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SOLUTION of LINEAR SYSTEM of PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

By

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ABSTRACT

In this paper we adopt operator method for solution of linear system of partial differential equations with constant coefficients. We deal with operator $L(D_x, D_y)$,where $D_x=\partial/\partial x$, that need not be homogeneous, where we introduce three possible solutions of the homogeneous equation in terms of basic (or arbitrary) functions in addition to the solution of the nonhomogeneous equation.

Keywords

Partial differential equation, operator method, separation of variables.

1. Introduction

Historically, the solution of a homogeneous linear partial differential equation with constant coefficients with homogeneous operator $L(D_x, D_y)\, u(x,y) = 0$, where

$$L(D_x, D_y) = D_x^n + a_1 D_x^{n-1} D_y + a_2 D_x^{n-2} D_y^2 + \ldots + a_n D_y^n$$

was obtained by considering $u(x,y)= F(y + \lambda x)$, where λ is the solution of the auxiliary equation

$$\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots + a_n = 0,$$

and F is an arbitrary function.
Unfortunately, this is valid only for homogeneous operator. We generalize this solution for non homogeneous operator by assuming exponential forms. Our method also generalize the method of separation of variables for cases where there is a separable solution in multiplication form, but the method of separation of variables cannot obtain it, for example the solution of the partial differential equation $u_{xx}(x,y) + u_{xy}(x,y) + u_{yy}(x,y) = 0$.

The particular integral solution of a nonhomogeneous linear partial differential equation with constant coefficients with homogeneous operator $L(D_x, D_y)$ was obtained by formulas like those that will be given in section (3), except for an arbitrary function φ (other than polynomial, sine, cosine, sinh, cosh, exponential):

$$\text{i)}\ \frac{1}{f(D_x, D_y)}\varphi^{(n)}(ax+by) = \frac{1}{f(a,b)}\varphi(ax+by)\quad,\ f(a,b)\neq 0,$$

where $\varphi^{(n)}(ax+by)$ is the n^{th} derivative of φ with respect to its argument, n is the order of the operator.

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$$\text{ii)}\ \frac{1}{(bD_x - aD_y)^r}\varphi(ax+by) = \frac{x^r}{b^r r!}\varphi(ax+by) \quad , \ f(a,b)=0.$$

All formulas given in this paper are direct consequence of differentiation, so their proofs will be omitted.

2. Solution of homogeneous equation

Consider the equation

$$L(Dx, Dy)\, u(x,y) = 0 . \qquad (2.1)$$

We assume the solution of the form $u(x,y) = \exp(\lambda_1 x + \lambda_2 y)$ **. Substituting in (2.1), we obtain**

$$L(\lambda_1, \lambda_2)\ \exp(\lambda_1 x + \lambda_2 y) = 0\,, \quad \exp(\lambda_1 x + \lambda_2 y) \neq 0$$

so that $$L(\lambda_1, \lambda_2) = 0. \qquad (2.2)$$

Equation (2.2) is called auxiliary equation of equation (2.1).
According to the roots λ_1 (λ_2) of (2.2), we have (for λ_2 arbitrary complex number which can be adjusted or chosen according to initial or boundary conditions i. e. λ_2 can take more than one value to generate different suitable solutions) the following cases:

a) **Real distinct roots**: $\lambda_{11}(\lambda_2), \lambda_{12}(\lambda_2), \ldots, \lambda_{1n}(\lambda_2)$.

The solution (with respect to roots λ_1 (λ_2)) will have the form:

$$u_1(x,y) = c_1 \exp(\lambda_{11}x + \lambda_2 y) + c_2 \exp(\lambda_{12}x + \lambda_2 y) + \ldots + c_n \exp(\lambda_{1n}x + \lambda_2 y),$$

where $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Example 2.1 **Solve the P.D.E. $u_{xx}(x,y) - u_{yy}(x,y) = 0$.**

Solution. **Auxiliary equation:** $(\lambda_1)^2 = (\lambda_2)^2$, $\lambda_1 = \pm\ \lambda_2$ **and the solution (with respect to roots $\lambda_1(\lambda_2)$) will be**

$$u_1(x,y) = c_1 \exp(\lambda_2(x+y)) + c_2 \exp(\lambda_2(-x+y)).$$

Note that on taking $\lambda_2 = i$ or $-i$, we obtain solutions of the form sin x cos y or cos x sin y or sinx siny and cosx cosy, and on taking $\lambda_2 = 1+i$, we obtain non separable solutions of the form exp(x+y) sin (x+y), exp(x+y) cos(x+y), exp(-x+y) sin (-x+y) and exp(-x+y) cos (-x+y).

b) **Real repeated roots**: $\lambda_1(\lambda_2), \lambda_1(\lambda_2), \ldots, \lambda_1(\lambda_2) \ldots$ k - times .

