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Indefinite q-integrals of quotients of q-hypergeometric functions

Gamela E. Heragy[1], Zeinab S.I. Mansour[2], and Karima M. Oraby[1]

[1] Mathematics Department, Faculty of Science, Suez University, Suez, Egypt.

[2] Mathematics Department, Faculty of Science, Cairo University, Giza, Egypt.

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ABSTRACT

This paper uses Heine contiguous relations for the basic hypergeometric function ${}_2\phi_1$, the q-integrating factor method for solving linear first order q-difference equations and an indefinite q-integral formula involving two arbitrary functions to derive indefinite q-integrals involving quotients of the hypergeometric functions ${}_2\phi_1$.

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1. Introduction

Heine in [1] introduced the basic hypergeometric function ${}_2\phi_1(q^a, q^b;\, q^c;\, q, x)$ as an extension of the hypergeometricfunction ${}_2F_1(a, b; c, x)$. His book [2], published in 1878, included contiguous relations of the function ${}_2\phi_1$ and continued fractions. We can transform Heine's contiguous relations to a first-order q-difference equation using the q-integrating factor method, a tool used to solve the first-order q-difference equation by multiplying with an appropriate function. In this paper, we apply the integrating factor method to Heine's contiguous relations of the q-hypergeometric function ${}_2\phi_1$, the relations (A1)-(A9) mentioned in the appendix, to obtain a set of first-order q-difference equations involving the function ${}_2\phi_1$. Then, we use the indefinite q-integral formula,

$$\int D_q\left(\frac{D_{q^{-1}}\, h(x)}{D_{q^{-1}}\, y(x)}\right) y(x)\, d_q x = D_{q^{-1}}\, h(x)\frac{y\left(\frac{x}{q}\right)}{D_{q^{-1}}\, y(x)} - h\left(\frac{x}{q}\right), \quad (1.1)$$

which holds for arbitrary functions h and y, which we shall prove in Theorem 3.1. We choose h and y so that $\frac{D_{q^{-1}}\, h(x)}{D_{q^{-1}}\, y(x)}$ is a reciprocal of a function g, $g := f \cdot {}_2\phi_1$, where f is one of the integrating factors we used to turn Heine's contiguous relations to first- order q-difference equations. Then $D_q\,(1/g)$ will give a quotient of functions ${}_2\phi_1$. We use this approach to derive many indefinite and, consequently, definite q-integrals of quotients of functions ${}_2\phi_1$. It is worth noting that the results of this paper are generalizations of Conway's results [3] for indefinite integrals of quotients of functions ${}_2F_1$.

* Corresponding author at Suez University

E-mail addresses: karima.mohamed@suezuniv.edu.eg (Karima M. Oraby)

Throughout this paper, let $q \in (0,1)$, $\mathbb{N}$ be the set of positive integers, and $\mathbb{N}_0$ be the set of non-negative integers. We follow Gasper and Rahman, see [4] for more details about q-notations and definitions. A q-natural number $[n]_q$ is defined by $[n]_q := \frac{1-q^n}{1-q}, n \in \mathbb{N}_0$. The indefinite q-integral

$$\int f(x)\, d_q x = F(x), \quad (1.2)$$

means that $D_q F(x) = f(x)$, where D_q is the Jackson's q-difference operator which is defined in (1.3) below. Jackson's q-derivative of a function f is denoted by $D_q f(x)$ and is defined as

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x}, & if\ x \neq 0; \\ f'(0) & if\ x = 0\,, \end{cases} \quad (1.3)$$

provided that $f(0)$ exists, see [2,5]. Jackson's q-integral of a function f is defined by

$$\int_0^a f(t)\ d_q t := (1-q)a \sum_{n=0}^{\infty} q^n f(aq^n), \ \ a \in \mathbb{R}, \quad (1.4)$$

provided that the corresponding series in (1.4) converges, see [6]. The q-hypergeometric series ${}_2\phi_1$ is defined by

$${}_2\phi_1\,(a, b; c; q, x) = \sum_{n=0}^{\infty} \frac{(a;q)_n\ (b;q)_n}{(c;q)_n\ (q;q)_n}\ x^n, \quad (1.5)$$

where $|x| < 1$, see [4]. We organize this paper as follows: In Section 2, we use the contiguous relations (A1)-(A9) and the integrating factor methods to derive a family of indefinite q-integrals of the q-hypergeometric functions ${}_2\phi_1$. In Section 3, we apply the results of Section 2 to the indefinite q-integral formula (1.1) with specific choices for the functions y and h to obtain new indefinite q-integrals involving quotients of the q-hypergeometric function.

2. Indefinite q-ntegrals involving ${}_2\phi_1$ functions

In this section, to save space, we use the following notations. The symbol ϕ shall always mean ${}_2\phi_1(q^a, q^b; q^c; q, x)$. We use $\phi(a\pm)$, $\phi(b\pm)$, or $\phi(c\pm)$ to denote ${}_2\phi_1(q^{a\pm1}, q^b; q^c; q, x)$, ${}_2\phi_1(q^a, q^{b\pm1}; q^c; q, x)$, or ${}_2\phi_1(q^a, q^b; q^{c\pm1}; q, x)$, respectively. We also use $\phi(qx)$ to denote ${}_2\phi_1(q^a, q^b; q^c; q, qx)$, $\phi(a\pm, b\pm)$, $\phi(a\pm, c\pm)$ to denote ${}_2\phi_1(q^{a\pm1}, q^{b\pm1}; q^c; q, x)$, ${}_2\phi_1(q^{a\pm1}, q^b; q^{c\pm1}; q, x)$. Substituting with

$$D_{q^{-1},x}\, {}_2\phi_1(q^a, q^b; q^c; q, x) = \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x/q), \tag{2.1}$$

or

$$D_{q,x}\, {}_2\phi_1(q^a, q^b; q^c; q, x) = \frac{(1-q^a)(1-q^b)}{(1-q)(1-q^c)}\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x), \tag{2.2}$$

