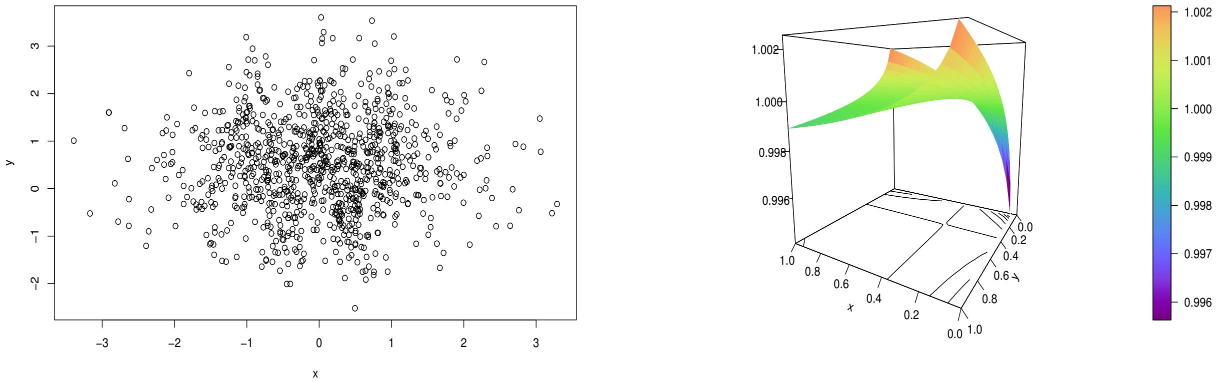

Figure 5. Scatterplot of the data for Artificial dataset 1 (left) and graphic of the ENSA10 copula density at the estimated parameters: $a = 5.51608312$ and $b = -0.02608018$, satisfying **C1** (right)

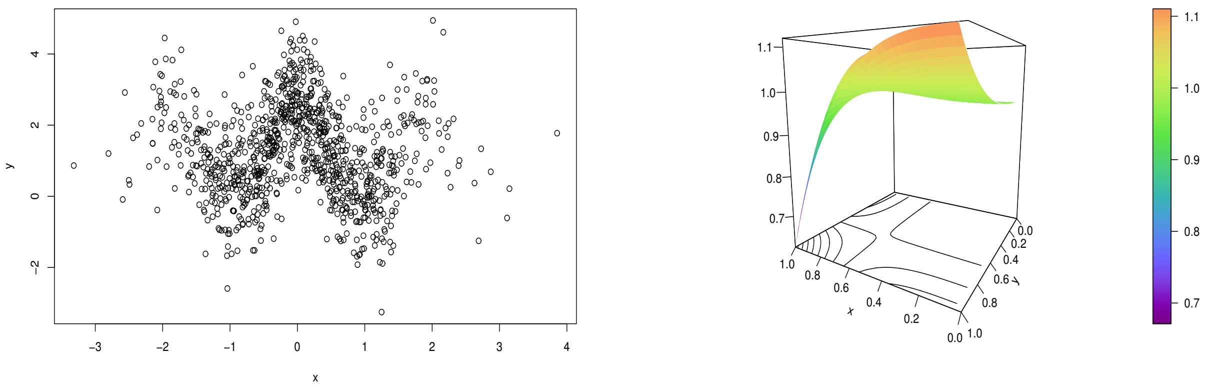

Figure 6. Scatterplot of the data for Artificial dataset 2 (left) and graphic of the ENSA10 copula density at the estimated parameters: $a = 3.1783218$ and $b = 0.1195665$, satisfying **C2** (right)

These figures show how flexible the ENSA10 copula density is in capturing the structure of dependence between two quantitative variables. We can note that the estimated copula related to Figure 5 is close to the independence copula in view of its range of values, which is not surprising in view of the relative homogeneousness of the set of points in the scatterplot.

We now explore the possible range of values for the tau of Kendall of the ENSA10 copula as defined in Equation (2.4). Table 1 determines the numerical values of this correlation measure for various parameter values.