The solution (with respect to roots λ_1 (λ_2)) will have the form:

$$u_1(x,y) = (\exp(\lambda_1(\lambda_2)x + \lambda_2 y))(c_1 + c_2 x + \ldots + c_k x^{k-1}),$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants.

Example 2.2 Solve the P.D.E. $u_{xx}(x,y) - 2u_{xy}(x,y) + u_{yy}(x,y) = 0$.

Solution. Auxiliary equation :$(\lambda_1)^2 - 2\lambda_1\lambda_2 + (\lambda_2)^2 = 0$, $\lambda_1 = \lambda_2, \lambda_2$ and the solution (with respect to roots $\lambda_1(\lambda_2)$) will be

$$u_1(x,y) = (\exp(\lambda_2(x+y)))(c_1 + c_2 x).$$

c) **Complex conjugate roots**: $\lambda_1 = h_1(\lambda_2) \pm i\, h_2(\lambda_2)$.

The solution (with respect to roots $\lambda_1(\lambda_2)$) will have the form:

$$u_1(x,y) = (\exp(h_1(\lambda_2)x + \lambda_2 y))\ (c_1 \cos(h_2(\lambda_2)x) + c_2 \sin(h_2(\lambda_2)x)),$$

where c_1, c_2 are arbitrary constants.

Example 2.3 Solve the P.D.E. $u_{xx}(x,y) + u_{yy}(x,y) = 0$.

Solution. Auxiliary equation :$(\lambda_1)^2 = -(\lambda_2)^2$, $\lambda_1 = \pm i\lambda_2$ and the solution (with respect to roots $\lambda_1(\lambda_2)$) will be

$$u_1(x,y) = (\exp(\lambda_2 y))(c_1 \cos(\lambda_2 x) + c_2 \sin(\lambda_2 x)).$$

Example 2.4 Solve the P.D.E. $\frac{\partial^8}{\partial x^8}u(x,y) + \frac{\partial^8}{\partial y^8}u(x,y) = 0$.

Solution. Auxiliary equation :$(\lambda_1)^8 = -(\lambda_2)^8$,

$$\lambda_1 = \lambda_2\left(-\frac{\sqrt{2-\sqrt{2}}}{2} \pm i\frac{\sqrt{2+\sqrt{2}}}{2}\right),$$

$$\lambda_1 = \lambda_2\left(-\frac{\sqrt{2+\sqrt{2}}}{2} \pm i\frac{\sqrt{2-\sqrt{2}}}{2}\right),$$

$$\lambda_1 = \lambda_2\left(\frac{\sqrt{2+\sqrt{2}}}{2} \pm i\frac{\sqrt{2-\sqrt{2}}}{2}\right),$$

$$\lambda_1 = \lambda_2\left(\frac{\sqrt{2-\sqrt{2}}}{2} \pm i\frac{\sqrt{2+\sqrt{2}}}{2}\right),$$

and the solution (with respect to roots $\lambda_1(\lambda_2)$) will be

$$\begin{aligned} u_1(x,y) = &(\exp(-\lambda_2(\frac{\sqrt{2-\sqrt{2}}}{2}x-y)))\,(c_1\ \cos(\frac{\sqrt{2+\sqrt{2}}}{2}\lambda_2 x) + c_2\ \sin(\frac{\sqrt{2+\sqrt{2}}}{2}\lambda_2 x))+ \\ &(\exp(-\lambda_2(\frac{\sqrt{2+\sqrt{2}}}{2}x-y)))\,(c_3\ \cos(\frac{\sqrt{2-\sqrt{2}}}{2}\lambda_2 x) + c_4\ \sin(\frac{\sqrt{2-\sqrt{2}}}{2}\lambda_2 x))+ \\ &(\exp(\lambda_2(\frac{\sqrt{2+\sqrt{2}}}{2}x+y)))\,(c_5\ \cos(\frac{\sqrt{2-\sqrt{2}}}{2}\lambda_2 x) + c_6\ \sin(\frac{\sqrt{2-\sqrt{2}}}{2}\lambda_2 x))+ \\ &(\exp(\lambda_2(\frac{\sqrt{2-\sqrt{2}}}{2}x+y)))\,(c_7\ \cos(\frac{\sqrt{2+\sqrt{2}}}{2}\lambda_2 x) + c_8\ \sin(\frac{\sqrt{2+\sqrt{2}}}{2}\lambda_2 x)). \end{aligned}$$

Example 2.5 Solve the P.D.E. $u_{xx}(x,y) + u_{xy}(x,y) + u_{yy}(x,y) = 0$, $u(0,y) = u(1,y) = 0$, $u(x,0) = x^2(1-x)$ and $u(x,1) = x(1-x)$.