see [4, Eq.(1.12.ii)], into Equations (A1)-(A9), we obtain

$$D_{q,x}\phi(x) + \frac{[a]_q}{q^a x}\phi(x) = \frac{[a]_q}{q^a x}\phi(a+)(x), \tag{2.3}$$

$$D_{q,x}\phi(x) + \frac{[b]_q}{q^b x}\,\phi(x) = \frac{[b]_q}{q^b x}\phi(b+)(x), \tag{2.4}$$

$$D_{q,x}\phi(x) + \frac{q^{1-c}[c-1]_q}{x}\phi(x) = \frac{q^{1-c}[c-1]_q}{x}\phi(c-)(x), \tag{2.5}$$

$$D_{q^{-1},x}\phi(x) - \frac{[a]_q}{(1-\frac{x}{q})}\phi(x) = \frac{q^c[a]_q[b-c]_q}{(1-\frac{x}{q})[c]_q}\phi(a+, c+)(x), \tag{2.6}$$

$$D_{q,x}\phi(x) - \frac{q^{a-c}[b]_q}{(1-q^{a+b-c}x)}\phi(x) = \frac{[b]_q[a-c]_q}{[c]_q(1-q^{a+b-c}x)}\phi(b+; c+)(x), \tag{2.7}$$

$$D_{q,x}\phi(x) + \frac{q[c-a]_q - q^a[b]_q x}{x(q^c - q^{a+b}x)}\,\phi(x) = \frac{q[c-a]_q}{x(q^c - q^{a+b}x)}\,\phi(a-)(x), \tag{2.8}$$

$$D_{q,x}\phi(x) + \frac{q[c-b]_q - q^b[a]_q x}{x(q^c - q^{a+b}x)}\phi(x) = \frac{q[c-b]_q}{x(q^c - q^{a+b}x)}\phi_1(b-)(x). \tag{2.9}$$

Now, we define the q-integrating factors for these Equations to be $f_i(x)$, where $i \in \{1, 2, 3, 4, 5, 6, 7\}$, then the q-integrating factors for (2.3)-(2.9) are

$$f_1(x) = x^a, \qquad f_2(x) = x^b, \qquad f_3(x) = x^{c-1},$$
$$f_4(x) = (x; q)_a, \qquad f_5(x) = (q^{a-c}x; q)_b, \qquad f_6(x) = x^{-\alpha}(q^{a-\alpha-c}x; q)_{\alpha+b}, \tag{2.10}$$
$$f_7(x) = x^{-\gamma}(q^{b-\gamma-c}x; q)_{\gamma+a},$$

where $q^\alpha = 1 + q^{1-c} - q^{1-a}$, and $q^\gamma = 1 + q^{1-c} - q^{1-b}$. The corresponding q-integrated recurrences for Equations (2.3)-(2.9) are

$$\begin{aligned}
D_{q,x}\Big(x^{a}\phi(x)\Big) &= [a]_q x^{a-1}\phi(a+)(x), && (2.11)\\
D_{q,x}\Big(x^{b}\,\phi(x)\Big) &= [b]_q x^{b-1}\phi(b+)(x), && (2.12)\\
D_{q,x}\Big(x^{c-1}\phi(x)\Big) &= [c-1]_q x^{c-2}\phi(c-)(x), && (2.13)\\
D_{q^{-1},x}\Big((x;q)_a\,\phi(x)\Big) &= \frac{[a]_q[b-c]_q}{q^{-c}[c]_q}(x;q)_{a-1}\phi(a+,c+)(x), && (2.14)\\
D_{q,x}\Big((q^{a-c}x;q)_b\phi(x)\Big) &= \frac{[a-c]_q[b]_q}{[c]_q}(q^{a-c+1}x;q)_{b-1}\phi(b+,c+)(x), && (2.15)\\
D_{q,x}\Big(x^{-\alpha}(q^{a-\alpha-c}x;q)_{\alpha+b}\phi(x)\Big) &= \frac{q^{1-\alpha-c}[c-a]_q}{x^{\alpha+1}(q^{a+b-c}x;q)_{1-\alpha-b}}\,\phi(a-)(x), && (2.16)\\
D_{q,x}\Big(x^{-\gamma}(q^{b-\gamma-c}x;q)_{\gamma+a}\,\phi(x)\Big) &= \frac{q^{1-\gamma-c}[c-b]_q}{x^{\gamma+1}(q^{a+b-c}x;q)_{1-\gamma-a}}\,\phi(b-)(x). && (2.17)
\end{aligned}$$

We obtain the following q-integrals from (2.11)-(2.17),

$$\int x^{a-1}\,{}_2\phi_1(q^{a+1},q^b;q^c;q,x)\,d_qx = \frac{x^a}{[a]_q}\,{}_2\phi_1(q^a,q^b;q^c;q,x),\ (\Re(a)>0\ \text{ and } |x|<1),$$

$$\int x^{b-1}\,{}_2\phi_1(q^{a},q^{b+1};q^c;q,x)\,d_qx = \frac{x^b}{[b]_q}\,{}_2\phi_1(q^a,q^b;q^c;q,x),\ (\Re(b)>0\ \text{ and } |x|<1),$$

$$\int x^{c-2}\,{}_2\phi_1(q^{a},q^{b};q^{c-1};q,x)\,d_qx = \frac{x^{c-1}}{[c-1]_q}\,{}_2\phi_1(q^a,q^b;q^c;q,x),\ (\Re(c)>1\ \text{ and } |x|<1),$$

$$\int (qx;q)_{a-1}\,{}_2\phi_1(q^{a+1},q^{b};q^{c+1};q,qx)\,d_qx = \frac{q^{-c}[c]_q}{[b-c]_q[a]_q}(x;q)_a\,{}_2\phi_1(q^a,q^b;q^c;q,x),$$
$$(b\neq c,\ a\neq 0,\ \text{ and } \ |x|<\frac{1}{q}),$$

