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A UNIQUENESS PROBLEM OF MEROMORPHIC FUNCTIONS WITH NORMAL FAMILIES

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Abstract. In this paper, we study the uniqueness of a transcendental meromorphic function contributing to a meromorphic function in sync with its first derivative and a linear differential polynomial of first order with two constant coefficients using the theory of normal families. Our result generalizes and supplements some previous results given by Jank-Mues-Volkmann [10], Chang-Fang [4], Chang [5], Lahiri-Ghosh [12], L*ü*-Yi [13] and L*ü*-Xu [15]. We also provide examples to demonstrate the correctness of our results.

1. Introduction, Definitions and Results

Consider that all the functions in this paper are in $\mathbb{C}$. The notations $m(r,f), N(r,f), T(r,f), m(r,\frac{1}{f-a}), N(r,\frac{1}{f-a})$.... are used here and are from the Nevanlinna value distribution theory; for references, see [9, 11, 24, 25]. We took it for granted that the reader is already familiar with all the notations. Here, $S(r,f)$ is used to represent any quantity that, possibly outside of a set with finite logarithmic measure, satisfies the formula $S(r,f) = o(T(r,f))$ as being $r \to \infty$.

Now, we recall the symbol $\rho(f)$ for the growth order of a meromorphic function f that are defined as follows:

$$\rho(f) = \limsup_{r\to\infty} \frac{\log^+ T(r,f)}{\log r}.$$

Assume that f and g are two non-constant meromorphic functions and that τ is a function or a finite complex value. If f and g have same τ-points with same multiplicity (neglect multiplicity), we say f and g share τ with CM (IM) and denoted by $f(z) = \tau(z) \rightleftharpoons g(z) = \tau(z)$ $(f(z) = \tau(z) \Leftrightarrow g(z) = \tau(z))$. If τ points of g whenever τ points of f, this is indicated by the symbol $f(z) = \tau(z) \Rightarrow g(z) = \tau(z)$. Consider the case where R is a rational function that asymptotically acts as cr^{α}, $r \to \infty$, where $c(\neq 0)$, and α are constants. The degree of R at the point of infinity is defined by $\deg_{\infty} R(z) = \max\{0, \alpha\}$.

In 1977, Rubel-Yang [22] first prove a sharing value problem for entire function. They proved: a non-constant entire function f and a, b two distinct finite values, if $f(z) = a(z) \rightleftharpoons f'(z) = a(z)$, $f(z) = b(z) \rightleftharpoons f'(z) = b(z)$, then $f(z) = f'(z)$ for all $z \in \mathbb{C}$. Following that, numerous authors looked into the sharing value problem for entire or meromorphic functions and came up with many significant results, see [1, 8, 15, 19].

In the theory of complex analytic functions, the normality criterion is an important part of the families of meromorphic functions. In 1907, Paul Montel's first introduced the notion of normal families. In the sense of Montel's, let Ω be a domain in $\mathbb{C}$ and let F be a family of holomorphic functions. The family F is said to be normal in Ω if every sequence $\{\xi_n\} \subseteq F$ contains either a subsequence which converges to a limit function $\xi(\not\equiv \infty)$ uniformly on each compact subset of Ω, or a subsequence which converge uniformly to ∞ on each compact subset.

To establish our theorem, we need the definitions listed below:

Definition 1.1. *Let $a \in \mathbb{C}$, we denote by $N_{(2}(r, \frac{1}{f-a})$ the counting function of those a-points of f whose multiplicities are not less than 2 where each a-point is counting according to their multiplicity.*

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Definition 1.2. *Let $a, b \in \mathbb{C}$, we denote by $N(r, f = a | g \neq b)$ the counting function of those a-points of f, counted according to their multiplicity, which is not the b-points of g.*

Definition 1.3. *Let $a, b \in \mathbb{C}$, we denote by $N(r, f = a | g = b)$ the counting function of those a-points of f, counted according to their multiplicity, which are the b-points of g.*

In 1986, Jank-Mues-Volkmann [10] examined the following theorem.

Theorem A. [10] Let f be an entire function, and a be a finite non-zero value. If

$$f(z) = a \Leftrightarrow f'(z) = a, \; f(z) = a \Rightarrow f''(z) = a,$$

then $f(z) = f'(z)$ for all $z \in \mathbb{C}$.

In 2002, Chang-Fang [4] investigated Theorem A, restoring a non-zero value a by a polynomial function z, and the result is as follows.

Theorem B. [4] Let f be a non-constant entire function. If

$$f(z) = z \Leftrightarrow f'(z) = z, \; f'(z) = z \Rightarrow f''(z) = z,$$

then $f(z) = f'(z)$ for all $z \in \mathbb{C}$.

In 2003, Chang [5] developed Theorem B, reconstitute z by a small function α as follows.

Theorem C. [5] Let f be a non-constant entire function and α be a meromorphic function satisfying $T(r, \alpha) = S(r, f)$ and $\alpha \neq \alpha'$. If

$$f(z) = \alpha(z) \Leftrightarrow f'(z) = \alpha(z), \; f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z),$$

then $f(z) = f'(z)$ for all $z \in \mathbb{C}$.

In 2009, Lahiri-Ghosh [12] reform Theorem C, substituting a first-degree polynomial for α, and obtain the following theorem.

Theorem D. [12] Let f be a non-constant entire function and $a(z) = \alpha z + \beta$, where $\alpha(\neq 0)$ and β are constants. If

$$f(z) = a(z) \Rightarrow f'(z) = a(z), \; f'(z) = a(z) \Rightarrow f''(z) = a(z),$$

then, either (i) $f(z) = Ae^z$, or (ii) $f(z) = \alpha z + \beta + (\alpha z + \beta - 2\alpha)e^{\frac{\alpha z + \beta - 2\alpha}{\alpha}}$ for all $z \in \mathbb{C}$.

In 2010, L$\ddot{u}$-Yi [13] proved the following important theorem, reconstruct a by a rational function and entire function by a transcendental meromorphic function in Theorem D.

Theorem E. [13] Let f be a non-constant transcendental meromorphic function with finitely many poles, and let R be a non-zero rational function. If

$$f(z) = R(z) \Rightarrow f'(z) = R(z), \; f'(z) = R(z) \Rightarrow f''(z) = R(z),$$

then, (i) $f(z) = f'(z)$, or (ii) $f'(z) = A[R(z) - R'(z)]e^z + R'(z)$ for all $z \in \mathbb{C}$, where A is a non-zero constant.

In 2012, L$\ddot{u}$-Xu [15] improved Theorem D and E and derive the next following result.

Theorem F. [15] Let f be a non-constant entire function, and let $\alpha = Pe^Q$ $(\alpha \neq \alpha')$ be an entire function satisfying $\rho(\alpha) < \rho(f)$, where $P(\neq 0)$ and Q are polynomials. If

$$f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z), \; f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z),$$

then, (i) $f(z) = f'(z)$, or (ii) $f'(z) = A[\alpha(z) - \alpha'(z)]e^z + \alpha'(z)$ for all $z \in \mathbb{C}$, and α reduces to a polynomial, where A is a non-zero constant.

The following questions emerge from Theorem F:

(1) Can we change the entire function with a meromorphic function?

(2) Is it possible to change Pe^Q to Re^P, where P, Q and R stand for polynomial and rational functions, respectively?

(3) Can we replace f'' with a first-order linear differential polynomial in f?

Using the idea of normal families, we analyze all of these concerns and produce a uniqueness theorem. Here, we use the notation

$$L(f, z) = af'(z) + bf(z), \tag{1.1}$$

where $a, b(\neq 0)$ are constants. We will now demonstrate the next theorem.

Theorem 1.1. *Let f be a non-constant transcendental meromorphic function with finitely many poles. Let $\tau = Re^P$ $(\tau \neq \tau')$ be a meromorphic function satisfying $\rho(\tau) < \rho(f)$, where R is a non-zero rational function and P is a polynomial function of degree n. Let $L(f, z)$ be defined as in (1) and $a + b \neq 1$. If*

$$f(z) = \tau(z) \Rightarrow f'(z) = \tau(z), \; f'(z) = \tau(z) \Rightarrow L(f, z) = \tau(z),$$

subsequently, (A) and (B) cases must materialize:

(A) *When* $a = 0$.
  (a) $f(z) = c_0 e^z$ *for all* $z \in \mathbb{C}$, *where* c_0 *is a non-zero constant and* $n = 0$.
  (b) $f(z) = \tau(z) + c_1 \tau(z)^{-(\frac{b}{1-b})} e^{(\frac{b}{1-b})z}$ *for all* $z \in \mathbb{C}$, *where* c_1 *is a non-zero constant and* τ *reduces to a rational function.*

(B) *When* $a \neq 0$.
  (c) $n = 0$,
    (1c) $f(z) = c_2 e^z$ *for all* $z \in \mathbb{C}$, *where* c_2 *is a non-zero constant.*
    (2c) $f(z) = R_3(z) + c_3 R_3(z)^{(\frac{b}{a+b-1})} e^{-(\frac{b}{a+b-1})z}$ *for all* $z \in \mathbb{C}$, *where* c_3 *is a non-zero constant.*

  (d) $n \geq 1$,
    (1d) $f(z) = \tau(z) + c_4 \tau(z)^{(\frac{b}{a+b-1})} e^{-(\frac{b}{a+b-1})z}$ *for all* $z \in \mathbb{C}$, *where* c_4 *is a non-zero constant and* τ *reduces to a rational function.*

Remark 1.1. *The requirement* $a + b \neq 1$ *is an essential case in this theorem. Otherwise, the assumption for* $b \neq 0$ *is* f, *and* f' *shares IM with a value of* τ. *It does not contain any* f *values.*

Remark 1.2. *The condition* f *is transcendental and cannot be ignored in Theorem* 1.1. *The following example demonstrates this.*

Example 1.1. *Let* $f(z) = cz^3 + z^2$ *and* $\tau(z) = z^2$ *and* $L(f,z) = 2z^2(1-z)$. *Then it is not difficult to prove that if* $f'(z) = \tau(z)$ *whenever* $f(z) = \tau(z)$ *and if* $L(f,z) = \tau(z)$ *whenever* $f'(z) = \tau(z)$. *But,* $f(z) \neq f'(z)$ *for all* $z \in \mathbb{C}$. *We have* $a = 0, b = 2, c = -1$.