Table 1. Values of the tau of Kendall of the ENSA10 copula for some parameter values

$a =\rightarrow$	0.1	0.6	1.1	1.6	2.1	2.6	3.1	3.6	4.1	4.6	5.1	5.6
$b = -0.9$	0.0426	0.1998	0.2931	0.3471	0.3765	0.3904	0.3948	0.3931	0.3878	0.3803	0.3714	0.3618

$a =\rightarrow$	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
$b = 2$	−0.0455	−0.0836	−0.116	−0.1438	−0.1678	−0.1887	−0.207	−0.223	−0.237	−0.2494

This table indicates that we can have $\tau \in [-0.25, 0.36]$, which significantly improves the range of the tau of Kendall obtained by the standard NSA10 copula.

Finally, in Figure 7, we illustrate the dependence versatility of the ENSA10 copula by plotting the associated DF of Kendall given in Equation (2.5), for various admissible values of the parameters.

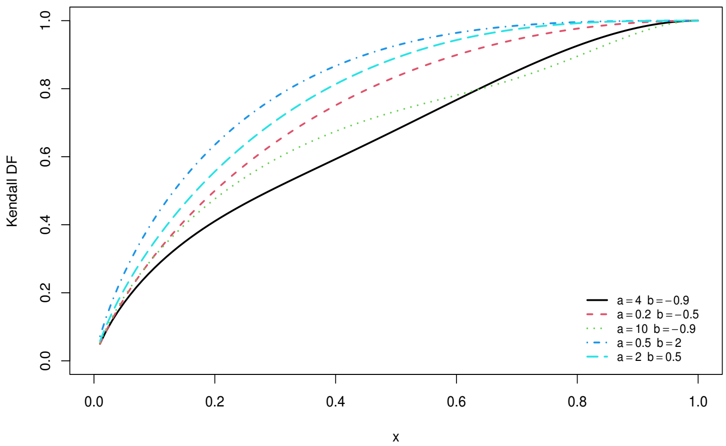

Figure 7. Graphics of the DF of Kendall of the ENSA10 copula for various admissible values of the parameters

Nuanced kinds of shapes (visually concave, or almost) are observed, and they are clearly influenced by the parameter b, supporting the interest of the ENSA10 copula for the bivariate dependence modeling.

3.2. *On the ENSA17 Copula*

Let us consider the ENSA17 copula as defined in Equation (2.6), Proposition 2.2, and the involved parameter conditions **C1** and **C2**. To begin, Figures 8 and 9 display the graphics of the ENSA17 copula for $b = 2$ and $a = 1$, satisfying **C1**, and for $b = -3/4$ and $a = -1/2$ satisfying **C2**, respectively.

From these figures, the classical shape characteristics of a copula can be seen. This illustrates Proposition 2.2 for the considered parameter values.