$$\int (q^{a-c+1}x;q)_{b-1}\,{}_2\phi_1(q^{a},q^{b+1};q^{c+1};q,x)\,d_qx = \frac{[c]_q}{[a-c]_q[b]_q}(q^{a-c}x;q)_b\,{}_2\phi_1(q^a,q^b;q^c;q,x),$$
$$(a\neq c,\ b\neq 0,\ \text{ and } \ |x|<1),$$

$$\int \frac{x^{-(\alpha+1)}}{(q^{a+b-c}x;q)_{1-\alpha-b}}\,{}_2\phi_1(q^{a-1},q^{b};q^{c};q,x)\,d_qx = \frac{q^{\alpha+c-1}}{[c-a]_q}x^{-\alpha}(q^{a-\alpha-c}x;q)_{\alpha+b}\,{}_2\phi_1(q^a,q^b;q^c;q,x),$$
$$(q^{\alpha}=1+q^{1-c}-q^{1-a}\ \text{ and } |x|<1),$$

and for $|x| < 1$ and $q^{\gamma} = 1 + q^{1-c} - q^{1-b}$

$$\int \frac{x^{-(\gamma+1)}}{(q^{a+b-c}x;q)_{1-\gamma-a}}\,{}_2\phi_1(q^a,q^{b-1};q^c;q,x)\,d_qx = \frac{q^{\gamma+c-1}}{[c-b]_q}x^{-\gamma}(q^{b-\gamma-c}x;q)_{\gamma+a}\,{}_2\phi_1(q^a,q^b;q^c;q,x).$$

3. Indefinite q-integrals involving quotients of ${}_2\phi_1$ functions

Theorem 3.1. *The indefinite q-integral formula (1.1) holds for arbitrary functions y and h.*

Proof. The proof follows by applying the q-integration by parts rule, see [7],

$$\int u(qx)D_qv(x)\,d_qx = (uv)(x) - \int D_qu(x)v(x)\,d_qx,$$

with $u(x) = y(x/q)$ and $v(x) = \dfrac{D_{q^{-1}}h(x)}{D_{q^{-1}}y(x)}$, and noting that from the fundamental theorem of q-Calculus, see [7], $\int D_qu(x)v(x)\,d_qx = \int D_qh(x/q)d_qx = h(x/q)$.

□

Theorem 3.2. *The following indefinite q-integrals hold true:*

(i) If $-c \notin \mathbb{N}_0$ and $|x| < q$, then

$$\int \frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^{a+2},q^{b+2};q^{c+2};q,x/q)}{{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x)\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x/q)}\,d_qx$$
$$= \frac{[c+1]_q}{[a+1]_q[b+1]_q}\left(\frac{[a]_q[b]_q}{q[c]_q}x - \frac{{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x/q)}\right).$$

(ii) If $\Re(c) < 0$, $-c \notin \mathbb{N}_0$, and $|x| < q$, then

$$\int x^{-1-c}\frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^{a+1},q^{b+1};q^c;q,x/q)}{{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x)\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x/q)}\,d_qx$$
$$= \frac{q^cx^{-c}}{[c]_q}\left(\frac{[a]_q[b]_qx}{q[c]_q[1-c]_q}x - \frac{{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x/q)}\right).$$

(iii) If $\Re(b) < -1$, $-c \notin \mathbb{N}_0$, and $|x| < q$, then

$$\int x^{-b-2}\frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^{a+1},q^{b+2};q^{c+1};q,x/q)}{{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x)\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x/q)}\,d_qx$$
$$= \frac{-q^{b+1}x^{-b}}{[b+1]_q}\left(\frac{q^{b-1}[a]_q}{[c]_q} + \frac{{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{x\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x/q)}\right).$$

(iv) If $\Re(a) < -1$, $-c \notin \mathbb{N}_0$, and $|x| < q$, then

$$\int x^{-a-2} \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+2}, q^{b+1}; q^{c+1}; q, x/q)}{{}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x)\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x/q)}\, d_qx$$
$$= \frac{-q^{a+1}x^{-a}}{[a+1]_q} \left(\frac{q^{a-1}[b]_q}{[c]_q} + \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{x\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x/q)} \right).$$

Proof. To prove the theorem, we substitute with $y(x) = {}_2\phi_1(q^a, q^b; q^c; q, x)$ into Theorem 3.1 with different choices of the function $h(x)$. To prove (i), we take $h(x) = x$, and substitute from (2.1) into (1.1). To prove (ii), we substitute with

$$h(x) = \frac{x^{1-c}}{[1-c]_q},\ c \neq 1,\ D_{q^{-1}}h(x) = \left(\frac{q}{x}\right)^c$$

into (1.1) then (ii) follows from (2.13). The proof of (iii) follows by substituting with

$$h(x) = \frac{-x^{-b}}{[b]_q},\ b \neq 0,\ \text{and } D_{q^{-1}}h(x) = qx^{-b-1}$$

into (1.1) and using (2.12). The identity (iv) follows from (iii) by swapping the parameters a and b. □

Theorem 3.3. *The following identities hold true:*

(i) If $-c \notin \mathbb{N}_0$ *and* $|x| < q$, *then*

$$\int \frac{1}{(x/q; q)_{a+2}} \cdot \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+2}, q^{b+1}; q^{c+2}; q, x)}{{}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x)\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x/q)}\, d_qx$$
$$= \frac{[c+1]_q}{q^{c+1}[a+1]_q[b-c]_q(x/q; q)_a} \left(\frac{[b]_q}{[c]_q} - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{(1-q^{a-1}x)\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x/q)} \right).$$

(ii) If $-c \notin \mathbb{N}_0$ *and* $|x| < q$, *then*

$$\int \frac{1}{(q^{a-c-1}x; q)_{b+2}} \cdot \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^{b+2}; q^{c+2}; q, x/q)}{{}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x/q)\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x)}\, d_qx$$
$$= \frac{[c+1]_q}{[b+1]_q[a-c]_q(q^{a-c-1}x; q)_b} \left(\frac{[a]_q}{q^{a-c}[c]_q} - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{(1-q^{a-c+b-1}x)\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x)} \right).$$