Remark 1.3. *The following four examples demonstrate that the cases (b), (*$2c$*), and (*$1d$*) cannot be deleted.*

Example 1.2. *Let* $f(z) = z^4 + c_1 z^2 e^{-\frac{z}{2}}$ *and* $\tau(z) = z^4$ *and* $L(f,z) = -z^4 - c_1 z^2 e^{-\frac{z}{2}}$. *It is easy to deduce that if* $f'(z) = \tau(z)$ *whenever* $f(z) = \tau(z)$ *and if* $L(f,z) = \tau(z)$ *whenever* $f'(z) = \tau(z)$, *we have* $a = 0, b = -1, c_1 = -32e^2$. *Thus the case (b) occured.*

Example 1.3. *Let* $f(z) = (z+1)^2 + c_1(z+1)^4 e^{-2z}$ *and* $\tau(z) = (z+1)^2$ *and* $L(f,z) = 2(z+1)^2 + 2c_1(z+1)^4 e^{-2z}$. *It satisfied the assumption if* $f'(z) = \tau(z)$ *whenever* $f(z) = \tau(z)$ *and if* $L(f,z) = \tau(z)$ *whenever* $f'(z) = \tau(z)$, *we have* $a = 0, b = 2, c_1 = -\frac{1}{8}e^2$. *Thus the case (b) occured.*

Example 1.4. *Let* $f(z) = z^2 + c_3 z^6 e^{-3z}$ *and* $R_3(z) = z^2$ *and* $L(f,z) = z(3z-2) + 6c_3 z^5 e^{-3z}(z-1)$. *It is easy to see that if* $f'(z) = \tau(z)$ *whenever* $f(z) = \tau(z)$ *and if* $L(f,z) = \tau(z)$ *whenever* $f'(z) = \tau(z)$, *we have* $a = -1, b = 3, c_3 = -\frac{1}{48}e^6$. *Thus the case (*$2c$*) occured.*

Example 1.5. *Let* $f(z) = (z+1)^3 + c_4(z+1)^6 e^{-2z}$ *and* $\tau(z) = (z+1)^3$ *and* $L(f,z) = (-2z^3 + 6z + 4) + c_4(-6z^6 - 24z^5 - 30z^4 + 30z^2 + 24z + 6)e^{-2z}$. *It is confirm the assumption if* $f'(z) = \tau(z)$ *whenever* $f(z) = \tau(z)$ *and if* $L(f,z) = \tau(z)$ *whenever* $f'(z) = \tau(z)$, *we have* $a = 2, b = -2, c_4 = -\frac{1}{54}e^4$. *Thus the case (*$1d$*) occured.*

Remark 1.4. *The condition* $\rho(\tau) < \rho(f)$ *takes an important part of our theorem. In Theorem* 1.2, *we will show that* $\rho(\tau) \leq \rho(f) \leq n$. *So, our condition is significant.*

Remark 1.5. *We attach two examples to demonstrate that when the constraint* $\rho(\tau) < \rho(f)$ *is changed to* $\rho(f) = \rho(\tau)$ *and the supporting conditions remain the same, the cases (b) and (*$1d$*) cannot be satisfied.*

Example 1.6. *Let* $f(z) = z^2 e^{2z} + c_1 z^4 e^{2z}$ *and* $\tau(z) = z^2 e^{2z}$ *and* $L(f,z) = 2z^2 e^{2z} + 2c_1 z^4 e^{2z}$. *Then, the assumption if* $f'(z) = \tau(z)$ *whenever* $f(z) = \tau(z)$ *and if* $L(f,z) = \tau(z)$ *whenever* $f'(z) = \tau(z)$ *holds, we have* $a = 0, b = 2, c_1 = -\frac{1}{8}$.

Example 1.7. *Let* $f(z) = z^4 e^{3z} + c_4 z^8 e^{4z}$ *and* $\tau(z) = z^4 e^{3z}$ *and* $L(f,z) = (\frac{3}{2}z^4 + 2z^3)e^{3z} + c_4(4z^7 + 2z^8)e^{4z}$. *Then, the assumption if* $f'(z) = \tau(z)$ *whenever* $f(z) = \tau(z)$ *and if* $L(f,z) = \tau(z)$ *whenever* $f'(z) = \tau(z)$ *occured, we have* $a = \frac{1}{2}, b = 1, c_4 = -\frac{1}{128}e^2$.

Remark 1.6. *This paper's key ideas are based on the* [16, 21].

We need f to be of finite order in order to prove the Theorem 1.1. We can acquire the result of independent interest by using normal families.

Theorem 1.2. *Let* f *be a non-constant transcendental meromorphic function with finitely many poles. Let* $\tau = Re^P(\tau \neq \tau')$ *be a meromorphic function, where* $R(\neq 0)$ *be a rational function and* P *be a polynomial function of degree* n. *If*

$$f(z) = \tau(z) \Rightarrow f'(z) = \tau(z),\ f'(z) = \tau(z) \Rightarrow L(f,z) = \tau(z),$$

then f *is of order atmost* n.

Remark 1.7. *In a related study, if* τ *is substituted with the* k*-th derivative* $\tau^{(k)}$, *the Theorem* 1.1 *remains valid.*

Remark 1.8. *The proof of Theorem* 1.2 *is based on* [6, 14, 18].

2. Some Important lemmas

We use some essential lemmas to show the Theorems 1.1 and 1.2. For convenience, we recall a few lemmas that play a crucial role in the argument.

Normal families are consistently used in operator theory on the space of holomorphic function. For example, {[23], lemma 3} and {[17], lemma 4}. The next lemma recover from famous Pang-Zalcman {[20], lemma 2}, Lu-Xu-Chen {[14], lemma 2.1}, respectively. This plays a significant part in a proof of the Theorem 1.1.

Lemma 2.1. [14, 20] *Let $\{f_n\}$ be a family of meromorphic (analytic) functions in the disc $D = \{z : |z| < 1\}$. If $(a_n) \to a$, $|a| < 1$, and $f_n^{\#}(a_n) \to \infty$, and if there exists $L \geq 1$ such that $|f_n'(z)| \leq L$, whenever $f_n(z) = 0$, then there exist*
(i) a subsequence of f_n (which we still write as f_n),
(ii) points $(z_n) \to z_0$, $|z_0| < 1$,
(iii) positive numbers $\rho_n \to 0$,
such that $\rho_n^{-1} f_n(z_n + \rho_n\zeta) = g_n(\zeta) \to g(\zeta)$ locally uniformly, where g is a non-constant meromorphic (resp. entire) function on $\mathbb{C}$, such that $\rho(g) \leq 2$ $(resp,\ \rho(g) \leq 1)$, $g^{\#}(\zeta) \leq g^{\#}(0) = L+1$, and

$$\rho_n \leq \frac{M_1}{f_n^{\#}(a_n)},$$

where M_1 is a constant which is independent of n. Here using $g^{\#}(\zeta) = \frac{g'(\zeta)}{1+|g(\zeta)|^2}$ is the spherical derivative.

Lemma 2.2. *{[7], Lem. 3} Let f be an entire function with $\rho(f) > 1$, then for each $0 < N < \rho(f) - 1$, there exist points $a_n \to \infty (n \to \infty)$, such that*

$$\lim_{n\to\infty} \frac{f^{\#}(a_n)}{|a_n|^N} = \infty.$$

Lemma 2.3. *{[25], Thm. 1.14} Suppose f and h be two non-constant meromorphic functions in the complex plane $\mathbb{C}$ with $\rho(f)$ and $\rho(h)$ as their orders, respectively. Then*

$$\rho(f.h) \leq \max(\rho(f), \rho(h)),$$
$$\rho(f + h) \leq \max(\rho(f), \rho(h)),$$

means the orders of products and sums of meromorphic functions are less than equal to the maximal order of the two functions.

Lemma 2.4. *Let f and τ be two non-constant meromorphic functions in the complex plane $\mathbb{C}$ with $\rho(f)$ as the order of f and $\rho(\tau)$ as the order of τ. If $\rho(\tau) < \rho(f)$, then $T(r_n, \tau) = o(T(r_n, f))$, as $n \to \infty$, where a set $J = (r_n), 1 \leq r_n \leq \infty$, as $r_n \to \infty$.*

Proof. From the definition of the order of meromorphic function in the complex plane $\mathbb{C}$, there exists a sequence $(r_n) \to \infty$, as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{\log^+ T(r_n, f)}{\log r_n} = \rho(f).$$

Now we take $0 < \varepsilon < k$, where $k = \frac{\rho(f)-\rho(\tau)}{2}$. Therefore, for any number $\varepsilon \in (0, k)$, there exists a positive integer l such that

$$T(r_n, f) \geq r_n^{\rho(f)-\varepsilon}$$

for any $n > l$. Also, for any number $\varepsilon \in (0, k)$, there exists a positive integer m such that

$$T(r_n, \tau) \leq r_n^{\rho(\tau)+\varepsilon}$$

for any $n > m$. Now, for any $n > \max\{l, m\}$, we have

$$\lim_{n\to\infty} \frac{T(r_n, \tau)}{T(r_n, f)} \leq \lim_{n\to\infty} r_n^{\rho(\tau)-\rho(f)+2\varepsilon} \leq \lim_{n\to\infty} r_n^{-2(k-\varepsilon)} = 0.$$

This conclusion the result holds. □

In Lemma 2.4, τ is referred to as a small function of f on J, and we define it as $T(r, \tau) = S(r, f)$, $(r \in J)$.

