مجلة التجارة والتمويل

https://caf.journals.ekb.eg/

كلية التجارة – جامعة طنطا

العدد : الثانى

يونيو ٢٠٢٣

Bayesian Estimation and Prediction for Discrete Zubair Weibull Distribution under Type-II Censoring Scheme

Dr. Mai Ahmed Hegazy
Lecturer of Statistics
Department of Statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Tafahna Al-Ashraf, Egypt
albraa.mohammad@yahoo.com

Prof. Abeer Abd-Alla EL-Helbawy
Professor of Statistics
Department of Statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo, Egypt
aah_elhelbawy@hotmail.com

Ayat Gamal AL-Kashlan
Assistant Lecturer
Department of Statistics, Faculty of Commerce, AL-Azhar University (Girls' Branch), Tafahna Al-Ashraf, Egypt
ayatgamal456@gmail.com

Abstract

This paper discussed the Bayesian estimation for the unknown parameters, survival, hazard rate, and alternative hazard rate functions of the discrete Zubair Weibull distribution. Informative priors (gamma and beta) for the parameters of the distribution are assumed. The Bayes estimators are derived under the squared error and linear-exponential loss functions based on Type-II censored sample. Credible intervals for the parameters, survival, hazard rate, and alternative hazard rate functions are obtained. The Bayes predictors (point and interval) for the future observation are obtained considering two-sample prediction. A simulation study is performed using the Markov Chain Monte Carlo algorithm for different sample sizes, and censoring rates to assess the performance of the estimators. Moreover, three real data sets were applied to investigate the flexibility and applicability of the distribution.

Keywords: *Three-parameter discrete Zubair Weibull distribution; Type-II censoring; Bayes estimators; Squared-error loss function; Linear exponential loss function; Credible interval; Bayesian prediction; Monte Carlo simulation.*

1. Introduction

Sometimes in survival analysis, the *survival function* (sf) may be a function of a count *discrete random variable* (drv) instead of being a function of a *continuous random variable* (crv). There are several well-known discrete distributions such as binomial, Poisson, and negative binomial distributions, etc., but they are inappropriate for some situations. Thus, there is a need to derive discrete distributions by discretizing the continuous distributions to fit different types of data.

Many researchers introduced some discrete distributions as an alternative to the continuous ones. For example, Nakagawa and Osaki (1975), Khan *et al.* (1989), Mudholkar and Srivastava (1993), Kemp (1997), Roy (2003), Krishna and Pundir (2009), Jazi *et al.* (2010), Gomez-Deniz and Calderin-Ojeda (2011), Al-Huniti and AL-Dayian (2012), Alamatsaz *et al*. (2016), Para and Jan (2018), Hegazy *et al*. (2018) and Almetwally (2020).

The *discrete Zubair Weibull* (DZW) distribution has three parameters α, θ and γ was proposed by AL-Kashlan *et al.* (2023). They discussed different properties and the estimation of its unknown parameters using the method of *maximum likelihood* (ML).

The DZW distribution is very flexible since it has different shapes of the *probability mass function* (pmf), *hazard rate function* (hrf) and *alternative hazard rate function* (ahrf). The plots of the pmf are increasing, decreasing, unimodal, left- skewed, right-skewed and decreasing, followed by unimodal shapes, which give it flexibility in handling most real data sets. The hrf and ahrf have various shapes, including decreasing, increasing, and unimodal shapes.

If the random variable X has a DZW distribution, then the pmf and *cumulative distribution function* (cdf) are given by

$$P(x;\alpha,\theta,\gamma)=\frac{e^{\alpha\left(1-\gamma^{(x+1)^{\theta}}\right)^{2}}-e^{\alpha\left(1-\gamma^{(x)^{\theta}}\right)^{2}}}{e^{\alpha}-1},\quad x=0,1,2,\dots,\ \alpha,\theta>0,\ 0<\gamma<1, \tag{1}$$

and

$$F(x;\alpha,\theta,\gamma)=\frac{e^{\alpha\left(1-\gamma^{(x+1)^{\theta}}\right)^{2}}-1}{e^{\alpha}-1},\qquad x=0,1,2,\dots,\ \alpha,\theta>0,\ 0<\gamma<1. \tag{2}$$