Solution. Auxiliary equation :$(\lambda_1)^2 + \lambda_1\lambda_2 + (\lambda_2)^2 = 0$, $\lambda_1 = \lambda_2(-1/2 \pm i\sqrt{3}/2)$ and the solution will be

$$\begin{aligned} u(x,y) = &(\exp(\lambda_2(y-x/2)))\,(c_1 \cos(\sqrt{3}\lambda_2 x/2) + c_2 \sin(\sqrt{3}\lambda_2 x/2)) + \\ &(\exp(\lambda_1(x-y/2)))\,(c_3 \cos(\sqrt{3}\lambda_1 y/2) + c_4 \sin(\sqrt{3}\lambda_1 y/2)). \end{aligned}$$

$u(0,y) = c_1 \exp(\lambda_2(y))) + (\exp((-\lambda_1 y/2)))\,(c_3 \cos(\sqrt{3}\lambda_1 y/2) + c_4 \sin(\sqrt{3}\lambda_1 y/2)) = 0$,
from which $c_1 = c_3 = c_4 = 0$.

$u(1,y) = c_2(\exp(\lambda_2(y-1/2)))\sin(\sqrt{3}\lambda_2/2)) = 0$, from which $\lambda_2 = \pm 2n\pi/\sqrt{3}$, $n = 1, 2, \ldots$
and the n^{th} solution will be

$u_n(x,y) = (A_n(\exp(2n\pi(y-x/2)/\sqrt{3})) + B_n(\exp(-2n\pi(y-x/2)/\sqrt{3})))\sin(n\pi x))$, $n = 1, 2, \ldots$
and the solution will be

$$u(x,y) = \sum_{n=1}^{\infty}(An(\exp(2n\pi(y-x/2)/\sqrt{3})) + Bn(\exp(-2n\pi(y-x/2)/\sqrt{3})))\sin(n\pi x)).$$

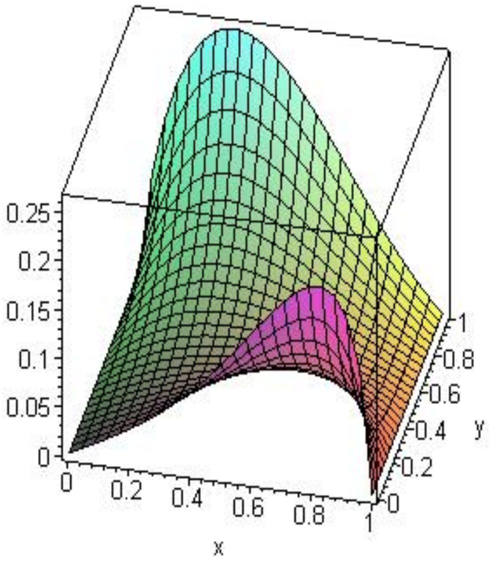
$$u(x,0) = \sum_{n=1}^{\infty}(A_n(\exp(-n\pi x/\sqrt{3})) + B_n(\exp(n\pi x/\sqrt{3})))\sin(n\pi x)) = x^2(1-x).$$

On replacing $\exp(-n\pi x/\sqrt{3})$ and $\exp(n\pi x/\sqrt{3})$ by their mean values $(\int_0^1 \exp(-n\pi x/\sqrt{3})\,dx/(1-0)$ and $(\int_0^1 \exp(n\pi x/\sqrt{3})\,dx/(1-0)$ respectively, we obtain the coefficients of $\sin(n\pi x)$ as Fourier sine coefficients of $x^2(1-x)$ on the interval (0,1).

We do the same for the second condition

$$u(x,1) = \sum_{n=1}^{\infty}(A_n(\exp(2n\pi(1-x/2)/\sqrt{3})) + B_n(\exp(-2n\pi(1-x/2)/\sqrt{3})))\sin(n\pi x)) = x(1-x),$$

so that A_n and B_n are found for every n and the solution is completely established.
The solution with n=6 has an error (for the boundary conditions) of the order of 10^{-3} while with n=60 has an error of the order of 10^{-6}.
The graph of the solution with n=60 (obtained by Maple 12) is



d) Repeated complex conjugate roots:

$\lambda_1 = h_1(\lambda_2) \pm i\, h_2(\lambda_2)$, $h_1(\lambda_2) \pm i\, h_2(\lambda_2)$, … , $h_1(\lambda_2) \pm i\, h_2(\lambda_2)$ … k-times.