Lemma 2.5. *{[9], pg. 60} Let f be a transcendental meromorphic function, and let a be a non-zero value. Then, for each positive integer k, either f or $f^{(k)} - a$ has infinitely many zeros.*

Lemma 2.6. *{[2], Lem. 2} Let f be a transcendental meromorphic function such that $f(z_0) \neq \infty$ at $z_0 = 0$ and the set of finite critical and asymptotic values of f is bounded. Then there exists $R_1 > 0$ such that*

$$|f'| \geq \frac{|f|}{2\pi|z|} \log \frac{|f|}{R_1},$$

for all $z \in \mathbb{C}\backslash\{0\}$ which are not poles of f.

Lemma 2.7. *{[3], Cor. 3} Let f be a meromorphic function with finite order. If f has only finitely many critical values, then it has only finitely many asymptotic values.*

The following lemma is the result of L*ü*-Yi [13], and it plays a crucial role in the proof of Theorem 1.1.

Lemma 2.8. *{[13]. Lem. 2.6} Let R and H be two non-zero rational functions, Q be a polynomial, and F be a transcendental meromorphic function with finite order. If F is a solution of the following differential equation*

$$\frac{F'}{F} - \frac{H}{F} = Re^Q,$$

then Q reduces to a constant.

3. **Proof of Theorem 1.2**

We prove the Theorem 1.2 using the method of Lu-Xu-Chen [14], Grahl-Meng {[6], Thm 1.1}, Pang-Zalcman [20]. We started our argument accurately in the interest of accommodation.

Let's say $\chi = f - \tau$. The implication then becomes

$$\begin{array}{ll} (I) & \chi(z) = 0 \Rightarrow \chi'(z) = \tau(z) - \tau'(z), \\ (II) & \chi'(z) = \tau(z) - \tau'(z) \Rightarrow a\chi'(z) + b\chi(z) = (1-b)\tau(z) - a\tau'(z). \end{array}$$

Of course, $\tau \neq \tau'$. First, we take into account

$$\varphi = \frac{\chi}{\tau - \tau'} = \frac{f - \tau}{\tau - \tau'}. \tag{3.1}$$

We now move on to the proof by separating the two cases.

Case 1. If $\rho(\varphi) > n$, then for each $0 < N < \frac{\rho(\varphi)-n}{n}$, it follows from Lemma 2.2 of Gu-Li-Yuan [7], that there exists a sequence w_n such that $w_n \to \infty$ and for every $N > 0$ (for n sufficiently large)

$$\varphi^{\#}(w_n) > |w_n|^N \quad i.e., \quad \lim_{n\to\infty} \frac{\varphi^{\#}(w_n)}{|w_n|^N} = \infty. \tag{3.2}$$

We create a family of holomorphic functions first. Naturally, $\tau - \tau' = (R - R' - RP')e^P = R_2 e^P$, where $R_2 = (R - R' - RP')$ is a rational function, has only a finite number of zeros. Then there exists a positive number r_1 such that for $|z| \geq r_1$, we have $\tau \neq \tau'$, with χ having a finite number of poles. Then, there exists a positive number $r_2 > 0$ such that χ is analytic in $\{z : |z| \geq r_2\}$. Let $r = \max\{r_1, r_2\}$ and $D = \{z : |z| \geq r\}$. Then φ is analytic in D.

In view of $w_n \to \infty$ as $n \to \infty$, without loss of generality, we may assume $|w_n| \geq r + 1$ for all n. Define $D_1 = \{z : |z| < 1\}$ and

$$\varphi_n(z) = \varphi(w_n + z) = \frac{\chi(w_n + z)}{\tau(w_n + z) - \tau'(w_n + z)}.$$

Now, for each $z \in D_1$, $|w_n + z| \geq r$. So, $(w_n + z) \in D$ for each $z \in D_1$. Then, $\chi(w_n + z)$ and $(\tau(w_n + z) - \tau'(w_n + z))$ both are analytic in D_1. Thus, we obtain a family of holomorphic functions $(\varphi_n)_n$.

Now, fix $z \in D$. If $\varphi(w_n + z) = 0$, then $\chi(w_n + z) = 0$. Noting, under the supposition (I) that $\chi'(w_n + z) = \tau(w_n + z) - \tau'(w_n + z)$. For comfort, we set $u_n = w_n + z$. Then, if $\varphi_n(z) = 0$ and n is large enough,

$$\begin{aligned} |\varphi_n'(z)| = |\varphi'(u_n)| &= \left| \frac{\chi'(u_n)}{\tau(u_n) - \tau'(u_n)} - \frac{\chi(u_n)}{\tau(u_n) - \tau'(u_n)} \frac{\tau'(u_n) - \tau''(u_n)}{\tau(u_n) - \tau'(u_n)} \right| \\ &\leq \left| \frac{\chi'(u_n)}{\tau(u_n) - \tau'(u_n)} \right| + \left| \frac{\chi(u_n)}{\tau(u_n) - \tau'(u_n))} \right| \left| \frac{\tau'(u_n) - \tau''(u_n)}{\tau(u_n) - \tau'(u_n)} \right| \\ &= 1. \end{aligned}$$

Next, we want to demonstrate that $(\varphi_n)_n$ is normal at $z = 0$. If this is not the case, we assume that $(\varphi_n)_n$ is not normal at $z = 0$. Applying Lemma 2.1, and choosing an appropriate subsequence of $(\varphi_n)_n$ if necessary, we may assume that there exists a sequence $(z_n)_n \in D_1$ and $(\rho_n)_n$ s.t $z_n \to 0$, $\rho_n \to 0$ and

$$g_n(\zeta) = \rho_n^{-1} \varphi_n(z_n + \rho_n\zeta) = \rho_n^{-1} \left(\frac{\chi(w_n + z_n + \rho_n\zeta)}{\tau(w_n + z_n + \rho_n\zeta) - \tau'(w_n + z_n + \rho_n\zeta)} \right) \to g(\zeta) \tag{3.3}$$

locally uniformly in $\mathbb{C}$, where $g(z)$ is a non-constant entire function, $\rho(g) \le 1$ and $g^{\#}(\zeta) \le g^{\#}(0) = L+1 = 2$ for all ζ in $\mathbb{C}$ and

$$\rho_n \le \frac{M_1}{\varphi_n^{\#}(0)} = \frac{M_1}{\varphi^{\#}(w_n)} \tag{3.4}$$

for a positive number M_1. For every $N > 0$ (for n sufficiently large), derived from (3.2) and (3.4), we have

$$\rho_n \le \frac{M_1}{\varphi_n^{\#}(0)} = \frac{M_1}{\varphi^{\#}(w_n)} \le M_1 |w_n|^{-N-\varepsilon}. \tag{3.5}$$

Set $\xi_n = (w_n + z_n + \rho_n \zeta)$. Differentiating on both sides (3.3), we obtain

$$\begin{aligned} g_n'(\zeta) &= \frac{(\tau(\xi_n) - \tau'(\xi_n))\chi'(\xi_n) - \chi(\xi_n)(\tau'(\xi_n) - \tau''(\xi_n))}{(\tau(\xi_n) - \tau'(\xi_n))^2} \\ &= \frac{\chi'(\xi_n)}{\tau(\xi_n) - \tau'(\xi_n)} - \frac{\chi(\xi_n)}{\tau(\xi_n) - \tau'(\xi_n)} . \frac{\tau'(\xi_n) - \tau''(\xi_n)}{\tau(\xi_n) - \tau'(\xi_n)}. \end{aligned}$$

Assume that $G_n(\zeta) = \frac{\chi'(\xi_n)}{\tau(\xi_n) - \tau'(\xi_n)}$. Then rewrite the above estimate using (3.3), we have

$$g_n'(\zeta) = G_n(\zeta) - \rho_n g_n(\zeta) . \frac{\tau'(\xi_n) - \tau''(\xi_n)}{\tau(\xi_n) - \tau'(\xi_n)} \tag{3.6}$$

locally uniformly in $\mathbb{C}$. we now have

$$\begin{aligned} \left| \frac{\tau' - \tau''}{\tau - \tau'} \right|_{z=\xi_n} &= \left| \frac{R_2' + R_2 P'^2}{R_2} \right|_{z=\xi_n} \\ &= \left| \frac{R' - R'' - 2R'P' - RP'' + RP' - RP'^2}{R - R' - RP'} \right|_{z=\xi_n} \\ &= O(|w_n|^{l_1}) \text{ (as } n \to \infty), \end{aligned} \tag{3.7}$$

where

$$\begin{aligned} l_1 &= \deg \left| \frac{R' - R'' - 2R'P' - RP'' + RP' - RP'^2}{R - R' - RP'} \right| \\ &= \deg \left| \frac{\frac{R'}{R} - \frac{R''}{R} - 2P'(\frac{R'}{R}) - P'' + P' - P'^2}{1 - \frac{R'}{R} - P'} \right| \\ &= \deg P' \end{aligned}$$

is a fixed constant.

Considering (3.2), (3.5), and (3.7), we now arrive at

$$\begin{aligned} \left| \frac{\chi(\xi_n)}{\tau(\xi_n) - \tau'(\xi_n)} . \frac{\tau'(\xi_n) - \tau''(\xi_n)}{\tau(\xi_n) - \tau'(\xi_n)} \right| &= \left| \frac{\rho_n g_n(\zeta)(\tau'(\xi_n) - \tau''(\xi_n))}{\tau(\xi_n) - \tau'(\xi_n)} \right| \\ &\le M_1 |w_n|^{-N-\varepsilon} |g_n(\zeta)| |w_n|^{l_1} \\ &= M_1 |g_n(\zeta)| |w_n|^{l_1 - N - \varepsilon} \\ &= M_1 |g_n(\zeta)| \frac{w_n^{l_1}}{|w_n|^{N+\varepsilon}} \to 0, \ as \ n \to \infty. \end{aligned} \tag{3.8}$$

We conclude from (3.6) and (3.8) that

$$g_n'(\zeta) = G_n(\zeta) = \frac{\chi'(\xi_n)}{\tau(\xi_n) - \tau'(\xi_n)} \to g'(\zeta) \tag{3.9}$$

locally uniformly in $\mathbb{C}$.