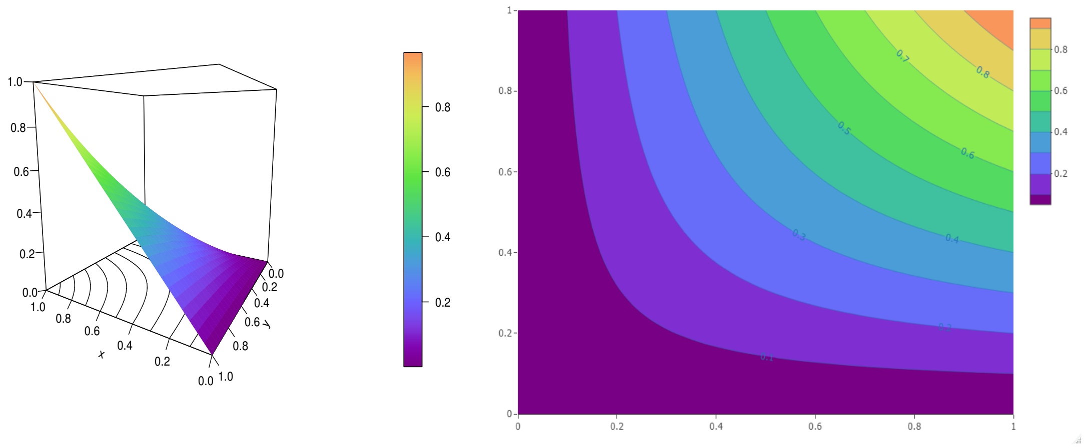

Figure 8. Graphics of the ENSA17 copula for $b = 2$ and $a = 1$: standard (left) and intensity/contour (right)

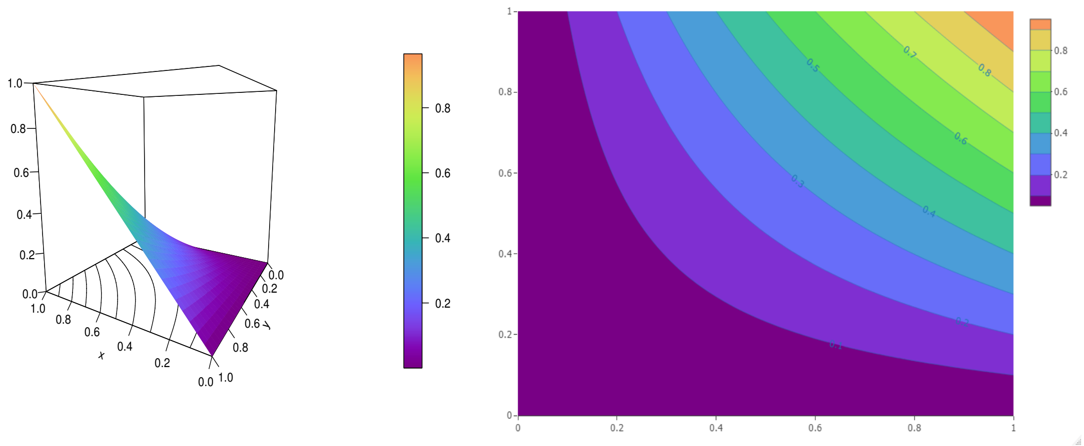

Figure 9. Graphics of the ENSA17 copula for $b = -3/4$ and $a = -1/2$: standard (left) and intensity/contour (right)

Figures 10 and 11 display the graphics of the ENSA17 copula density under the same parameter configurations than above, i.e., for $b = 2$ and $a = 1$, satisfying **C1**, and for $b = -3/4$ and $a = -1/2$ satisfying **C2**, respectively.

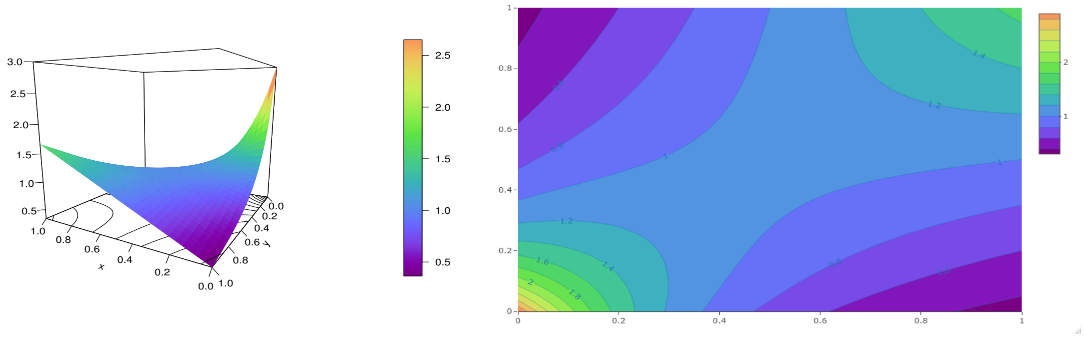

Figure 10. Graphics of the ENSA17 copula density for $b = 2$ and $a = 1$: standard (left) and intensity/contour (right)