(iii) Let $q^\alpha = 1 + q^{-c} - q^{-a}$. *If* $-c \notin \mathbb{N}_0$, $|x| < \min\{q,\, q^{1+c-a}\}$, *and* $\Re(\alpha) > 0$, *then*

$$\int \frac{x^{\alpha-1}}{(q^{a-c-\alpha-1}x;q)_{\alpha+b+2}} \cdot \frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^a,q^{b+1};q^{c+1};q,x/q)}{{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x/q)\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x)}\,d_qx$$
$$= \frac{q^c x^\alpha}{[c-a]_q}\left(\frac{[a]_q[b]_q x}{q[c]_q[\alpha+1]_q}\,{}_2\phi_1(q^{b+\alpha+1},q^{\alpha+1};q^{\alpha+2};q,q^{a-\alpha-c-1}x) - \frac{(q^{a+b-c}x;q)_{-(b+\alpha+1)}\,{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x/q)}\right).$$

(iv) Let $q^\gamma = 1+q^{-c}-q^{-b}$. *If* $-c \notin \mathbb{N}_0$, $|x| < \min\{q,\, q^{1+c-b}\}$, *and* $\Re(\gamma) > 0$, *then*

$$\int \frac{x^{\gamma-1}}{(q^{b-c-\gamma-1}x;q)_{\gamma+a+2}} \cdot \frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^{a+1},q^b;q^{c+1};q,x/q)}{{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x/q)\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x)}\,d_qx$$
$$= \frac{q^c x^\gamma}{[c-b]_q}\left(\frac{[a]_q[b]_q x}{q[c]_q[\gamma+1]_q}\,{}_2\phi_1(q^{a+\gamma+1},q^{\gamma+1};q^{\gamma+2};q,q^{b-\gamma-c-1}x) - \frac{(q^{a+b-c}x;q)_{-(a+\gamma+1)}\,{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x/q)}\right).$$

Proof. To prove the theorem, we substitute with $y(x) = {}_2\phi_1(q^a,q^b;q^c;q,x)$ into Theorem 3.1 with different choices of the function $h(x)$. To prove (i), we substitute with

$$h(x) = \frac{1}{[a]_q(x;q)_a},\ a \neq 0, \quad \text{and } D_{q^{-1}}h(x) = \frac{1}{(\frac{x}{q};q)_{a+1}}$$

into (1.1) and using (2.14). To prove (ii), we substitute with

$$h(x) = \frac{1}{[b]_q(q^{a-c}x;q)_b},\ b \neq 0, \quad \text{and } D_{q^{-1}}h(x) = \frac{q^{a-c}}{(q^{a-c-1}x;q)_{a+1}}$$

into (1.1) and using (2.15). To prove (iii), we take

$$h(x) = \frac{x^{\alpha+1}}{[\alpha+1]_q}\,{}_2\phi_1(q^{b+\alpha+1},q^{\alpha+1};q^{\alpha+2};q,q^{a-\alpha-c}x),$$

$q^\alpha = 1+q^{-c}-q^{-a}$and $|q^{a-\alpha-c}x| < 1$, and substitute with

$$D_{q^{-1}}h(x) = \left(\frac{x}{q}\right)^\alpha (q^{a+b-c}x;q)_{-b-\alpha-1}$$

into (1.1) and using (2.16). Finally, (iv) follows from (iii) by swapping the parameters a and b. □

Theorem 3.4. *We have the following indefinite q-integrals* *(i) If $\Re(c) > 1$ and $|x| < q$, then*

$$\int x^{c-1} \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^c; q, x/q)}{{}_2\phi_1(q^a, q^b; q^{c-1}; q, x)\, {}_2\phi_1(q^a, q^b; q^{c-1}; q, x/q)}\, d_qx = \frac{q^{2-c}[c-1]_q x^{c-1}}{[a]_q[b]_q} \left(1 - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^a, q^b; q^{c-1}; q, x/q)}\right).$$

(ii) If $-c \notin \mathbb{N}_0 \cup \{-1, -2\}$ and $|x| < q$, then

$$\int \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^a, q^b; q^{c-2}; q, x/q)}{{}_2\phi_1(q^a, q^b; q^{c-1}; q, x)\, {}_2\phi_1(q^a, q^b; q^{c-1}; q, x/q)}\, d_qx = \frac{x}{q[c-2]_q} \left([c-1]_q - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^a, q^b; q^{c-1}; q, x/q)}\right).$$

(iii) If $\Re(c-b) > 1$, $-c \notin \mathbb{N}_0 \cup \{-1\}$, and $|x| < q$, then

$$\int x^{c-b-2} \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^a, q^{b+1}; q^{c-1}; q, x/q)}{{}_2\phi_1(q^a, q^b; q^{c-1}; q, x)\, {}_2\phi_1(q^a, q^b; q^{c-1}; q, x/q)}\, d_qx = \frac{q^{b-c+1} x^{c-b-1}}{[b]_q} \left(\frac{[c-1]_q}{[c-b-1]_q} - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^a, q^b; q^{c-1}; q, x/q)}\right).$$

(iv) If $\Re(c-a) > 1$, $-c \notin \mathbb{N}_0 \cup \{-1\}$, and $|x| < q$, then

$$\int x^{c-a-2} \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^b; q^{c-1}; q, x/q)}{{}_2\phi_1(q^a, q^b; q^{c-1}; q, x)\, {}_2\phi_1(q^a, q^b; q^{c-1}; q, x/q)}\, d_qx = \frac{q^{a-c+1} x^{c-a-1}}{[a]_q} \left(\frac{[c-1]_q}{[c-a-1]_q} - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^a, q^b; q^{c-1}; q, x/q)}\right).$$