We claim that: $g(\zeta) = 0 \Rightarrow g'(\zeta) = 1$. Suppose that $g(\zeta_0) = 0$. Then according to Hurwitz's theorem, there exists a sequence $(\zeta_n)_n$, $\zeta_n \to \zeta_0$ such that (for n sufficiently large)

$$g_n(\zeta_n) = \rho_n^{-1} \left(\frac{\chi(w_n + z_n + \rho_n \zeta_n)}{\tau(w_n + z_n + \rho_n \zeta_n) - \tau'(w_n + z_n + \rho_n \zeta_n)} \right) = 0.$$

Thus $\chi(w_n + z_n + \rho_n \zeta_n) = 0$ and by the assumption (I), we have $\chi'(w_n + z_n + \rho_n \zeta_n) = \tau(w_n + z_n + \rho_n \zeta_n) - \tau'(w_n + z_n + \rho_n \zeta_n)$. Then, by using (3.9), we conclude that

$$g'(\zeta_0) = \lim_{n \to \infty} G_n(\zeta_n) = \lim_{n \to \infty} \frac{\chi'(w_n + z_n + \rho_n \zeta_n)}{\tau(w_n + z_n + \rho_n \zeta_n) - \tau'(w_n + z_n + \rho_n \zeta_n)} = 1.$$

Thus, this proves our claim.

Next, we prove that $g'(\zeta) \neq 1$. Suppose that there exists a point η_0 such that $g'(\eta_0) = 1$. Obviously, $g' \not\equiv 1$. Otherwise, $g^{\#}(0) \leq g'(0) = 1$. But $g^{\#}(0) = 2$, a contradiction. Therefore, again by Hurwitz's theorem, there exists a sequence $(\eta_n)_n$, $\eta_n \to \eta_0$ s.t (for n large enough)

$$g'(\eta_n) = 1,$$

and that provides

$$\chi'(w_n + z_n + \rho_n \eta_n) = \tau(w_n + z_n + \rho_n \eta_n) - \tau'(w_n + z_n + \rho_n \eta_n).$$

It is clear from the assumption (II) that

$$\chi(w_n + z_n + \rho_n \eta_n) = \left(\frac{1-a-b}{b}\right)\tau(w_n + z_n + \rho_n \eta_n).$$

Setting $\nu_n = w_n + z_n + \rho_n \eta_n$ now, and using (3.3) and (3.5), we have

$$\begin{aligned} g(\eta_0) = \lim_{n\to\infty} g_n(\eta_n) &= \lim_{n\to\infty} \rho_n^{-1}\left(\frac{\chi(\nu_n)}{\tau(\nu_n) - \tau'(\nu_n)}\right) \\ &= \lim_{n\to\infty} \rho_n^{-1} \frac{(\frac{1-a-b}{b})\tau(\nu_n)}{\tau(\nu_n) - \tau'(\nu_n)} \\ &\geq \lim_{n\to\infty} \frac{|w_n|^{N+\varepsilon}}{M_1}\left(\frac{1-a-b}{b}\right)\frac{\tau(\nu_n)}{\tau(\nu_n) - \tau'(\nu_n)} \\ &= \lim_{n\to\infty} \frac{|w_n|^{N+\varepsilon}}{M_1}\left(\frac{1-a-b}{b}\right)\frac{R(\nu_n)}{R(\nu_n) - R'(\nu_n) - R(\nu_n)P'(\nu_n)}. \end{aligned}$$

Then, $g(\eta_0) \to \infty$ *as* $n \to \infty$. Thus $g'(\eta) = 1 \Rightarrow g(\eta) = \infty$, which contradicts. So, $g'(\eta) \neq 1$ on $\mathbb{C}$. Since $\rho(g) \leq 1$, so g' also. Consequently, $g'(\eta)$ can be expressed as

$$\begin{aligned} \textbf{(1.1)}\ g'(\eta) &= 1 + c_1, \\ or\ \textbf{(1.2)}\ g'(\eta) &= 1 + e^{c_2\eta + c_3}, \end{aligned}$$

where $c_1, c_2 (\neq 0)$, and c_3 are constants.

Subcase 1.1. In the event that $g'(\eta) = 1 + c_1$, we have

$$g(\eta) = (1 + c_1)\eta + c_4,$$

where c_4 is a constant. As a result of $g = 0 \Rightarrow g' = 1$, the result above produces $c_1 = 0$. With a straightforward calculation, we arrive at $g^{\#}(0) < 2$, which contradicts the condition.

Subcase 1.2. Whenever $g'(\eta) = 1 + e^{c_2\eta + c_3}$. Due to the fact that $c_2 \neq 0$, g is a transcendental meromorphic function with the order at most one. Since $g' \neq 1$, by Lemma 2.5, we know that $g(z)$ has infinitely many zeros $z_1, z_2, ..., z_n, ..$ and $|z_n| \to \infty$ as $n \to \infty$. Define $H(z) = g(z) - z$, then $H'(z) = g'(z) - 1 \neq 0$. Therefore, there are no critical values for H. According to Lemma 2.7, H has a finite number of asymptotic values. Now, using Lemma 2.6 to H, we have

$$|H'(z_n)| \geq \frac{|H(z_n)|}{2\pi|z_n|}\log\frac{|H(z_n)|}{R},$$

and this gives

$$\frac{|z_n H'(z_n)|}{|H(z_n)|} \geq \frac{1}{2\pi}\log\frac{|H(z_n)|}{R}.$$

It deduces that

$$\frac{|z_n H'(z_n)|}{|H(z_n)|} \to \infty, \text{ as } n \to \infty. \tag{3.10}$$

Now $g(z) = 0 \Rightarrow g'(z) = 1$, we have

$$\frac{|z_n H'(z_n)|}{|H(z_n)|} = 0. \tag{3.11}$$

We obtain a contradiction from (3.10) and (3.11). All of the aforementioned discussion demonstrates that at $z=0$, $(\varphi_n)_n$ is normal. On the other hand, it follows that

$$\begin{aligned}\varphi_n^{\#}(0) &= \frac{|\varphi_n'(0)|}{1+|\varphi_n(0)|^2}\\ &= \frac{|\varphi'(w_n)|}{1+|\varphi(w_n)|^2}\\ &= \varphi^{\#}(w_n).\end{aligned}$$

So,

$$\varphi_n^{\#}(0)=\varphi^{\#}(w_n)\to\infty,\ as\ n\to\infty.$$

According to Marty's criterion, $(\varphi_n)_n$ is not normal at $z=0$, which is a contradiction. As a result, Case 1 is ruled out.

Case 2. If $\rho(\varphi)\le n$. Next, we will demonstrate this

$$\rho(f)\le\rho(\varphi)\le n. \tag{3.12}$$

So, for further investigation, we separate three subcases.

Subcase 2.1. If $\rho(\tau)<\rho(f)$, then according to (3.1), we have

$$f=\varphi(\tau-\tau')+\tau.$$

Using Lemma 2.3, we obtain

$$\begin{aligned}\rho(f)=\rho(\varphi(\tau-\tau')+\tau) &\le \max\{\rho(\varphi(\tau-\tau')),\rho(\tau)\}\\ &\le \max\{\rho(\varphi),\rho(\tau)\}.\end{aligned}$$

Since

$$\rho(\tau-\tau')\le\max\{\rho(\tau),\rho(\tau')\}\le\rho(\tau),$$

and also

$$\rho(\varphi(\tau-\tau'))\le\max\{\rho(\varphi),\rho(\tau-\tau')\}\le\max\{\rho(\varphi),\rho(\tau)\}.$$

The conclusion (3.12) follows from assuming the condition $\rho(\tau)<\rho(f)$.

Subcase 2.2. If $\rho(\tau)=\rho(f)$. Now $\tau=Re^P$. Consequently, $\rho(\tau)=\deg P$. As a result, we have

$$\rho(\tau)=\rho(f)=\deg P.$$

Subcase 2.3. If $\rho(\tau)>\rho(f)$. By (3.1), we write

$$\varphi=\frac{f-\tau}{\tau-\tau'}=\frac{f-Re^P}{R_2e^P}=\frac{f}{R_2e^P}-\frac{R}{R_2}.$$

Since $\frac{R}{R_2}$ is a rational function, so $\rho(\frac{R}{R_2})=0$ and

$$n=\deg P=\rho(R_2e^P)=\rho(\tau)>\rho(f),$$

we obtain that

$$\rho(f)<\deg P=\rho(\varphi).$$

Thus, we obtain the conclusion (3.12). The Theorem 1.2 is now complete.

4. Proof of Theorem 1.1

From the consequence of the Theorem 1.2, it is clear that f is of finite order.

Part A: When $a=0$. We take into account that $\xi=f-\tau$ for the sake of simplicity. Based on the assumption we have

$$(I)\quad \xi(z)=0\Rightarrow\xi'(z)=\tau(z)-\tau'(z),$$

$$(II)\quad \xi'(z)=\tau(z)-\tau'(z)\Rightarrow\xi(z)=\left(\frac{1-b}{b}\right)\tau(z).$$

Now that we have defined

$$\lambda=\frac{b\xi(\tau-\tau')-(1-b)\tau\xi'}{\xi}. \tag{4.1}$$

It follows from the Lemma 2.4 that τ is a small function of f and ξ on J, where $J=\{r_n\},1\le n\le\infty$. Without losing generality, if $T(r,g)=o(T(r,f))$ on J, we can omit J and only state that g is a small function of f and $T(r,g)=S(r,f)$. We are now separating the two cases below.

Case 1. Suppose that $\lambda = 0$. From (4.1), it is evident

$$b\xi(\tau-\tau') = (1-b)\tau\xi'.$$

Integrating the above differential equation, this yields

$$\xi(z) = c_1 e^{(\frac{b}{1-b})z}\tau(z)^{-(\frac{b}{1-b})},$$

and this implies

$$f(z) = R(z)e^{P(z)} + c_1 R(z)^{-(\frac{b}{1-b})}e^{(\frac{b}{1-b})(z-P(z))}, \tag{4.2}$$

where c_1 is a non-zero constant. The form of ξ now leads us to conclude that $\deg P = n = \rho(\tau) < \rho(f) = \rho(\xi) = \deg[(\frac{b}{1-b})(z-P(z))] = \deg(z-P(z))$. P must therefore be a constant given this.