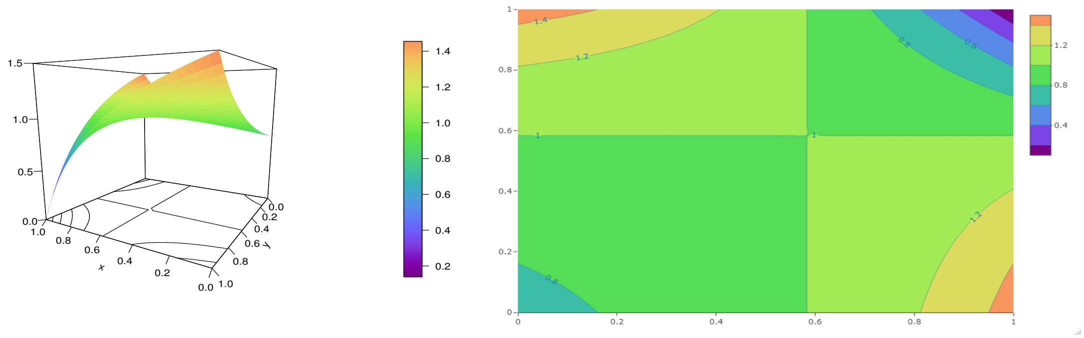

Figure 11. Graphics of the ENSA17 copula density for $b = -3/4$ and $a = -1/2$: standard (left) and intensity/contour (right)

Completely different shapes are observed, demonstrating versatile functionalities.

In order to illustrate the importance of the findings and the role of the parameter b in particular, a short statistical and graphical work is now done. We generate two pairs of $n = 1000$ related artificial data with the following configurations:

Artificial dataset 1: $(u_1, \cos(\pi u_1) + v_1), \ldots, (u_n, \cos(\pi u_n) + v_n)$,

Artificial dataset 2: $(u_1, \cos^2(\pi u_1) + v_1), \ldots, (u_n, \cos^2(\pi u_n) + v_n)$,

where, for $i = 1, \ldots, n$, u_i and v_i are the observations of two independent random variables U and V that follow the standard normal distribution.

For each data set, we estimate the parameters a and b by the omnibus method and show how the shapes of the ENSA17 copula density defined with these estimated parameters are affected.

Figures 12 and 13 display the scatterplots of the data and the shape of the estimated ENSA17 copula density for Artificial datasets 1 and 2, respectively.

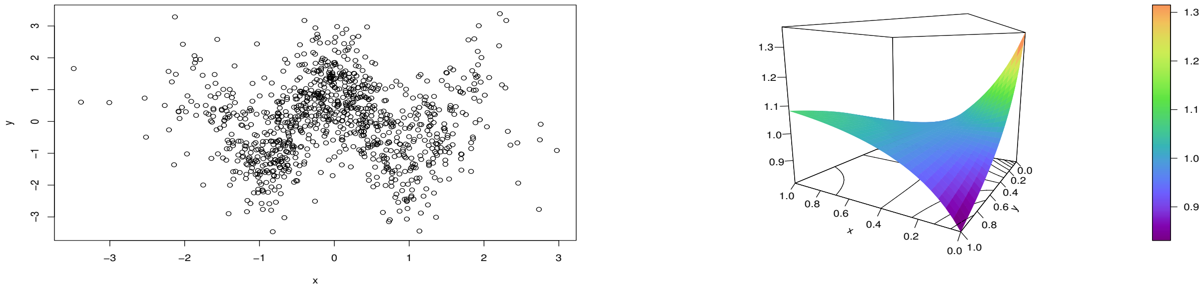

Figure 12. Scatterplot of the data for Artificial dataset 1 (left) and graphic of the ENSA17 copula density at the estimated parameters: $a = -0.6504814$ and $b = 3.4228576$, satisfying **C1** (right)