The solution (with respect to roots λ_1 ($\lambda 2$)) will have the form:

$$u_1(x,y) = (\exp(h_1(\lambda_2)x + \lambda_2 y))\ ((\cos(h_2(\lambda_2)\,x))\ (c_1 + c_2 x + \dots + c_k x^{k-1})$$
$$+ (\sin(h_2(\lambda_2)\,x))\ (d_1 + d_2 x + \dots + d_k x^{k-1})),$$

where $c_1, c_2, \dots, c_k$, $d_1, d_2, \dots, d_k$ are arbitrary constants.

Example 2.6 Solve the P.D.E. $(D_{xx} + D_{yy})^2\, u(x,y) = 0$.

Solution. Auxiliary equation $((\lambda_1)^2 + (\lambda_2)^2)^2 = 0$, $\lambda_1 = \pm i\, \lambda_2$, $\pm i\, \lambda_2$ and the solution (with respect to roots λ_1 (λ_2)) will be

$$u_1(x,y) = (\exp(\lambda_2\, y))\ ((c_1 + c_2 x)\cos(\lambda_2\, x) + (d_1 + d_2 x)\ \sin(\lambda_2\, x)).$$

e) Real valued roots of λ_1 (Auxiliary equation: $L(\lambda_1, \lambda_2) = L_1(\lambda_1)$):

$\lambda_1 = k_1, k_2, \dots, k_n$.

The solution will have the form:

$$u_1(x,y) = \varphi_1(y)\ \exp(k_1 x) + \varphi_2(y)\ \exp(k_2 x)\ + \dots + \varphi_n(y)\ \exp(k_n x),$$

where $\varphi_1, \varphi_2, \dots, \varphi_n$ are arbitrary functions of y. Also, this solution can be generalized as before for repeated roots, complex roots and repeated complex roots.

The same can be done for $L(\lambda_1, \lambda_2) = L_2(\lambda_2)$.

Example 2.7 Solve the P.D.E. $(D_x - 3)\,(D_y + 5)\, u(x,y) = 0$.

Solution. Auxiliary equation $(\lambda_1 - 3)\,(\lambda_2 + 5) = 0$, $\lambda_1 = 3$, $\lambda_2 = -5$, and the solution will be

$$u(x,y) = \varphi(y)\ \exp(3x) + \psi(x)\ \exp(-5y).$$

<u>Example 2.8</u> Solve the P.D.E. $(D_x\, D_y)\, u(x,y) = 0$.

<u>Solution</u>. Auxiliary equation $\lambda_1\, \lambda_2 = 0$, $\lambda_1 = 0$, $\lambda_2 = 0$, and the solution will be $u(x,y) = \varphi(x) + \psi(y)$, where $\varphi(x)$, $\psi(y)$ are arbitrary functions of x and y respectively.

<u>Note</u>. In all previous cases, there are two other possible solutions:

1) **$u_2(x,y)$ which is the solution with respect to the roots λ_2 (λ_1).**
2) **$u_3(x,y)$ the Polynomial solution which can be obtained in the form $u_3(x,y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 + \ldots + c_ky^m$, where m = max(order of D_x in the operator , order of D_y in the operator) after canceling unnecessary constants by substituting $u_3(x,y)$ in the partial differential equation. For example, for the partial differential equation $u_{xx}(x,y) + u_{yy}(x,y) = 0$, m=2 so we suppose $u_3(x,y) = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2$. Substituting in the equation, we obtain $2c_4+2c_6 = 0$, that is $u_3(x,y) = d_1 + d_2x + d_3y + d_4xy + d_5(x^2 - y^2)$.**

We can select one or more (may be of the same type, $u_1(x,y)$ or $u_2(x,y)$ or $u_3(x,y)$) of these possible solutions to form the solution of the homogenous equation according to conditions given in the problem. For example, the two components of the solution

$$u(x,y) = (\exp(x))\,(c_1\cos(y) + c_2\sin(y)) + (\exp(-x))\,(d_1\cos(y) + d_2\sin(y))$$

of the p.d.e. $u_{xx}(x,y) + u_{yy}(x,y) = 0$ are of the type $u_2(x,y)$ with $\lambda_1=1$, $\lambda^*_1= -1$.