Proof. To prove the theorem, we substitute with $y(x) = x^{c-1}\, {}_2\phi_1(q^a, q^b; q^c; q, x)$ into Theorem 3.1 with different choices of the function $h(x)$. To prove (i), we substitute with

$$h(x) = \frac{x^{c-1}}{[c-1]_q}, \quad c \neq 1, \quad \text{and } D_{q^{-1}} h(x) = q^{2-c} x^{c-2}$$

into (1.1) and use (2.2) and (2.13) . To prove (ii), we set $h(x) = x$ in (1.1) and using (2.13). The proof of (iii) follows by substituting with

$$h(x) = \frac{x^{c-b-1}}{[c-b-1]_q}, \quad \text{and } D_{q^{-1}} h(x) = q^{2+b-c} x^{c-b-2}$$

into (1.1) and using (2.12). The identity (iv) follows from (iii) by swapping the parameters a and b. □

Theorem 3.5. *We have the following indefinite q-integrals* *(i) If* $\Re(b) > -1$, $-c \notin \mathbb{N}_0$, *and* $|x| < q$, *then*

$$\int x^b \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^{b+2}; q^{c+1}; q, x/q)}{{}_2\phi_1(q^a, q^{b+1}; q^c; q, x)\, {}_2\phi_1(q^a, q^{b+1}; q^c; q, x/q)}\, d_qx$$
$$= \frac{q^{1-b}[c]_q x^b}{[a]_q[b+1]_q}\left(1 - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^a, q^{b+1}; q^c; q, x/q)}\right).$$

(ii) If $\Re(b-c) > -1$, $-c \notin \mathbb{N}_0 \cup \{-1\}$, *and* $|x| < q$, *then*

$$\int x^{b-c} \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^a, q^{b+1}; q^{c-1}; q, x/q)}{{}_2\phi_1(q^a, q^{b+1}; q^c; q, x)\, {}_2\phi_1(q^a, q^{b+1}; q^c; q, x/q)}\, d_qx$$
$$= \frac{q^{c-b-1}x^{b-c+1}}{[c-1]_q}\left(\frac{[b]_q}{[b-c+1]_q} - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^a, q^{b+1}; q^c; q, x/q)}\right).$$

(iii) If $\Re(b-a) > 0$, $-c \notin \mathbb{N}_0$, *and* $|x| < q$, *then*

$$\int x^{b-a-1} \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^c; q, x/q)}{{}_2\phi_1(q^a, q^{b+1}; q^c; q, x)\, {}_2\phi_1(q^a, q^{b+1}; q^c; q, x/q)}\, d_qx$$
$$= \frac{q^{a-b}x^{b-a}}{[a]_q}\left(\frac{[b]_q}{[b-a]_q} - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^a, q^{b+1}; q^c; q, x/q)}\right).$$

(iv) If $\Re(b) > -1$, $-c \notin \mathbb{N}_0$, *and* $|x| < q$, *then*

$$\int \frac{x^b}{(q^{a-c-1}x; q)_{b+2}} \cdot \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^a, q^{b+2}; q^{c+1}; q, x/q)}{{}_2\phi_1(q^a, q^{b+1}; q^c; q, x/q)\, {}_2\phi_1(q^a, q^{b+1}; q^c; q, x)}\, d_qx$$
$$= \frac{q^{-b}[c]_q x^b}{[a-c]_q[b+1]_q(q^{a-c-1}x; q)_b}\left(1 - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{(1 - q^{a-c+b-1}x)\, {}_2\phi_1(q^a, q^{b+1}; q^c; q, x/q)}\right).$$

Proof. To prove the theorem, we substitute with $y(x) = x^b\, {}_2\phi_1(q^a, q^b; q^c; q, x)$ into Theorem 3.1 with different choices of the function $h(x)$. The proof of (i) follows by substituting with $h(x) = \frac{x^b}{[b]_q}$, $b \neq 0$ into (1.1) and using (2.2). To prove (ii), we substitute with $h(x) = \frac{x^{b-c+1}}{[b-c+1]_q}$ and $D_{q^{-1}}h(x) = q^{c-b}x^{b-c}$ into (1.1) and using (2.13). The proof of (iii) follows by substituting with

$$h(x) = \frac{x^{b-a}}{[b-a]_q},\ b \neq a \ \text{ and } D_{q^{-1}}h(x) = q^{a-b+1}x^{b-a-1}$$

into (1.1) and using (2.11). Finally, The proof of (iv) follows by substituting with

$$h(x) = \frac{x^b}{[b]_q(q^{a-c}x; q)_b},\ b \neq 0 \ \text{ and } D_{q^{-1}}h(x) = \frac{q^{1-b}x^{b-1}}{(q^{a-c-1}x; q)_{b+1}}$$

into (1.1) and using (2.15). □

Theorem 3.6. *We have the following indefinite q-integrals*

(i) If $\Re(a) > -1$, $-c \notin \mathbb{N}_0$, *and* $|x| < q$, *then*

$$\int x^a \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+2}, q^{b+1}; q^{c+1}; q, x/q)}{{}_2\phi_1(q^{a+1}, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^b; q^c; q, x/q)}\, d_qx = \frac{q^{1-a}[c]_q x^a}{[b]_q[a+1]_q}\left(1 - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^{a+1}, q^b; q^c; q, x/q)}\right).$$

(ii) If $\Re(a-c) > -1$, $-c \notin \mathbb{N}_0 \cup \{-1\}$, *and* $|x| < q$, *then*

$$\int x^{a-c} \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^b; q^{c-1}; q, x/q)}{{}_2\phi_1(q^{a+1}, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^b; q^c; q, x/q)}\, d_qx = \frac{q^{c-a-1}x^{a-c+1}}{[c-1]_q}\left(\frac{[a]_q}{[a-c+1]_q} - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^{a+1}, q^b; q^c; q, x/q)}\right).$$

(iii) If $\Re(a-b) > 0$, $-c \notin \mathbb{N}_0$, *and* $|x| < q$, *then*

$$\int x^{a-b-1} \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^{b+1}; q^c; q, x/q)}{{}_2\phi_1(q^{a+1}, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^b; q^c; q, x/q)}\, d_qx = \frac{q^{b-a}x^{a-b}}{[b]_q}\left(\frac{[a]_q}{[a-b]_q} - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^{a+1}, q^b; q^c; q, x/q)}\right).$$