Following that, change (4.2) to

$$f(z) = \tau(z) + c_1\tau(z)^{-(\frac{b}{1-b})}e^{(\frac{b}{1-b})z},$$

where c_1 is a non-zero constant and τ reduces to a rational function.

Case 2. Assume that $\lambda \neq 0$. Now, by the lemma of logarithmic derivative, we have

$$m(r,\lambda) \le m\left(r, b(\tau-\tau')\frac{\xi}{\xi}\right) + m\left(r,(1-b)\tau\frac{\xi'}{\xi}\right) + \log 2 = S(r,\xi).$$

If w is a simple zero of ξ. Then, with (I) and (II) we have $\tau = 0$. Therefore, $\xi(\tau-\tau') - \xi'\tau = 0$. So, λ has no pole at a simple zero of ξ.

Next, we establish that

$$N_{(2}\left(r,\frac{1}{\xi}\right) = S(r,\xi). \tag{4.3}$$

Let w_1 be a multiple zero points of $\xi = f-\tau$ with multiplicity $m \ge 2$. Then, by $f(z) = \tau(z) \Rightarrow f'(z) = \tau(z)$, we have in the victinity at $w_1 \in \mathbb{C}$,

$$f(z) - \tau(z) = (z-w_1)^m f_1(z);\ \text{ where } f_1(w_1) \neq 0. \tag{4.4}$$

$$f'(z) - \tau(z) = (z-w_1)^l f_2(z);\ \text{where } f_2(w_1) \neq 0. \tag{4.5}$$

Through (4.4) we acquire,

$$f'(z) - \tau'(z) = (z-w_1)^{m-1} f_3(z);\ \text{where } f_3(w_1) \neq 0. \tag{4.6}$$

Thus we write for $n \ge \min\{l, m-1\} \ge 1$,

$$\tau(z) - \tau'(z) = (z-w_1)^n f_4(z);\ \text{where } f_4(w_1) \neq 0. \tag{4.7}$$

Then the aforementioned suggests that

$$N_{(2}\left(r,\frac{1}{f-\tau}\right) = N_{(2}\left(r,\frac{1}{\xi}\right) \le N\left(r,\frac{1}{\tau-\tau'}\right) = S(r,\xi).$$

Thus, (4.3) is established.

Furthermore, $\xi = f-\tau$ has finitely many poles. It is easy to see that all the possible poles of λ come from the multiple zero points and poles of ξ. Then, by (4.3) we get

$$N(r,\lambda) \le N_{(2}\left(r,\frac{1}{\xi}\right) = S(r,\xi).$$

It follows from this that

$$T(r,\lambda) = m(r,\lambda) + N(r,\lambda) = S(r,\xi). \tag{4.8}$$

As a result, λ is a small function of ξ. Now, (4.1) can be rewritten as

$$\xi[b(\tau-\tau') - \lambda] = (1-b)\tau\xi'.$$

Put $\beta = \tau-\tau'$, $c = 1-b$. The above then turns into

$$\xi = \frac{c\tau\xi'}{b\beta-\lambda}. \tag{4.9}$$

By differentiating (4.9), we have

$$\xi' = \left(\frac{c\tau}{b\beta-\lambda}\right)\xi'' + \left(\frac{c\tau}{b\beta-\lambda}\right)'\xi',$$

which implies that

$$\left[1-\left(\frac{c\tau}{b\beta-\lambda}\right)'\right]\xi' = \left(\frac{c\tau}{b\beta-\lambda}\right)\xi''. \tag{4.10}$$

In case that, $\left(\frac{c\tau}{b\beta-\lambda}\right)' \neq 1$. Then, we recast (4.10) as the following way

$$\left[1-\left(\frac{c\tau}{b\beta-\lambda}\right)'\right](\xi'-\beta) = \left(\frac{c\tau}{b\beta-\lambda}\right)(\xi''-\beta') + \left(\frac{c\tau}{b\beta-\lambda}\right)\beta' - \left[1-\left(\frac{c\tau}{b\beta-\lambda}\right)'\right]\beta,$$

and deduce that

$$\Gamma_1(\xi'-\beta) = \Gamma_2(\xi''-\beta') + \Gamma_3. \tag{4.11}$$

Denote

$$\Gamma_1 = 1-\left(\frac{c\tau}{b\beta-\lambda}\right)';\ \Gamma_2 = \left(\frac{c\tau}{b\beta-\lambda}\right);\ \Gamma_3 = \left(\frac{c\tau}{b\beta-\lambda}\right)\beta' - \left[1-\left(\frac{c\tau}{b\beta-\lambda}\right)'\right]\beta.$$

It is clear that Γ_i $(i = 1,2,3)$ are small functions of ξ. After that, we can transform the equation (4.11) into

$$\Gamma_3 = \Gamma_1(\xi'-\beta) - \Gamma_2(\xi''-\beta'). \tag{4.12}$$

Next we examine the following two subcases.

Subcase 2.1. In subcase that $\Gamma_3 = 0$. Then, we have

$$\left(\frac{c\tau}{b\beta-\lambda}\right)\beta' = \left[1-\left(\frac{c\tau}{b\beta-\lambda}\right)'\right]\beta.$$

Let $Q = \frac{c\tau}{b\beta-\lambda}$. Differentiating the above case on both sides and we get

$$Q\beta'' + Q'\beta' + Q''\beta + Q'\beta' = \beta'.$$

The above estimate can then be written in the form

$$\left(\frac{\frac{\beta''}{\beta} + 2\frac{Q'}{Q}.\frac{\beta'}{\beta} + \frac{Q''}{Q}}{\frac{\beta'}{\beta}}\right) = \frac{1}{Q}.$$

Now, based on the logarithmic derivative lemma, we have

$$m\left(r, \frac{1}{Q}\right) = S(r,Q) + S(r,e^P).$$

Furthermore, by (4.8), we have

$$\begin{aligned} N\left(r,\frac{1}{Q}\right) &\leq N\left(r,\frac{Q'}{Q}\right) + N\left(r,\frac{Q''}{Q}\right) + S(r,\xi) \\ &\leq 3\overline{N}(r,Q) + 3\overline{N}\left(r,\frac{1}{Q}\right) + S(r,\xi) = S(r,\xi). \end{aligned}$$

Consequently, based on the first fundamental theorem and the two observations above, we have

$$T(r,Q) = S(r,Q) + S(r,\xi). \tag{4.13}$$

Again, on another hand of (4.9) we have

$$\frac{c\tau}{b\beta-\lambda} = Q = \frac{\xi}{\xi'}.$$

After that, (4.13) becomes

$$T\left(r,\frac{\xi}{\xi'}\right) = S\left(r,\frac{\xi}{\xi'}\right) + S(r,\xi) = S\left(r,\frac{\xi}{\xi'}\right),$$

which is a contradiction.

Subcase 2.2. In that subcase, $\Gamma_3 \neq 0$. Then, (4.12) can be changed to

$$\frac{\Gamma_3}{\xi'-\beta} = \Gamma_1 - \Gamma_2\left(\frac{\xi''-\beta'}{\xi'-\beta}\right).$$

By the lemma of logarithmic derivative, we write

$$m\left(r,\frac{1}{\xi'-\beta}\right) \leq m\left(r,\frac{\Gamma_3}{\xi'-\beta}\right) + m\left(r,\frac{1}{\Gamma_3}\right) = S(r,\xi). \tag{4.14}$$

Supposing that w_2 is zero of $\xi'-(\tau-\tau')$ along multiplicity m_1, and not the zero of ξ. Then, by (II) and (4.1), we deduce that w_2 is a zero of λ. Then, we have $N\left(r,\frac{1}{\lambda}\right)=S(r,\xi)$. Moreover, it follows from the assumption (II), that $\xi'-(\tau-\tau')$ has finitely many multiple zeros, it means that $N_{(2}(r,\frac{1}{\xi'-(\tau-\tau')})=O(\log r)=S(r,\xi)$. Therefore,

$$N(r,\tau-\tau'=\xi'/\xi\neq 0)\leq N\left(r,\frac{1}{\lambda}\right)+N_{(2}\left(r,\frac{1}{\xi'-(\tau-\tau')}\right)=S(r,\xi).$$

So,

$$\begin{aligned} N\left(r,\frac{1}{\xi'-(\tau-\tau')}\right) &= N\left(r,\frac{1}{\xi}\right)+N(r,\tau-\tau'=\xi'/\xi\neq 0)\\ &= N\left(r,\frac{1}{\xi}\right)+S(r,\xi). \end{aligned}$$

That is,

$$N\left(r,\frac{1}{\xi'-\beta}\right)=N\left(r,\frac{1}{\xi}\right)+S(r,\xi). \tag{4.15}$$

Denote

$$\Lambda=\frac{\xi-\xi'+\tau-\tau'}{\xi}. \tag{4.16}$$

The following two subcases are now further examined:

Subcase 2.2.1. Assuming that $\Lambda=0$. Then, (4.16) yields $f(z)=f'(z)$, resulting in $f(z)=c_0e^z$, where c_0 is a non-zero constant. In this case, $\rho(f)=1$. If $n=0$, this had to be the case. Alternatively, contradiction as $\rho(\tau)<\rho(f)$.