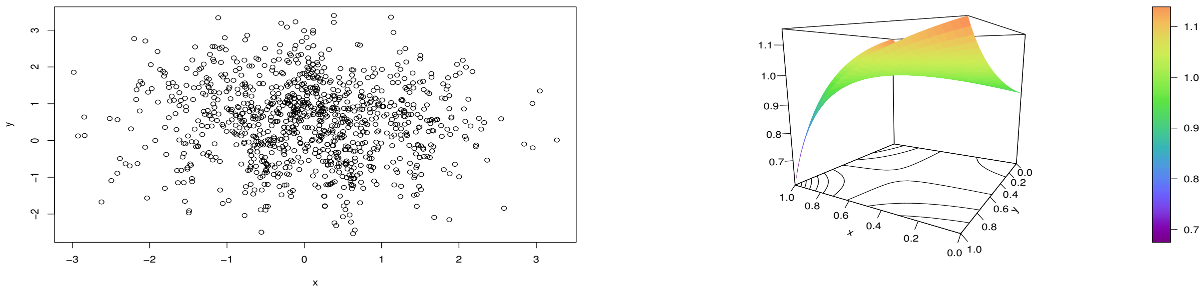

Figure 13. Scatterplot of the data for Artificial dataset 2 (left) and graphic of the ENSA17 copula density at the estimated parameters: $a = -0.8501710$ and $b = -0.7840553$, satisfying **C2** (right)

Thus, these figures illustrate the adaptability of the ENSA17 copula density to capture the dependence structure between two quantitative variables at its best.

We now explore the possible range of values for the tau of Kendall of the ENSA17 copula as defined in Equation (2.7). Table 2 determines the numerical values of this correlation measure for various parameter values.

In particular, this table indicates that, for $b = 0.05$, then we can have $\tau \in [-0.97, 1]$, which significantly improves the range of the tau of Kendall obtained by the standard NSA17 copula. One can note that this correlation's performance is similar to that of the Frank copula, which is semi-surprising due to the relation that exists between them when a is large and b is small.

Finally, we demonstrate the flexibility of the ENSA17 copula in Figure 14 by plotting the associated DF of Kendall given in Equation (2.8) for a range of admissible parameter values.

It is evident that the parameter b has an impact on a variety of shapes (some of which are visually concave or nearly so), which supports the utility of the ENSA17 copula in the bivariate dependence modeling.

Table 2. Values of the tau of Kendall of the ENSA17 copula for some parameter values

$a =\rightarrow$	−10000	−1000	−100	−50	−12	−2	2	12	50	100	1000	10000
$b = 0.05$	−0.9601	−0.8923	−0.4361	−0.2473	−0.0587	−0.0054	0.0161	0.0694	0.2592	0.4505	0.9384	0.9994

$a =\rightarrow$	−1000	−100	−50	−12	−2	2	12	50	100	1000
$b = 0.5$	−0.7498	−0.6928	−0.6324	−0.3478	−0.0398	0.1219	0.4574	0.8181	0.911	0.9994
$b = 1$	−0.6109	−0.5861	−0.5588	−0.399	−0.0606	0.1931	0.6161	0.8872	0.9778	0.9998

$a =\rightarrow$	−0.9	−0.7	−0.5	−0.3	−0.1
$b = -0.6$	−0.0121	−0.0365	−0.0613	−0.0863	−0.1116

4. Conclusion

Bivariate copula creation is crucial for the objectives of dependence-type modeling. In this article, we offered theoretical developments in the area of the creation of strict Archimedean copulas. The functional scope of two understudied copulas of this type was in fact expanded. The key innovation remains the accurate placement of a parameter denoted by b to improve their modeling capabilities. These extended copulas, as well as the admissible values of their parameters, are listed in Table 3.

Table 3. The two extended strict Archimedean copulas studied in this article.