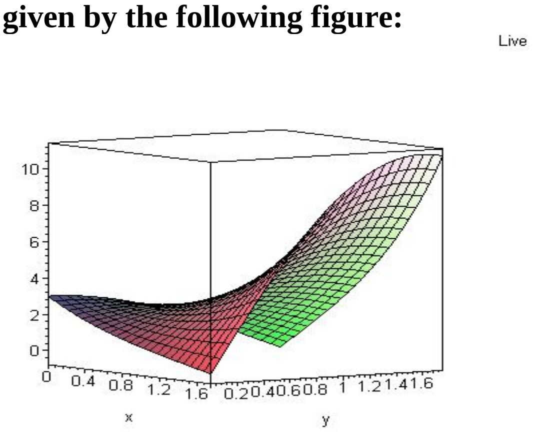
<u>Example 2.9</u> Solve the P.D.E. $u_{xx}(x,y) + u_{yy}(x,y) = 0$, $0 \le x \le \pi/2$, $0 \le y \le \pi/2$, $u(0,y) = 3\cos(y)$, $u(\pi/2,y) = 4\sinh(\pi/2)\sin(y)$, $u(x,0) = -3\sinh(x-\pi/2) / \sinh(\pi/2)$, $u(x,\pi/2) = 4\sinh(x)$.
<u>Solution</u>. Applying the boundary conditions on the solution

$$u(x,y) = (\exp(x))\,(c_1\cos(y) + c_2\sin(y)) + (\exp(-x))\,(d_1\cos(y) + d_2\sin(y)),$$

we obtain

$$u(x,y) = (\exp(x))\left(\frac{-3\exp(-\pi/2)}{2\sinh(\pi/2)}\cos(y) + 2\sin(y)\right) + (\exp(-x))\left(\frac{3\exp(\pi/2)}{2\sinh(\pi/2)}\cos(y) - 2\sin(y)\right).$$

The graph of the solution is given by the following figure:



2. <u>Solution of nonhomogeneous equation</u>

Having the equation L(Dx , Dy) u(x,y) = f(x,y), the particular integral solution can be obtained by operator method in the form u(x,y) = (1/ L(Dx , Dy)) f(x,y). According to the type of the function f(x,y), we have the following cases:

a) $$\frac{1}{f(D_x,D_y)}\exp(\alpha x+\beta y)=\frac{1}{f(\alpha,\beta)}\exp(\alpha x+\beta y)\quad ,\ f(\alpha,\beta)\neq 0.$$

Example 3.1 $$\frac{1}{D_x^{\,2}+2D_x+3D_y}\exp(2x+3y)=\frac{1}{17}\exp(2x+3y)\ .$$

b) $$\frac{1}{f(D_x,D_y)}(\exp(\alpha x+\beta y))v(x,y)=\exp(\alpha x+\beta y)\frac{1}{f(D_x+\alpha,D_y+\beta)}v(x,y)\quad ,\ f(\alpha,\beta)=0.$$

Example 3.2 $$\frac{1}{2D_x+3D_y}\exp(-3x+2y).1=\exp(-3x+2y)\frac{1}{2(D_x-3)+3(D_y+2)}.1\ .$$

$$=\exp(-3x+2y)\frac{1}{2D_x+3D_y}.1=\exp(-3x+2y)\frac{1}{2D_x}\left(\frac{1}{1+3D_y/2D_x}\right).1$$

$$=\frac{x}{2}\exp(-3x+2y).$$

c) $$\frac{1}{f(D_x^{\,2},D_y^{\,2})}\begin{cases}\cos(\alpha x+\beta y)\\ \sin(\alpha x+\beta y)\end{cases}=\frac{1}{f(-\alpha^2,-\beta^2)}\begin{cases}\cos(\alpha x+\beta y)\\ \sin(\alpha x+\beta y)\end{cases}\quad ,\ f(-\alpha^2,-\beta^2)\neq 0.$$

Example 3.3 $$\frac{1}{D_x+3D_y^{\,2}}\cos(x+y)=\frac{1}{D_x-3}\cos(x+y)$$

$$=\frac{D_x+3}{D_x^{\,2}-9}\cos(x+y)=\frac{D_x+3}{-10}\cos(x+y)$$

$$=\frac{1}{10}\sin(x+y)-\frac{3}{10}\cos(x+y).$$

d) $$\frac{1}{f(D_x^{\,2},D_y^{\,2})}\begin{cases}\cos(\alpha x+\beta y)\\ \sin(\alpha x+\beta y)\end{cases}=\frac{1}{f(D_x^{\,2},D_y^{\,2})}\begin{cases}\mathrm{Re}\,\exp(i(\alpha x+\beta y))\\ \mathrm{Im}\,\exp(i(\alpha x+\beta y))\end{cases}\quad ,\ f(-\alpha^2,-\beta^2)=0.$$

Example 3.4 $\frac{1}{D_x^2 - D_y^2}\cos(x+y) = \mathrm{Re}\frac{1}{(D_x - D_y)(D_x + D_y)}\exp(i(x+y))$