(iv) If $\Re(a) > -1$, $-c \notin \mathbb{N}_0$, *and* $|x| < q$, *then*

$$\int \frac{x^a}{(x/q;q)_{a+2}} \cdot \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+2}, q^b; q^{c+1}; q, x)}{{}_2\phi_1(q^{a+1}, q^b; q^c; q, x/q)\, {}_2\phi_1(q^{a+1}, q^b; q^c; q, x)}\, d_qx = \frac{q^{-c-a}[c]_q x^a}{[a+1]_q[b-c]_q(x/q;q)_a}\left(1 - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{(1-q^{a-1}x)\, {}_2\phi_1(q^{a+1}, q^b; q^c; q, x/q)}\right).$$

Proof. The proofs of (i), (ii), and (iii) follow from (i), (ii), and (iii) in Theorem 3.5 by swapping the parameters a and b. The proof of (iv) follows by substituting with $y(x) = x^a\, {}_2\phi_1(q^a, q^b; q^c; q, x)$ $(a \neq 0)$, $h(x) = \dfrac{x^a}{[a]_q(x;q)_a}$, and $D_{q^{-1}}h(x) = \frac{q^{1-a}x^{a-1}}{(x/q;q)_{a+1}}$ into (1.1) and using (2.14). □

Theorem 3.7. *The following indefinite q-integrals formula hold true:*

(i) If $-c \notin \mathbb{N}_0$ *and* $|x| < \min\{1, q^{-\Re(a)}\}$, *then*

$$\int (x;q)_a \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+2}, q^{b+1}; q^{c+2}; q, x)}{{}_2\phi_1(q^{a+1}, q^b; q^{c+1}; q, x)\, {}_2\phi_1(q^{a+1}, q^b; q^{c+1}; q, qx)}\, d_qx = \frac{[c+1]_q(x/q;q)_a}{[b]_q[a+1]_q}\left(\frac{q^b[c-b]_q}{[c]_q} - \frac{{}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^{a+1}, q^b; q^{c+1}; q, x)}\right).$$

(ii) If $\Re(a) < -1$, $-c \notin \mathbb{N}_0$, *and* $|x| < 1$, *then*

$$\int x^{-a-2}(x;q)_a \frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^{a+2},q^b;q^{c+1};q,x)}{{}_2\phi_1(q^{a+1},q^b;q^{c+1};q,x)\,{}_2\phi_1(q^{a+1},q^b;q^{c+1};q,qx)}\,d_qx$$
$$= \frac{-x^{-a}(x/q;q)_a}{q^a[a+1]_q}\left(\frac{q^{c+a}[b-c]_q}{[c]_q} + \frac{q\,{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{x\,{}_2\phi_1(q^{a+1},q^b;q^{c+1};q,x)}\right).$$

Proof. To prove the theorem, we substitute with $y(x) = (x;q)_a\,{}_2\phi_1(q^a,q^b;q^c;q,x)$ $(a \neq 0)$ into Theorem 3.1 with different choices of the function $h(x)$. The proof of (i) follows by substituting with $h(x) = \dfrac{-(x;q)_a}{[a]_q}$ into (1.1) and using (2.2). To prove (ii), we substitute with $h(x) = \dfrac{-(x;q)_a}{[a]_q x^a}$ and $D_{q^{-1}}h(x) = qx^{-a-1}(x;q)_{a-1}$ into (1.1) and using (2.11). □

Theorem 3.8. *We have the following indefinite q-integrals*

(i) If $-c \notin \mathbb{N}_0$, *and* $|x| < \min\{q, q^{\Re(c-a-b)}\}$, *then*

$$\int (q^{a-c}x;q)_b \frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^{a+1},q^{b+2};q^{c+2};q,x/q)}{{}_2\phi_1(q^a,q^{b+1};q^{c+1};q,x)\,{}_2\phi_1(q^a,q^{b+1};q^{c+1};q,x/q)}\,d_qx$$
$$= \frac{q[c+1]_q(q^{a-c-1}x;q)_b}{[a]_q[b+1]_q}\left(\frac{[c-a]_q}{[c]_q} - \frac{{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{{}_2\phi_1(q^a,q^{b+1};q^{c+1};q,x/q)}\right).$$

(ii) If $\Re(b) < -1$, $-c \notin \mathbb{N}_0$, *and* $|x| < \min\{q, q^{\Re(c-a-b)}\}$, *then*

$$\int x^{-b-2}(q^{a-c}x;q)_b \frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^a,q^{b+2};q^{c+1};q,x/q)}{{}_2\phi_1(q^a,q^{b+1};q^{c+1};q,x)\,{}_2\phi_1(q^a,q^{b+1};q^{c+1};q,x/q)}\,d_qx$$
$$= \frac{-q^{b+1}(q^{a-c-1}x;q)_b}{[b+1]_q x^b}\left(\frac{q^b[a-c]_q}{[c]_q} + \frac{q\,{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{x\,{}_2\phi_1(q^a,q^{b+1};q^{c+1};q,x/q)}\right).$$

Proof. To prove the theorem, we substitute with $y(x) = (q^{a-c}x;q)_b\,{}_2\phi_1(q^a,q^b;q^c;q,x)$ $(b \neq 0)$ into Theorem 3.1 with different choices of the function $h(x)$. The proof of (i) follows by substituting with $h(x) = \dfrac{-(q^{a-c}x;q)_b}{[b]_q}$ into (1.1) and using (2.2). To prove (ii), we substitute with $h(x) = \dfrac{-(q^{a-c}x;q)_b}{[b]_q x^b}$ and $D_{q^{-1}}h(x) = qx^{-b-1}(q^{a-c}x;q)_{b-1}$ into (1.1) and using (2.12). □

Theorem 3.9. *If* $q^\alpha := 1 + q^{1-c} - q^{1-a}$, *then the following indefinite q-integrals hold true.*