Names	$C(x, y)$	Parameter conditions
ENSA10	$\left\{\frac{b+1}{[(b+1)x^{-a}-b][(b+1)y^{-a}-b]+b}\right\}^{1/a}$	$b \in (-1, 0]$, $a > 0$ $b > 0$, $ab \in (0, 1]$
ENSA17	$\frac{1}{b}\left[\left\{1+\frac{[(1+bx)^{-a}-1]\,[(1+by)^{-a}-1]}{(1+b)^{-a}-1}\right\}^{-1/a}-1\right]$	$b > 0$, $a \in \mathbb{R}/\{0\}$ $a \in (-1, +\infty)/\{0\}$, $b \in \left[(a+1)^{-1/a}-1, 0\right)$

In addition to the admissible values of the two involved parameters, we discussed the main properties of the ENSA10 and ENSA17 copulas. In a nutshell, in addition to being strict Archimedean, these

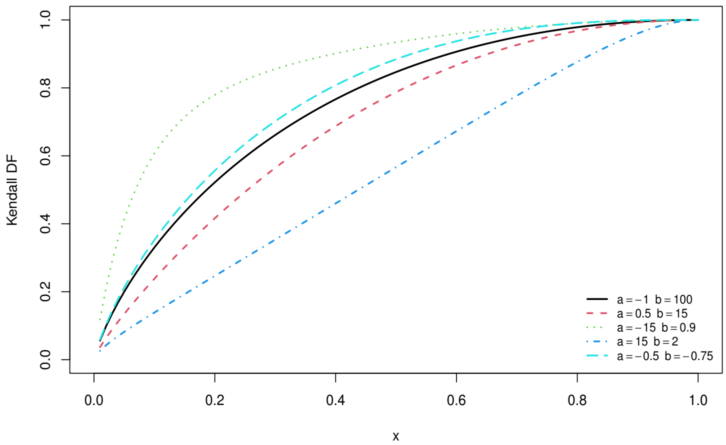

Figure 14. Graphics of the DF of Kendall of the ENSA17 copula for various admissible values of the parameters

copulas are diagonally symmetric, are not radially symmetric, satisfy the Fréchet-Hoeffding bounds, have no tail dependence, have manageable medial correlation, have univariate expression of the tau of Kendall, have a simple DF of Kendall and can serve to generate bivariate distributions. It is also proved that the ENSA17 copula has the Frank copula as the limit case under a special parameter configuration. A numerical work has illustrated the shapes of the copulas, copula density functions, and DFs of Kendall, and a short statistical work has demonstrated the usefulness of the parameter extension strategy.

New bivariate copulas can be directly derived from the ENSA10 and ENSA17 copulas, such as their x-flipping and y-flipping versions (see [3]). For particular applied scenarios involving the dependence of two quantitative variables, each of them might be of interest. So, this work presents fresh viewpoints in this sense. On the other hand, the idea of extending existing generator functions and the related Archimedean copulas with the addition of one or more parameters can also be inspiring. For broader perspectives, the multivariate versions of the ENSA10 and ENSA17 copulas are also of interest. For any integer $n \geq 3$, they can be logically defined as

$$C(x_1, \ldots, x_n) = \left\{ \frac{b+1}{\prod_{i=1}^{n}[(b+1)x_i^{-a} - b] + b} \right\}^{1/a}, \quad (x_1, \ldots, x_n) \in [0,1]^n$$

and

$$C(x_1, \ldots, x_n) = \frac{1}{b}\left[\left\{ 1 + \frac{\prod_{i=1}^{n}[(1+bx_i)^{-a} - 1]}{[(1+b)^{-a} - 1]^{n-1}} \right\}^{-1/a} - 1 \right], \quad (x_1, \ldots, x_n) \in [0,1]^n,$$

respectively, but the new admissible ranges of values of a and b remain to be discovered. In addition, the use of the proposed copulas for modern data analysis is another logical angle of view, with the development of packages, and others. However, this applied component necessitates a high level of practical understanding, which we leave to subsequent research.

Conflict of interest

The authors state that they have no financial or other conflicts of interest to disclose with connection to this research.

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