The sf, hrf and ahrf of the DZW distribution are

$$s(x;\alpha,\theta,\gamma)=\frac{e^{\alpha}-e^{\alpha\left(1-\gamma^{(x)^{\theta}}\right)^{2}}}{e^{\alpha}-1},\qquad x=0,1,2,\dots,\ \alpha,\theta>0,\ 0<\gamma<1, \tag{3}$$

$$h(x,\alpha,\theta,\gamma)=\frac{e^{\alpha\left(1-\gamma^{(x+1)^{\theta}}\right)^{2}}-e^{\alpha\left(1-\gamma^{(x)^{\theta}}\right)^{2}}}{e^{\alpha}-e^{\alpha\left(1-\gamma^{(x)^{\theta}}\right)^{2}}},\quad x=0,1,2,\dots,\ \alpha,\theta>0,\ 0<\gamma<1, \tag{4}$$

and

$$ah(x;\alpha,\theta,\gamma)=ln\left[\frac{e^{\alpha}-e^{\alpha\left(1-\gamma^{(x)^{\theta}}\right)^{2}}}{e^{\alpha}-e^{\alpha\left(1-\gamma^{(x+1)^{\theta}}\right)^{2}}}\right],\qquad x=0,1,2,\dots,\ \alpha,\theta>0,\ 0<\gamma<1. \tag{5}$$

In Bayesian analysis, the unknown parameter is regarded as being the value of a random variable from a given probability distribution, with some prior the knowledge about its value. [For more details, see Bhattacharya (1967)].

Prediction of the lifetimes of future samples based on an informative sample is very interesting for researchers, engineers, statistician, and other applied scientists that use the prediction techniques for various purposes. The future prediction problem can be classified into two types: one-sample and the two-sample prediction, which is a special case of the multiple-sample prediction problem. In this paper, two-sample prediction is considered.

Some references in the field of Bayesian estimation and prediction of future observations for different lifetime models include Migdadi (2015), who derived the Bayes estimators for the scale parameter of the discrete Rayleigh distribution, Kamari *et al.* (2016) obtained a Bayesian analysis of the discrete Burr distribution. Also, Ashour and Muiftah (2019) introduced Bayesian estimation of the parameters of the discrete Weibull Type-I distribution, Hegazy *et al*. (2021) studied Bayesian estimation and prediction of the discrete Gompertz distribution and El-Morshedy *et al.* (2021) presented a discrete analogue of the odd Weibull-G family of distributions: properties, classical and Bayesian estimation with applications to count data.

This paper aims to obtain the Bayesian estimation (point and interval) for the unknown parameters, sf, hrf, and ahrf of the DZW (α, θ, γ) distribution in Section 2. The Bayes estimators are derived under two types of loss function: *squared error* (SE) as a symmetric loss function and *linear exponential* (LINEX) as an asymmetric loss function. In Section 3, Bayesian prediction (point and interval) is considered for a future observation of the DZW(α, θ, γ) distribution under two-sample prediction. A numerical illustration is presented in Section 4 to illustrate the theoretical results developed in this paper.

2. Bayesian Estimation

In this section, the Bayes estimators for the parameters, sf, hrf, and ahrf of the DZW (α, θ, γ) distribution based on Type-II censored samples are derived, under two types of loss functions SE and LINEX loss functions. Also, *credible intervals* (CIs) for the parameters, sf, hrf, and ahrf are obtained.

Let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ denote a Type-II censored sample of size r obtained from a life-test on n items of DZW (α, θ, γ) distribution, then the likelihood is

$$L\left(\underline{\varphi}|\underline{x}\right) \propto \left\{\prod_{i=1}^{r} P(x_i)\right\}\left[S\left(x_{(r)}\right)\right]^{n-r}, \qquad (6)$$

The likelihood function can be derived by substituting (1) and (3) in (6) as follows:

$$L\left(\underline{\varphi}|\underline{x}\right) \propto \left\{\prod_{i=1}^{r} (w_1 - w_2)\right\} (e^{\alpha} - 1)^{-n}[e^{\alpha} - w_r]^{n-r}, \quad x = 0, 1, 2, \ldots, \underline{\varphi} > \underline{0}, \qquad (7)$$

where $\underline{\varphi}$ is the vector of parameters α, θ and γ,

$$w_1 = e^{\alpha\left(1-\gamma^{(x_i+1)^{\theta}}\right)^2},\ w_2 = e^{\alpha\left(1-\gamma^{(x_i)^{\theta}}\right)^2}\ and\ w_r = e^{\alpha\left(1-\gamma^{(x_r)^{\theta}}\right)^2}. \qquad (8)$$