Subcase 2.2.2. Conceding that $\Lambda\neq 0,1$. Following that, (4.16) can be expressed as

$$1-\Lambda=\frac{\xi'-\tau+\tau'}{\xi}.$$

Put $\Xi=1-\Lambda$, and so

$$\Xi=\frac{\xi'-\tau+\tau'}{\xi}. \tag{4.17}$$

The lemma of the logarithmic derivative allows us to arrive at

$$m(r,\Xi)\leq m\left(r,\frac{\xi'}{\xi}\right)+m\left(r,\frac{\tau-\tau'}{\xi}\right)\leq m\left(r,\frac{1}{\xi}\right)+S(r,\xi).$$

It follows that Ξ also has a finite number of poles because ξ has multiple zeros and a finite number of poles. Then, we have $N(r,\Xi)=O(\log r)$. Thus, it follows that

$$T(r,\Xi)=m(r,\Xi)+N(r,\Xi)\leq m\left(r,\frac{1}{\xi}\right)+S(r,\xi). \tag{4.18}$$

As a result of (4.8) and (4.9), we have

$$\begin{aligned} T(r,\xi)=T\left(r,\frac{c\tau\xi'}{b\beta-\lambda}\right) &\leq T(r,\xi')+T(r,\lambda)+O(1)\\ &\leq T(r,\xi')+S(r,\xi). \end{aligned} \tag{4.19}$$

The lemma of the logarithmic derivative also provides us with

$$\begin{aligned} T(r,\xi')=m(r,\xi') &\leq m\left(r,\frac{\xi'}{\xi}\right)+m(r,\xi)+O(1)\\ &= T(r,\xi)+S(r,\xi). \end{aligned} \tag{4.20}$$

Connecting (4.19) and (4.20), we get

$$T(r,\xi)=T(r,\xi')+S(r,\xi). \tag{4.21}$$

We obtain using (4.14), (4.15), and (4.21)

$$\begin{aligned} m\left(r,\frac{1}{\xi}\right) &= T(r,\xi)-N\left(r,\frac{1}{\xi}\right)+S(r,\xi)\\ &= T(r,\xi')-N\left(r,\frac{1}{\xi}\right)+S(r,\xi)\\ &= m\left(r,\frac{1}{\xi'-\beta}\right)+N\left(r,\frac{1}{\xi'-\beta}\right)-N(r,\frac{1}{\xi})+S(r,\xi)\\ &= N\left(r,\frac{1}{\xi}\right)-N\left(r,\frac{1}{\xi}\right)+S(r,\xi)\\ &= S(r,\xi). \end{aligned} \tag{4.22}$$

Now, taking into account of (4.22) in (4.18), we obtain

$$T(r,\Xi)=S(r,\xi). \tag{4.23}$$

Combining (4.1) and (4.17), leads to

$$[b(\tau-\tau')-\lambda-(1-b)\tau\ \Xi]\xi=(1-b)\tau(\tau-\tau').$$

It is clear that $b(\tau-\tau')-\lambda-(1-b)\tau\Xi\neq 0$. Following that, using (4.8) and (4.23), we obtain

$$T(r,\xi)\leq T(r,\lambda)+T(r,\Xi)=S(r,\xi),$$

and there is a contradiction here. The proof of Theorem 1.1(A) is thus finished.

Part B (c): When $a\neq 0$ and $n=0$. Then, τ becomes a rational function, say R_3. We start by considering $\Theta=f-R_3$ for our comfort. The assumption can be expressed as

$$\begin{aligned} &(III) \quad \Theta(z)=0\Rightarrow\Theta'(z)=R_3(z)-R_3'(z).\\ &(IV) \quad \Theta'(z)=R_3(z)-R_3'(z)\Rightarrow L(\Theta,z)=R_3(z)-aR_3'(z)-bR_3(z). \end{aligned}$$

Now we set

$$\begin{aligned} \Delta &= \frac{L(\Theta,z)(R_3-R_3')-(R_3-aR_3'-bR_3)\Theta'}{\Theta}\\ &= \frac{(a+b-1)R_3\Theta'+b(R_3-R_3')\Theta}{\Theta}. \end{aligned} \tag{4.24}$$

Now we discuss about two important cases:

Case 3. Whenever $\Delta=0$. Then, from (4.24) we deduce that

$$(a+b-1)R_3\Theta'+b(R_3-R_3')\Theta=0.$$

Integrating this results in

$$\Theta(z)=c_2R_3(z)^{(\frac{b}{a+b-1})}e^{-(\frac{b}{a+b-1})z},$$

and that provides

$$f(z)=R_3(z)+c_2R_3(z)^{(\frac{b}{a+b-1})}e^{-(\frac{b}{a+b-1})z},$$

where c_2 is a non-zero constant.

Case 4. Wherever $\Delta\neq 0$. By the caption of the logarithmic derivative lemma, we have

$$m(r,\Delta)\leq m\left(r,R_3\frac{\Theta'}{\Theta}\right)+m\left(r,\frac{(R_3-R_3')\Theta}{\Theta}\right)+\log 2=O(\log r).$$

Observing equation (4.24), we appriciate that the possible poles of Δ appear from the multiple zeros and the poles of Θ. Additionally, Θ has a finite number of poles, and (III) assumes that Θ has a finite number of multiple zeros. Consequently, $N(r,\Delta)=O(\log r)$ follows from the conclusion stated above. Therefore, $T(r,\Delta)=O(\log r)$, which views that Δ is a rational function.

Next, we consider

$$\Phi=\frac{\Theta-\Theta'+R_3-R_3'}{\Theta}. \tag{4.25}$$

We discuss the following two subcases:

Subcase 4.1. If $\Phi=0$. Now, (4.25) yields $f(z)=f'(z)$, i.e., $f(z)=c_2e^z$, where c_2 is a non-zero constant.

Subcase 4.2. If $\Phi \neq 0, 1$. Consequently, (4.25) can be expressed as

$$1 - \Phi = \frac{\Theta' - R_3 + R_3'}{\Theta}.$$

Put $\Pi(z) = 1 - \Phi(z)$, and so

$$\Pi = \frac{\Theta' - R_3 + R_3'}{\Theta}. \tag{4.26}$$

Let $w_3 \in \mathbb{C}$ be a zero of $\Theta' - (R_3 - R_3')$ with multiplicity m_2, and not the zero of Θ. Then using the assumption (IV), and (4.25) we can claim that w_3 is also a zero of Δ. Also, note that Δ is a rational function. So, we have $N(r, \frac{1}{\Delta}) = O(\log r)$. Meanwhile, the assumption (IV) conclude that $\Theta' - (R_3 - R_3')$ has finitely many multiple zeros. Thus, we have $N_{(2}\left(r, \frac{1}{\Theta' - (R_3 - R_3')}\right) = O(\log r)$. Then,

$$N(r, R_3 - R_3' = \Theta'/\Theta \neq 0) \leq N\left(r, \frac{1}{\Delta}\right) + N_{(2}\left(r, \frac{1}{\Theta' - (R_3 - R_3')}\right) = O(\log r). \tag{4.27}$$

Since Θ has a finite number of poles and multiple zeros, we can conclude that Π has a finite number of poles. Together with the fact of (4.26) and (4.27), we have

$$N\left(r, \frac{1}{\Pi}\right) \leq N(r, R_3 - R_3' = \Theta'/\Theta \neq 0) + N_{(2}\left(r, \frac{1}{\Theta' - (R_3 - R_3')}\right) = O(\log r),$$

which indicates that Π has finitely many zeros. Then Π can be expressed as

$$\Pi(z) = R_2(z) e^{P_1(z)},$$

where R_2, P_1 are rational and polynomial functions, respectively. Rewriting (4.26) as follows

$$\Theta'(z) - R_2(z) e^{P_1(z)} \Theta(z) = R_3(z) - R_3'(z).$$

By applying Lemma 2.8, we can state that P_1 is a constant, say d. Let $\Pi(z) = R_4(z)$. Then the equation stated above becomes

$$\Theta' - R_4 \Theta = R_3 - R_3'.$$

Together with (4.24) and the above estimate, we get

$$[\Delta - b(R_3 - R_3') - (a + b - 1) R_3 R_4] \Theta = (a + b - 1) R_3 (R_3 - R_3').$$

Evidently, $\Delta - b(R_3 - R_3') - (a + b - 1) R_3 R_4 \neq 0$. The result above then suggests that Θ is a rational function, which is impossible. Thus, the proof (c) of (B) is done.

(d): When $a \neq 0$ and $n \geq 1$, and $\tau = Re^P$, where R, P are rational and polynomial functions, respectively. For the sake of simplicity, let's assume $\Omega = f - \tau = f - Re^P$. The assumption turns into

$$\begin{aligned} &(V) \quad \Omega(z) = 0 \Rightarrow \Omega'(z) = \tau(z) - \tau'(z), \\ &(VI) \quad \Omega'(z) = \tau(z) - \tau'(z) \Rightarrow L(\Omega, z) = \tau(z) - a\tau'(z) - b\tau(z). \end{aligned}$$

Now we illustrate

$$\eta = \frac{L(\Omega, z)(\tau - \tau') - (\tau - a\tau' - b\tau)\Omega'}{\Omega}. \tag{4.28}$$

The next two cases are now distinguished.

Case 5. In that case, $\eta = 0$. Then, (4.28) implies that

$$(a + b - 1)\tau\Omega' + b\Omega(\tau - \tau') = 0.$$

Integrating the above differential equation and we get

$$\Omega(z) = c_4 R(z)^{\left(\frac{b}{a+b-1}\right)} e^{-\left(\frac{b}{a+b-1}\right)(z - P(z))},$$

therefore, it follows

$$f(z) = R(z) e^{P(z)} + c_4 R(z)^{\left(\frac{b}{a+b-1}\right)} e^{-\left(\frac{b}{a+b-1}\right)(z - P(z))},$$

where c_4 is a non-zero constant. Now, we can deduce from the form of Ω that $\deg P = n = \rho(\tau) < \rho(f) = \rho(\Omega) = \deg(z - P(z))$. This suggests that P is a constant. Then

$$f(z) = \tau(z) + c_4 \tau(z)^{\left(\frac{b}{a+b-1}\right)} e^{-\left(\frac{b}{a+b-1}\right)z},$$

where c_4 is a non-zero constant and τ reduces to a rational function.