$= \mathrm{Re}\frac{1}{(D_x - D_y)(2i)}\exp(i(x+y)) = \mathrm{Re}\ \exp(i(x+y))\frac{-i}{2(D_x - D_y)}1$

$= \mathrm{Re}\ \exp(i(x+y))\frac{-i}{2D_x(1-\frac{D_y}{D_x})}1 = \mathrm{Re}\,(\cos(x+y) + i\sin(x+y))(\frac{-ix}{2})$

$= \frac{x}{2}\sin(x+y).$

e) $\frac{1}{f(D_x^2, D_y^2)}\begin{cases}\cosh(\alpha x+\beta y)\\ \sinh(\alpha x+\beta y)\end{cases} = \frac{1}{f(\alpha^2,\beta^2)}\begin{cases}\cosh(\alpha x+\beta y)\\ \sinh(\alpha x+\beta y)\end{cases}$, $f(\alpha^2,\beta^2)\neq 0.$

Example 3.5 $\frac{1}{D_x^2 + D_y}\cosh(x+2y) = \frac{1}{1+ D_y}\cosh(x+2y)$

$= \frac{1- D_y}{1- D_y^2}\cosh(x+2y) = \frac{1- D_y}{-3}\cosh(x+2y)$

$= \frac{-1}{3}\cosh(x+2y) + \frac{2}{3}\sinh(x+2y).$

Example 3.6 $\frac{1}{D_x^2 + D_y}\cosh(x) = \frac{1}{1+ D_y}\cosh(x)$

$= \frac{1- D_y}{1- D_y^2}\cosh(x) = \frac{1- D_y}{1}\cosh(x)$

$= \cosh(x).$

f) $\frac{1}{f(D_x^2, D_y^2)}\begin{cases}\cosh(\alpha x+\beta y)\\ \sinh(\alpha x+\beta y)\end{cases} = \frac{1}{f(D_x^2, D_y^2)}\begin{cases}\frac{\exp(\alpha x+\beta y)+\exp(-(\alpha x+\beta y))}{2}\\ \frac{\exp(\alpha x+\beta y)-\exp(-(\alpha x+\beta y))}{2}\end{cases}$, $f(\alpha^2,\beta^2)=0.$

Example 3.7 $\frac{1}{D_x^2 - D_y^2}\cosh(x+y) = \frac{1}{(D_x - D_y)(D_x + D_y)}\frac{\exp(x+y)+\exp(-(x+y))}{2}$

$= \frac{1}{(D_x - D_y)}\frac{\exp(x+y)-\exp(-(x+y))}{4} = \frac{\exp(x+y)-\exp(-(x+y))}{4}\frac{1}{(D_x - D_y)}.1$

$= \frac{\exp(x+y)-\exp(-(x+y))}{4}x = \frac{x}{2}\sinh(x+y).$

$$g)\ \frac{1}{1-D_x}p_{n,m}(x,y)=(1+D_x+D_x^{\ 2}+\ldots+D_x^{\ n})p_{n,m}(x,y),$$

$$\frac{1}{1-D_y}p_{n,m}(x,y)=(1+D_y+D_y^{\ 2}+\ldots+D_y^{\ m})p_{n,m}(x,y),$$

where $p_{n,m}(x,y)$ is a polynomial of degree n in x , m in y.

Example 3.8 Solve the P.D.E. $u_{xx}(x,y) - u_{yy}(x,y) = \cosh(x+y)$, $0 \le x \le 1$, $0 \le y \le 1$, $u(0,y) = 4\exp(y)$, $u(1,y) = 2+3y+4\exp(1+y)+(1/2)\sinh(1+y)$, $u(x,0) = 2x+4\exp(x)+(x/2)\sinh(x)$, $u(x,1) = 5x+4\exp(1+x)+(x/2)\sinh(1+x)$.

Solution. Taking the general solution as the summation of $u_1(x,y)$, $u_3(x,y)$ and $u_{p.i.}(x,y)$, we obtain

$u(x,y) = c_1 + c_2x + c_3y + c_4(x^2 + y^2) + c_5xy + c_6\exp(a(x+y))+ c_7\exp(b(-x+y))+ (x/2)\sinh(x+y)$.