(i) *If* $\Re(\alpha+b)<0$, $-c \notin \mathbb{N}_0$, *and* $|x|<\min\{q, q^{\Re(c-a-b)}\}$, *then*

$$\int \frac{(q^{a-\alpha-c}x;q)_{\alpha+b}}{x^{\alpha+b+1}} \cdot \frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^{a-1},q^{b+1};q^c;q,x/q)}{{}_2\phi_1(q^{a-1},q^b;q^c;q,x)\,{}_2\phi_1(q^{a-1},q^b;q^c;q,x/q)}\,d_qx = \frac{(q^{a-\alpha-c-1}x;q)_{\alpha+b}}{q^{-b}x^{\alpha+b}[b]_q}\left(\frac{q^{1-c}[c-a]_q}{[-\alpha-b]_q} - \frac{q^{\alpha}\,{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{{}_2\phi_1(q^{a-1},q^b;q^c;q,x/q)}\right).$$

(ii) *If* $\Re(\alpha+a)<1$, $-c \notin \mathbb{N}_0$, *and* $|x|<q$, *then*

$$\int \frac{(q^{a-\alpha-c}x;q)_{\alpha+b}}{x^{\alpha+a}} \cdot \frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{{}_2\phi_1(q^{a-1},q^b;q^c;q,x)\,{}_2\phi_1(q^{a-1},q^b;q^c;q,x/q)}\,d_qx = \frac{x^{1-\alpha-a}}{q^{-a}[a-1]_q}\left(\frac{[c-a]_q}{q^c[1-\alpha-a]_q}\,{}_2\phi_1(q^{1-\alpha-b},q^{1-\alpha-a};q^{2-\alpha-a};q,q^{a+b-c-1}x) - \frac{(q^{a-\alpha-c-1}x;q)_{\alpha+b}\,{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{q^{-\alpha+1}\,{}_2\phi_1(q^{a-1},q^b;q^c;q,x/q)}\right).$$

(iii) *If* $\Re(\alpha+c)<1$, $-c \notin \mathbb{N}_0 \cup \{-1\}$, *and* $|x|<q$, *then*

$$\int \frac{(q^{a-\alpha-c}x;q)_{\alpha+b}}{x^{\alpha+c}} \cdot \frac{{}_2\phi_1(q^a,q^b;q^c;q,x)\,{}_2\phi_1(q^{a-1},q^b;q^{c-1};q,x/q)}{{}_2\phi_1(q^{a-1},q^b;q^c;q,x)\,{}_2\phi_1(q^{a-1},q^b;q^c;q,x/q)}\,d_qx = \frac{x^{1-\alpha-c}}{[c-1]_q}\left(\frac{[c-a]_q}{[1-\alpha-c]_q}\,{}_2\phi_1(q^{1-\alpha-b},q^{1-\alpha-c};q^{2-\alpha-c};q,q^{a+b-c-1}x) - \frac{(q^{a-\alpha-c-1}x;q)_{\alpha+b}\,{}_2\phi_1(q^a,q^b;q^c;q,x/q)}{q^{1-c-\alpha}\,{}_2\phi_1(q^{a-1},q^b;q^c;q,x/q)}\right).$$

Proof. To prove the theorem, we substitute with

$$y(x) = x^{-\alpha}(q^{a-\alpha-c}x;q)_{\alpha+b}\,{}_2\phi_1(q^a,q^b;q^c;q,x)$$

into Theorem 3.1 with different choices of the function $h(x)$. The proof of (i) follows by substituting with

$$h(x) = \frac{q^{-(\alpha+b+1)}}{[-\alpha-b]_q}x^{-(\alpha+b)}(q^{a-\alpha-c}x;q)_{\alpha+b},\ D_{q^{-1}}h(x) = x^{-\alpha-b-1}(q^{a-\alpha-c}x;q)_{\alpha+b-1}$$

into (1.1) and using (2.12). To prove (ii), we set

$$h(x) = \frac{q^{-(\alpha+a)}}{[1-\alpha-a]_q}x^{-(\alpha+a-1)}\,{}_2\phi_1(q^{1-\alpha-b},q^{1-\alpha-a};q^{2-\alpha-a};q,q^{a+b-c}x)$$

and substitute with $D_{q^{-1}}h(x) = x^{-\alpha-a}(q^{a-\alpha-c}x;q)_{\alpha+b-1}$ into (1.1) using (2.11). Finally, to prove (iii), we set

$$h(x) = \frac{q^{-(\alpha+c)}}{[1-\alpha-c]_q} x^{-(\alpha+c-1)} \, {}_2\phi_1(q^{1-\alpha-b}, q^{1-\alpha-c}; q^{2-\alpha-c}; q, q^{a+b-c}x)$$

and substitute with $D_{q^{-1}} h(x) = x^{-\alpha-c}(q^{a-\alpha-c}x; q)_{\alpha+b-1}$ into (1.1) and using (2.13). $\square$

Theorem 3.10. *If $q^\gamma = 1 + q^{1-c} - q^{1-b}$, the the following indefinite q-integrals hold true:*

(i) If $\Re(\gamma + a) < 0$, $-c \notin \mathbb{N}_0$, and $|x| < q$, then

$$\begin{aligned} &\int \frac{(q^{b-\gamma-c}x; q)_{\gamma+a}}{x^{\gamma+a+1}} \cdot \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^{a+1}, q^{b-1}; q^c; q, x/q)}{{}_2\phi_1(q^a, q^{b-1}; q^c; q, x)\, {}_2\phi_1(q^a, q^{b-1}; q^c; q, x/q)} \, d_qx \\ &= \frac{(q^{b-\gamma-c-1}x; q)_{\gamma+a}}{q^{-a}[a]_q x^{\gamma+a}} \left(\frac{q^{1-c}[c-b]_q}{[-\gamma-a]_q} - \frac{q^\gamma \, {}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^a, q^{b-1}; q^c; q, x/q)} \right). \end{aligned}$$