Case 6. In this case, $\eta \neq 0$. The logarithmic derivative lemma provides us with the following

$$m(r,\eta) \leq m\left(r, \frac{\Omega'}{\Omega}\right) + O(1) = S(r,\Omega).$$

At the time that Ω has finitely many poles and from the assumption (V) that Ω has finitely many multiple zeros. Observe equation (4.28), the possible poles of η come from the multiple zeros and poles of Ω. All of the above discussion thus suggests that η has a finite number of poles. Then, we write $N(r,\eta) = O(\log r) = S(r,\Omega)$. Thus, we have

$$T(r,\eta) = m(r,\eta) + N(r,\eta) = S(r,\Omega). \tag{4.29}$$

This demonstrates that η is a small function of Ω. Now, we transform (4.28) into

$$\left[1 - \frac{b(\tau-\tau')}{\eta}\right]\Omega = \left[\frac{(a+b-1)\tau}{\eta}\right]\Omega'.$$

Differentiating on both sides, we obtain

$$\left[1 - \frac{b(\tau-\tau')}{\eta}\right]\Omega' + \left[1 - \frac{b(\tau-\tau')}{\eta}\right]'\Omega = \left[\frac{(a+b-1)\tau}{\eta}\right]\Omega'' + \left[\frac{(a+b-1)\tau}{\eta}\right]'\Omega',$$

which suggests that

$$\left[1 - \frac{b(\tau-\tau')}{\eta}\right]'\Omega = \left[\left(\frac{(a+b-1)\tau}{\eta}\right)' - 1 + \frac{b(\tau-\tau')}{\eta}\right]\Omega' + \left[\frac{(a+b-1)\tau}{\eta}\right]\Omega''. \tag{4.30}$$

Set

$$\beta_1 = 1 - \frac{b(\tau-\tau')}{\eta},\ \beta_2 = \frac{(a+b-1)\tau}{\eta},\ \beta = \tau - \tau'.$$

Consequently, (4.30) becomes

$$\beta_1'\Omega = (\beta_2' - \beta_1)\Omega' + \beta_2\Omega''. \tag{4.31}$$

If $\beta_1' = 0$, i.e., $\beta_1 = c_6$. This follows that $1 - \frac{b(\tau-\tau')}{\eta} = c_6$, where c_6 is a constant. If $c_6 = 0$. By (4.31), we have the following

$$\left(\frac{\tau}{\tau-\tau'}\right)'\Omega' + \left(\frac{\tau}{\tau-\tau'}\right)\Omega'' = 0.$$

Integrating the above result and we get

$$\Omega' = c_7\left(\frac{\tau-\tau'}{\tau}\right),$$

where c_7 is a non-zero constant. This results in $f' = c_7\left(\frac{\tau-\tau'}{\tau}\right) + \tau'$, which is a contradiction due to $\rho(\tau) < \rho(f)$. Therefore, $c_6 \neq 0$. Afterwards, by (4.31), we have

$$(\beta_2' - c_6)\Omega' + \beta_2\Omega'' = 0.$$

On integration, it deduces that

$$\beta_2\Omega' = c_6\Omega + c_8,$$

where c_8 is an integrating constant. The equality shown above suggests that

$$\frac{c_8}{\Omega} = -c_6 + \beta_2\frac{\Omega'}{\Omega}. \tag{4.32}$$

Now, using the first fundamental theorem and (4.32), we have

$$T\left(r, \frac{\Omega'}{\Omega}\right) = T\left(r, \frac{1}{\Omega}\right) = T(r,\Omega) + O(1),$$

also, as follows

$$S\left(r, \frac{\Omega'}{\Omega}\right) = S(r,\Omega).$$

If $c_8 = 0$, from (4.32) we write

$$\frac{\Omega'}{\Omega} = \frac{c_6}{\beta_2}.$$

Now, from the above result, we get

$$m\left(r,\frac{\Omega'}{\Omega}\right)=m\left(r,\frac{c_6}{\beta_2}\right)\le m\left(r,\frac{\eta}{\tau}\right)=S(r,\Omega).$$

Also follows

$$N\left(r,\frac{\Omega'}{\Omega}\right)=N\left(r,\frac{\eta}{\tau}\right)=O(\log r)=S(r,\Omega).$$

Thus we get

$$T\left(r,\frac{\Omega'}{\Omega}\right)=m\left(r,\frac{\Omega'}{\Omega}\right)+N\left(r,\frac{\Omega'}{\Omega}\right)=S(r,\Omega)=S\left(r,\frac{\Omega'}{\Omega}\right),$$

which is a contradiction. Therefore, $c_8\ne 0$. From, (4.32) we written as

$$\frac{1}{\Omega}=-\frac{c_6}{c_8}+\frac{\beta_2}{c_8}\frac{\Omega'}{\Omega}.$$

Now, by (4.29) and the lemma of logarithmic derivative, we deduce that

$$m\left(r,\frac{1}{\Omega}\right)\le T(r,\eta)+S(r,\Omega)=S(r,\Omega).$$

Furthermore, it follows from (4.29) that

$$\begin{aligned}N\left(r,\frac{1}{\Omega}\right) &\le N\left(r,\frac{1}{\eta}\right)+N\left(r,\frac{\Omega'}{\Omega}\right)+S(r,\Omega)\\ &\le \overline{N}(r,\Omega)+\overline{N}\left(r,\frac{1}{\Omega}\right)+S(r,\Omega)\\ &\le \overline{N}_{(2}\left(r,\frac{1}{\Omega}\right)+S(r,\Omega)=S(r,\Omega).\end{aligned}$$

Thus

$$T(r,\Omega)=m\left(r,\frac{1}{\Omega}\right)+N\left(r,\frac{1}{\Omega}\right)=S(r,\Omega),$$

a contradiction. Therefore, $\beta_1'\ne 0$. Following that, (4.31) can be expressed as

$$\Omega=\left(\frac{\beta_2'-\beta_1}{\beta_1'}\right)\Omega'+\left(\frac{\beta_2}{\beta_1'}\right)\Omega''.$$

Differentiating again the equality above and we get

$$\Omega'=\left(\frac{\beta_2'-\beta_1}{\beta_1'}\right)\Omega''+\left(\frac{\beta_2'-\beta_1}{\beta_1'}\right)'\Omega'+\left(\frac{\beta_2}{\beta_1'}\right)\Omega'''+\left(\frac{\beta_2}{\beta_1'}\right)'\Omega'',$$

this yields

$$\left[1-\left(\frac{\beta_2'-\beta_1}{\beta_1'}\right)'\right]\Omega'=\left[\frac{\beta_2'-\beta_1}{\beta_1'}+\left(\frac{\beta_2}{\beta_1'}\right)'\right]\Omega''+\left(\frac{\beta_2}{\beta_1'}\right)\Omega'''.$$

Rewrite the above as

$$\begin{aligned}\left[1-\left(\frac{\beta_2'-\beta_1}{\beta_1'}\right)'\right](\Omega'-\beta) &= \left[\frac{\beta_2'-\beta_1}{\beta_1'}+\left(\frac{\beta_2}{\beta_1'}\right)'\right](\Omega''-\beta')+\left(\frac{\beta_2}{\beta_1'}\right)(\Omega'''-\beta'')\\ &\quad+\left[\frac{\beta_2'-\beta_1}{\beta_1'}+\left(\frac{\beta_2}{\beta_1'}\right)'\right]\beta'-\left[1-\left(\frac{\beta_2'-\beta_1}{\beta_1'(z)}\right)'\right]\beta\\ &\quad+\left(\frac{\beta_2}{\beta_1'}\right)\beta'',\end{aligned}$$

and deduce that

$$\Upsilon_1(\Omega'-\beta)+\Upsilon_2(\Omega''-\beta')+\Upsilon_3(\Omega'''-\beta'')=\Upsilon_4, \tag{4.33}$$

where and in what follows

$$\Upsilon_1=1-\left(\frac{\beta_2'-\beta_1}{\beta_1'}\right)';\ \Upsilon_2=\frac{\beta_1-\beta_2'}{\beta_1'}-\left(\frac{\beta_2}{\beta_1'}\right)';\ \Upsilon_3=-\frac{\beta_2}{\beta_1'};$$

$$\Upsilon_4=\left[\frac{\beta_2'-\beta_1}{\beta_1'}+\left(\frac{\beta_2}{\beta_1'}\right)'\right]\beta'-\left[1-\left(\frac{\beta_2'-\beta_1}{\beta_1'}\right)'\right]\beta+\left(\frac{\beta_2}{\beta_1'}\right)\beta''.$$

Obviously, Υ_i $(i = 1, 2, 3, 4)$ are small functions of Ω.

Next if $\Upsilon_4 = 0$, then

$$\left[\frac{\beta_2' - \beta_1}{\beta_1'} + \left(\frac{\beta_2}{\beta_1'}\right)'\right]\beta' - \left[1 - \left(\frac{\beta_2' - \beta_1}{\beta_1'}\right)'\right]\beta + \left(\frac{\beta_2}{\beta_1'}\right)\beta'' = 0,$$

and so

$$\left[\left(\frac{\beta_2' - \beta_1}{\beta_1'}\right)\beta\right]' + \left[\left(\frac{\beta_2}{\beta_1'}\right)\beta'\right]' = \beta.$$

Put $\beta = K'$, where K is a primitive function of β. After integrating the above result twice, we obtain

$$\beta_2 K' = \beta_1 K + c_9\beta_1 + c_{10}, \tag{4.34}$$

where c_9, c_{10} are two non-zero constants. Since $\beta_1 \neq 0$, then, (4.34) can be written as

$$K = \frac{\beta_2}{\beta_1}K' - c_9 - \frac{c_{10}}{\beta_1}.$$

Differentiating the above estimate and we write

$$\left[1 - \left(\frac{\beta_2}{\beta_1}\right)'\right]K' = \frac{\beta_2}{\beta_1}K'' - c_{10}\left(\frac{1}{\beta_1}\right)'.$$

After some calculation, it follows that

$$\begin{aligned}\eta^2 K' &= \{2b(\tau - \tau')K' + (a + b - 1)\tau' K' + (a + b - 1)\tau K'' - c_6 b(\tau' - \tau'')\}\eta \\ &\quad + \{c_1 b(\tau - \tau') - (a + b - 1)\tau\}\eta' + b(\tau - \tau')^2\{(1 - a)\tau' - b\tau\}.\end{aligned}$$