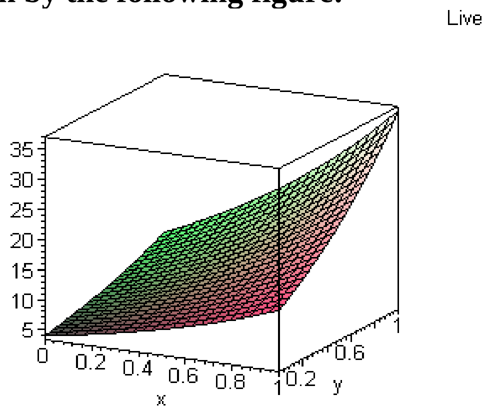
Choosing $c_1 = c_3 = c_4 = c_7 = 0$, $a=1$, we obtain

$$u(x,y) = c_2x + c_5xy + c_6\exp(x+y) + (x/2)\sinh(x+y).$$

Applying the boundary conditions on the solution, we obtain

$$u(x,y) = 2x + 3xy + 4\exp(x+y) + (x/2)\sinh(x+y).$$

The graph of the solution is given by the following figure:



4. Solution of a system of partial differential equations with constant coefficients

Consider the system

$$f_1(D_x, D_y)\,u(x,y) + f_2(D_x, D_y)\,v(x,y) = g_1(x,y), \qquad (4.1)$$
$$f_3(D_x, D_y)\,u(x,y) + f_4(D_x, D_y)\,v(x,y) = g_2(x,y). \qquad (4.2)$$

The equations of u(x,y) , v(x,y) are:

$$\begin{vmatrix} f_1(D_x,D_y) & f_2(D_x,D_y) \\ f_3(D_x,D_y) & f_4(D_x,D_y) \end{vmatrix} u(x,y) = \begin{vmatrix} g_1(x,y) & f_2(D_x,D_y) \\ g_2(x,y) & f_4(D_x,D_y) \end{vmatrix},$$

$$\begin{vmatrix} f_1(D_x,D_y) & f_2(D_x,D_y) \\ f_3(D_x,D_y) & f_4(D_x,D_y) \end{vmatrix} v(x,y) = \begin{vmatrix} f_1(D_x,D_y) & g_1(x,y) \\ f_3(D_x,D_y) & g_2(x,y) \end{vmatrix}.$$

The right hand sides of the first and second equations are $(f_4(D_x , D_y))g_1(x,y)$ - $(f_2(D_x , D_y))g_2(x,y)$ and $(f_1(D_x , D_y))g_2(x,y)$ - $(f_3(D_x , D_y))g_1(x,y)$ respectively. We can solve one of the equations to find u(x,y) or v(x,y) and substitute in the other equation to find the other function (as in example 4.1) or solve each of them to find u(x,y) and v(x,y) and then substitute in one of the equations (4.1 or 4.2) to eliminate the excess arbitrary constants.

Example 4.1 **Solve the system of partial differential equations**

$$(D_x+3)\, u(x,y) + D_y^2 v(x,y) = \exp(x-2y), \qquad (4.3)$$

$$u(x,y) - (D_x+ D_y)v(x,y) = 6x. \qquad (4.4)$$

Solution. The equation of v(x,y) is

$$\begin{vmatrix} D_x+3 & D_y^2 \\ 1 & -D_x-D_y \end{vmatrix} v(x,y) = \begin{vmatrix} D_x+3 & \exp(x-2y) \\ 1 & 6x \end{vmatrix},$$

$$(-D_x^2 - 3D_x - D_xD_y - 3D_y - D_y^2)v(x,y) = 6 + 18x - \exp(x-2y).$$

Auxiliary equation:

$$-\lambda_1^2 - 3\lambda_1 - \lambda_1\lambda_2 - 3\lambda_2 - \lambda_2^2 = 0.$$

$$\lambda_1 = -\frac{\lambda_2}{2} - \frac{3}{2} \pm \frac{\sqrt{-3\lambda_2^2 - 6\lambda_2 + 9}}{2},$$

$$\lambda_2 = -\frac{\lambda_1}{2} - \frac{3}{2} \pm \frac{\sqrt{-3\lambda_1^2 - 6\lambda_1 + 9}}{2}.$$

λ_1, λ_2 are real in the interval [-3, 1].

$$v_{1c.f.}(x,y) = c_1 \exp(\lambda_2 y + (-\frac{\lambda_2}{2} - \frac{3}{2} + \frac{\sqrt{-3\lambda_2^2 - 6\lambda_2 + 9}}{2})x) +$$
$$c_2 \exp(\lambda_2 y + (-\frac{\lambda_2}{2} - \frac{3}{2} - \frac{\sqrt{-3\lambda_2^2 - 6\lambda_2 + 9}}{2})x),$$

$$v_{2c.f.}(x,y) = c_3 \exp(\lambda_1 x + (-\frac{\lambda_1}{2} - \frac{3}{2} + \frac{\sqrt{-3\lambda_1^2 - 6\lambda_1 + 9}}{2})y) +$$
$$c_4 \exp(\lambda_1 x + (-\frac{\lambda_1}{2} - \frac{3}{2} - \frac{\sqrt{-3\lambda_1^2 - 6\lambda_1 + 9}}{2})y).$$