(ii) If $\Re(\gamma + b) < 1$, $-c \notin \mathbb{N}_0$, and $|x| < q$, then

$$\begin{aligned} &\int \frac{(q^{b-\gamma-c}x; q)_{\gamma+a}}{x^{\gamma+b}} \cdot \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^a, q^b; q^c; q, x/q)}{{}_2\phi_1(q^a, q^{b-1}; q^c; q, x)\, {}_2\phi_1(q^a, q^{b-1}; q^c; q, x/q)} \, d_qx = \\ &\frac{x^{1-\gamma-b}}{q^{-b}[b-1]_q} \left(\frac{[c-b]_q}{q^c[1-\gamma-b]_q} \, {}_2\phi_1(q^{1-\gamma-a}, q^{1-\gamma-b}; q^{2-\gamma-b}; q, q^{a+b-c-1}x) \right. \\ &\left. - \frac{(q^{b-\gamma-c-1}x; q)_{\gamma+a} \, {}_2\phi_1(q^a, q^b; q^c; q, x/q)}{q^{-\gamma+1} \, {}_2\phi_1(q^a, q^{b-1}; q^c; q, x/q)} \right). \end{aligned}$$

(iii) If $\Re(\gamma + c) < 0$, $-c \notin \mathbb{N}_0 - \{1\}$, and $|x| < q$, then

$$\begin{aligned} &\int \frac{(q^{b-\gamma-c}x; q)_{\gamma+a}}{x^{\gamma+c}} \cdot \frac{{}_2\phi_1(q^a, q^b; q^c; q, x)\, {}_2\phi_1(q^a, q^{b-1}; q^{c-1}; q, x/q)}{{}_2\phi_1(q^a, q^{b-1}; q^c; q, x)\, {}_2\phi_1(q^a, q^{b-1}; q^c; q, x/q)} \, d_qx = \\ &\frac{x^{1-\gamma-c}}{[c-1]_q} \left(\frac{[c-b]_q}{[1-\gamma-c]_q} \, {}_2\phi_1(q^{1-\gamma-a}, q^{1-\gamma-c}; q^{2-\gamma-c}; q, q^{a+b-c-1}x) \right. \\ &\left. - \frac{(q^{b-\gamma-c-1}x; q)_{\gamma+a} \, {}_2\phi_1(q^a, q^b; q^c; q, x/q)}{q^{1-c-\gamma} \, {}_2\phi_1(q^a, q^{b-1}; q^c; q, x/q)} \right). \end{aligned}$$

Proof. The proof follows similarly as the proof of Theorem 3.9 by swapping the parameters a and b. $\square$

Appendix A. Contiguous relations

See [4,8]

$$\begin{aligned} &{}_2\phi_1(q^{a+1}, q^{b+1}; q^c; q, x) - {}_2\phi_1(q^{a+1}, q^{b+1}; q^{c+1}; q, x) \\ &- \frac{(1-q^{a+1})(1-q^{b+1})}{(1-c)(1-cq)} \, {}_2\phi_1(q^{a+2}, q^{b+2}; q^{c+2}; q, x) = 0, \end{aligned} \tag{A1}$$

$$ {}_2\phi_1(q^a,q^b;q^c;q,x)-(1-q^a)\,{}_2\phi_1(q^{a+1},q^b;q^c;q,x)-q^a\,{}_2\phi_1(q^a,q^b;q^c;q,qx)=0, \tag{A2}$$

$$ {}_2\phi_1(q^a,q^b;q^c;q,x)+\frac{q^c-q^b}{1-q^c}\,{}_2\phi_1(q^a,q^b;q^{c+1};q,qx)-\frac{1-q^b}{1-q^c}\,{}_2\phi_1(q^a,q^{b+1};q^{c+1};q,x)=0, \tag{A3}$$

$$ \begin{aligned} &q^{a+1}(1-q^c)\,{}_2\phi_1(q^{a+1},q^{b+1};q^c;q,x)+(q^c-q^{a+1})\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x)\\ &-q^c(1-q^{a+1})\,{}_2\phi_1(q^{a+2},q^{b+1};q^{c+1};q,x)=0, \end{aligned} \tag{A4}$$

$$ {}_2\phi_1(q^{a+1},q^b;q^c;q,x)-\,{}_2\phi_1(q^a,q^{b+1};q^c;q,x)-\frac{(q^a-q^b)x}{(1-q^c)}\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x)=0, \tag{A5}$$

$$ {}_2\phi_1(q^a,q^b;q^c;q,x)+\frac{q^{b-a}[a]_q}{[b-a]_q}\,{}_2\phi_1(q^{a+1},q^b;q^c;q,x)-\frac{[b]_q}{[b-a]_q}\,{}_2\phi_1(q^a,q^{b+1};q^c;q,x)=0, \tag{A6}$$

$$ {}_2\phi_1(q^{a+1},q^b;q^c;q,x)-\,{}_2\phi_1(q^a,q^b;q^c;q,x)=q^a x\frac{[b]_q}{[c]_q}\,{}_2\phi_1(q^{a+1},q^{b+1};q^{c+1};q,x), \tag{A7}$$

$$ \begin{aligned} &[q^c(1+q-q^a)-q^{a+1}+q^a x(q^a-q^b)]\,{}_2\phi_1(q^a,q^b;q^c;q,x)\\ &-(q^c-q^{a+b}x)(1-q^a)\,{}_2\phi_1(q^{a+1},q^b;q^c;q,x)=q^{1+c}(1-q^{a-c})\,{}_2\phi_1(q^{a-1},q^b;q^c;q,x), \end{aligned} \tag{A8}$$

$$ \begin{aligned} &[q^{a+b}(q^c+q)-q^c(q^{a+1}+q^b)]\,{}_2\phi_1(q^a,q^b;q^c;q,x)\\ &+q^b(q^c-q^{a+b}x)(1-q^a)\,{}_2\phi_1(q^{a+1},q^b;q^c;q,x)=q^{a+b+1}(1-q^{c-b})\,{}_2\phi_1(q^a,q^{b-1};q^c;q,x). \end{aligned} \tag{A9}$$

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