Set

$$F_1 = \beta;\ F_2 = 2b\beta^2 + (a + b - 1)\tau'\beta + (a + b - 1)\tau\beta' - c_6 b\beta';$$
$$F_3 = c_1 b\beta - (a + b - 1)\tau;\ F_4 = b\beta^2\{(1 - a)\tau' - b\tau\}$$

are small functions of Ω. Therefore, we rewrite

$$F_1\eta^2 = F_2\eta + F_3\eta' + F_4. \tag{4.35}$$

On the other hand of (4.28), we deduce that

$$\eta = \frac{b\beta}{1 - \beta_2\frac{\Omega'}{\Omega}}. \tag{4.36}$$

Now substituting (4.36) in (4.35), we have

$$F_1\left(\frac{b\beta}{1 - \beta_2\frac{\Omega'}{\Omega}}\right)^2 = F_2\left(\frac{b\beta}{1 - \beta_2\frac{\Omega'}{\Omega}}\right) + F_3\left(\frac{b\beta}{1 - \beta_2\frac{\Omega'}{\Omega}}\right)' + F_4.$$

A methodical calculation shows that

$$\begin{aligned}F_4\beta_2^2\left(\frac{\Omega'}{\Omega}\right)^2 &= \left(F_1 b^2\beta^2 - F_2 b\beta - F_3 b\beta' - F_4\right) + \left(F_2 b\beta\beta_2 + F_3 b\beta'\beta_2\right. \\ &\quad \left. - F_3 b\beta\beta_2' + 2F_4\beta_2\right)\left(\frac{\Omega'}{\Omega}\right) - F_3 b\beta\beta_2\left(\frac{\Omega'}{\Omega}\right)'.\end{aligned}$$

The above estimate, stated in the form

$$\aleph_1\left(\frac{\Omega'}{\Omega}\right)^2 = \aleph_2 + \aleph_3\left(\frac{\Omega'}{\Omega}\right) + \aleph_4\left(\frac{\Omega'}{\Omega}\right)', \tag{4.37}$$

where

$$\begin{aligned}\aleph_1 &= F_4\beta_2^2;\ \aleph_2 = F_1 b^2\beta^2 - F_2 b\beta - F_3 b\beta' - F_4; \\ \aleph_3 &= F_2 b\beta\beta_2 + F_3 b\beta'\beta_2 - F_3 b\beta\beta_2' + 2F_4\beta_2;\ \aleph_4 = -F_3 b\beta\beta_2\end{aligned}$$

are small functions of Ω. Then, based on (4.37), we conclude that

$$\begin{aligned} 2m\left(r,\frac{\Omega'}{\Omega}\right) &= m\left[r,\left(\frac{\Omega'}{\Omega}\right)^2\right] \\ &\leq m\left[r,\aleph_1\left(\frac{\Omega'}{\Omega}\right)^2\right]+m\left(r,\frac{1}{\aleph_1}\right)+O(1) \\ &\leq m\left[r,\aleph_2+\aleph_3\left(\frac{\Omega'}{\Omega}\right)+\aleph_4\left(\frac{\Omega'}{\Omega}\right)'\right]+S(r,\Omega) \\ &\leq m\left(r,\frac{\Omega'}{\Omega}\right)+S\left(r,\frac{\Omega'}{\Omega}\right). \end{aligned}$$

Then

$$m\left(r,\frac{\Omega'}{\Omega}\right)=S\left(r,\frac{\Omega'}{\Omega}\right).$$

Moreover, we have

$$\begin{aligned} N\left(r,\frac{\Omega'}{\Omega}\right) &\leq \overline{N}(r,\Omega)+\overline{N}\left(r,\frac{1}{\Omega}\right)+S(r,\Omega) \\ &\leq \overline{N}_{(2}\left(r,\frac{1}{\Omega}\right)+S(r,\Omega)=S(r,\Omega)=S\left(r,\frac{\Omega'}{\Omega}\right). \end{aligned}$$

Thus

$$T\left(r,\frac{\Omega'}{\Omega}\right)=m\left(r,\frac{\Omega'}{\Omega}\right)+N\left(r,\frac{\Omega'}{\Omega}\right)=S\left(r,\frac{\Omega'}{\Omega}\right),$$

which is a contradiction. So, $\Upsilon_4\neq 0$. Then, rewrite (4.33) as

$$\frac{\Upsilon_4}{\Omega'-\beta}=\Upsilon_1+\Upsilon_2\frac{\Omega''-\beta'}{\Omega'-\beta}+\Upsilon_3\frac{\Omega'''-\beta''}{\Omega'-\beta}.$$

Now, using the above result and the lemma for the logarithmic derivative, we have

$$\begin{aligned} m\left(r,\frac{1}{\Omega'-\beta}\right) &\leq m\left(r,\frac{\Upsilon_4}{\Omega'-\beta}\right)+m\left(r,\frac{1}{\Upsilon_4}\right) \\ &= m\left(r,\Upsilon_1+\Upsilon_2\frac{\Omega''-\beta'}{\Omega'-\beta}+\Upsilon_3\frac{\Omega'''-\beta''}{\Omega'-\beta}\right)+S(r,\Omega) \\ &= S(r,\Omega). \end{aligned} \tag{4.38}$$

Supposing that w_4 is a zero of $\Omega'-\beta$ along multiplicity m_3, and not the zero of Ω. According to (VI) and (4.28), we deduce that w_4 is a zero of η. Moreover, by the hypothesis (VI) we can conclude that, $\Omega'-\beta$ has finitely many multiple zeros, so $N_{(2}(r,\frac{1}{\Omega'-\beta})=O(\log r)=S(r,\Omega)$. Then,

$$N(r,\beta=\Omega'|\Omega\neq 0)\leq N\left(r,\frac{1}{\eta}\right)+N_{(2}\left(r,\frac{1}{\Omega'-\beta}\right)=S(r,\Omega).$$

However, by (VI), we get

$$\begin{aligned} N\left(r,\frac{1}{\Omega'-\beta}\right) &= N(r,\beta=\Omega'|\Omega\neq 0)+N(r,\beta=\Omega'|\Omega=0) \\ &\leq N\left(r,\frac{1}{\Omega}\right)+N_{(2}\left(r,\frac{1}{\Omega'-\beta}\right)+S(r,\Omega) \\ &= N\left(r,\frac{1}{\Omega}\right)+S(r,\Omega). \end{aligned} \tag{4.39}$$

Define

$$\zeta=\frac{\Omega'-\beta}{\Omega}. \tag{4.40}$$

Clearly, $\zeta \neq 0$. It is clear from (V) that $N(r,\zeta) = S(r,\Omega)$, and according to the logarithmic derivative lemma, we have

$$\begin{aligned} T(r,\zeta) = m(r,\zeta) + S(r,\Omega) &\leq m\left(r, \frac{\Omega'}{\Omega}\right) + m\left(r, \frac{\beta}{\Omega}\right) + S(r,\Omega) \\ &\leq m\left(r, \frac{1}{\Omega}\right) + S(r,\Omega). \end{aligned} \tag{4.41}$$

On the other hand, (4.28) can be expressed as

$$\Omega = \frac{(a+b-1)\tau\Omega'}{[1-b(\tau-\tau')]\eta}.$$

Consequently, by (4.29), it follows from the estimation above that

$$\begin{aligned} T(r,\Omega) &\leq T(r,\Omega') + T(r,\eta) + O(1) \\ &\leq T(r,\Omega') + S(r,\Omega). \end{aligned}$$

Also, we have from the lemma of the logarithmic derivative

$$\begin{aligned} T(r,\Omega') = m(r,\Omega') &\leq m\left(r, \frac{\Omega'}{\Omega}\right) + m(r,\Omega) + O(1) \\ &= T(r,\Omega) + S(r,\Omega). \end{aligned}$$

By combining the two observations mentioned above, we get

$$T(r,\Omega) = T(r,\Omega') + S(r,\Omega). \tag{4.42}$$

Then, from (4.38), (4.39) and (4.42), it yields that

$$\begin{aligned} m\left(r, \frac{1}{\Omega}\right) &= T(r,\Omega) - N\left(r, \frac{1}{\Omega}\right) + S(r,\Omega) \\ &= T(r,\Omega') - N\left(r, \frac{1}{\Omega}\right) + S(r,\Omega) \\ &= m\left(r, \frac{1}{\Omega'-\beta}\right) + N\left(r, \frac{1}{\Omega'-\beta}\right) - N(r, \frac{1}{\Omega}) + S(r,\Omega) \\ &= S(r,\Omega). \end{aligned}$$

Following that, (4.41) becomes

$$T(r,\zeta) = S(r,\Omega). \tag{4.43}$$

From (4.40) we written as

$$\frac{1}{\Omega} = \frac{1}{\beta}\frac{\Omega'}{\Omega} - \frac{\zeta}{\beta}.$$

Now, by applying the logarithmic derivative lemma and using (4.43), we obtain

$$m\left(r, \frac{1}{\Omega}\right) \leq m\left(r, \frac{1}{\beta}\frac{\Omega'}{\Omega} - \frac{\zeta}{\beta}\right) = S(r,\Omega).$$

Combining the above result and

$$\begin{aligned} N\left(r, \frac{1}{\Omega}\right) &\leq 2N\left(r, \frac{1}{\beta}\right) + N\left(r, \frac{\Omega'}{\Omega}\right) + N(r,\zeta) \\ &\leq \overline{N}(r,\Omega) + \overline{N}\left(r, \frac{1}{\Omega}\right) + S(r,\Omega) \\ &\leq \overline{N}_{(2}\left(r, \frac{1}{\Omega}\right) + S(r,\Omega) = S(r,\Omega), \end{aligned}$$

it deduces that

$$T(r,\Omega) = m\left(r, \frac{1}{\Omega}\right) + N\left(r, \frac{1}{\Omega}\right) = S(r,\Omega),$$

which shows a contradiction. This completes the proof (d) of (B). Thus, the proof of Theorem 1.1 is completed.

5. Compliance with ethical standards

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Human participants: This article does not contain any studies with human participants or animals performed by any of the authors.

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