$$v_{p.i.} = \frac{-1}{D_x^2 + 3D_x + D_x D_y + 3D_y + D_y^2}(6 + 18x - \exp(x - 2y))$$

$$= \exp(x - 2y)\frac{1}{D_x^2 + 3D_x + D_x D_y + 3D_y + D_y^2}.1 -$$

$$\frac{1}{(D_x^2 + 3D_x)(1 + \frac{D_x D_y + 3D_y + D_y^2}{D_x^2 + 3D_x})}(6 + 18x)$$

$$= \frac{x}{3}\exp(x - 2y) - \frac{1}{D_x^2 + 3D_x}(6 + 18x) = \frac{x}{3}\exp(x - 2y) - 3x^2 .$$

Then

$$v(x, y) = c_1\exp(\lambda_2 y + (-\frac{\lambda_2}{2} - \frac{3}{2} + \frac{\sqrt{-3\lambda_2^2 - 6\lambda_2 + 9}}{2})x) +$$

$$c_2\exp(\lambda_2 y + (-\frac{\lambda_2}{2} - \frac{3}{2} - \frac{\sqrt{-3\lambda_2^2 - 6\lambda_2 + 9}}{2})x) +$$

$$c_3\exp(\lambda_1 x + (-\frac{\lambda_1}{2} - \frac{3}{2} + \frac{\sqrt{-3\lambda_1^2 - 6\lambda_1 + 9}}{2})y) +$$

$$c_4\exp(\lambda_1 x + (-\frac{\lambda_1}{2} - \frac{3}{2} - \frac{\sqrt{-3\lambda_1^2 - 6\lambda_1 + 9}}{2})y) + \frac{x}{3}\exp(x - 2y) - 3x^2.$$

From equation (4.4):

$$u(x,y) = 6x + (D_x + D_y)v(x,y)$$

$$u(x, y) = (\exp(\lambda_2 y + (-\frac{\lambda_2}{2} - \frac{3}{2} + \frac{\sqrt{-3\lambda_2^2 - 6\lambda_2 + 9}}{2})x))(\frac{\lambda_2 c_1}{2}$$

$$-\frac{3c_1}{2} + \frac{c_1}{2}\sqrt{-3\lambda_2^2 - 6\lambda_2 + 9}) +$$

$$(\exp(\lambda_2 y + (-\frac{\lambda_2}{2} - \frac{3}{2} - \frac{\sqrt{-3\lambda_2^2 - 6\lambda_2 + 9}}{2})x))(\frac{\lambda_2 c_2}{2}$$

$$-\frac{3c_2}{2} - \frac{c_2}{2}\sqrt{-3\lambda_2^2 - 6\lambda_2 + 9}) +$$

$$\exp(\lambda_1 x + (-\frac{\lambda_1}{2} - \frac{3}{2} + \frac{\sqrt{-3\lambda_1^2 - 6\lambda_1 + 9}}{2})y))(\frac{\lambda_1 c_3}{2}$$

$$-\frac{3c_3}{2} + \frac{c_3}{2}\sqrt{-3\lambda_1^2 - 6\lambda_1 + 9}\ ) +$$

$$(\exp(\lambda_1 x + (-\frac{\lambda_1}{2} - \frac{3}{2} - \frac{\sqrt{-3\lambda_1^2 - 6\lambda_1 + 9}}{2})y))(\frac{\lambda_1 c_4}{2}$$

$$-\frac{3c_4}{2} - \frac{c_4}{2}\sqrt{-3\lambda_1^2 - 6\lambda_1 + 9}\ ) + \frac{\exp(x-2y)}{3}(1-x).$$

For $\lambda_1 = \lambda_2 = 0$, the solution (using $u(x,y)=u_1(x,y) + u_2(x,y) + u_{p.i.}(x,y)$, $v(x,y)=v_1(x,y) + v_2(x,y) + v_{p.i.}(x,y)$) is given by:

$$u(x,y) = -3c_2\exp(-3x) - 3c_4\exp(-3y) + \frac{1-x}{3}\exp(x-2y),$$

$$v(x,y) = c_1 + c_3 + c_2\exp(-3x) + c_4\exp(-3y) + \frac{x}{3}\exp(x-2y) - 3x^2,$$

while $u_3(x,y)$ and $v_3(x,y)$ can be considered in the form $k_1 + k_2(x-y)$.

Note. The method can be applied to ordinary differential equations as well ($u=u(x)$) when D_y is absent and in this case λ_{1k} (λ_2) will be constant and $\lambda_2=0$.

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