If the parameters $\underline{\varphi} = (\varphi_1, \varphi_2, \varphi_3) = (\alpha, \theta, \gamma)$ of the DZW distribution are unknown and the joint prior of the parameters α and θ is independent of the prior of the parameter γ. Then the joint prior distribution of $\underline{\varphi}$ is

$$\pi\left(\underline{\varphi}\right) = \pi(\alpha, \theta)\pi(\gamma). \qquad (9)$$

Assuming that the unknown parameters α and θ are dependent, and using the joint bivariate prior distribution which was used by AL-Hussaini and Jaheen (1992) as follows:

$$\pi(\alpha, \theta) = g_1(\alpha|\theta) g_2(\theta)\ , \qquad \alpha, \theta > 0. \qquad (10)$$

Hence, the joint prior density function of α and θ is given by

$$g_1(\alpha|\theta) = \frac{\theta^a}{\Gamma(a)} \alpha^{a-1} e^{-\theta\alpha}, \qquad \alpha, \theta, a > 0, \qquad (11)$$

and

$$g_2(\theta) = \frac{b^c}{\Gamma(c)}\, \theta^{c-1}\, e^{-b\theta}. \qquad \theta, b, c > 0. \qquad (12)$$

Considering that the prior of γ is beta distribution, then

$$\pi(\gamma) \propto \gamma^{d-1}(1-\gamma)^{q-1}. \qquad d, q > 0, \quad 0 < \gamma < 1. \qquad (13)$$

Substituting (11)-(13) in (9), then the joint prior distribution of $\underline{\varphi}$ is

$$\pi\left(\underline{\varphi}\right) \propto \alpha^{a-1}\, \theta^{a+c-1}\, e^{-\theta(\alpha+b)}\, \gamma^{d-1}\, (1-\gamma)^{q-1},$$
$$\alpha, \theta, a, b, c, d, q > 0, \quad 0 < \gamma < 1, \qquad (14)$$

where a, b, c, d, q are the hyper parameters of the joint prior distribution.

The joint posterior distribution of $\underline{\varphi}$ can be derived by combining the likelihood function in (7) and the joint prior distribution in (14) as given below

$$\pi\left(\underline{\varphi}|\underline{x}\right) \propto L\left(\underline{\varphi}|\underline{x}\right)\pi\left(\underline{\varphi}\right) \tag{15}$$

$$= k\,\alpha^{a-1}\,\theta^{a+c-1}\,\gamma^{d-1}\,e^{-\theta(\alpha+b)}\,(1-\gamma)^{q-1}$$
$$\times\left\{\prod_{i=1}^{r}(w_1-w_2)\right\}(e^{\alpha}-1)^{-n}\,[e^{\alpha}-w_r]^{n-r}, \tag{16}$$

where w_1, w_2, w_r are given in (8) and k is the normalizing constant defined by

$$k^{-1} = \int_{\underline{\varphi}} L\left(\underline{\varphi}|\underline{x}\right)\pi\left(\underline{\varphi}\right)\, d\underline{\varphi}$$

$$= \int_{\underline{\varphi}} \alpha^{a-1}\,\theta^{a+c-1}\,e^{-\theta(\alpha+b)}\,\gamma^{d-1}\,(1-\gamma)^{q-1}$$

$$\times\left\{\prod_{i=1}^{r}(w_1-w_2)\right\}(e^{\alpha}-1)^{-n}\,[e^{\alpha}-w_r]^{n-r}d\underline{\varphi}, \tag{17}$$

where

$$\int_{\underline{\varphi}} = \int_{\alpha}\int_{\theta}\int_{\gamma} \quad and \quad d\underline{\varphi} = d\alpha\, d\theta\, d\gamma.$$

The marginal posterior distributions for the parameter can be obtained as follows:

$$\pi(\varphi_{\ell}|\underline{x}) = \int_{\underline{\varphi}_j} \pi\left(\underline{\varphi}|\underline{x}\right) d\underline{\varphi}_j, \qquad \ell \neq j\,,\ \ \ell, j = 1, 2, 3. \tag{18}$$

Point estimation

In this subsection, the Bayes estimators of the parameters, sf, hrf, and ahrf based on Type-II censoring samples are considered under two different loss functions, the SE and LINEX loss functions.

2.1.1 Bayesian estimation under squared error loss function

a. Bayesian estimation for the parameters

The Bayes estimators of the parameters $\underline{\varphi}$ under the SE loss function are the means of their marginal posterior distributions, which can be obtained using (18) as follows:

$$\tilde{\varphi}_{\ell(SE)} = E(\varphi_\ell|\underline{x}) = \int_{\underline{\varphi}} \varphi_\ell \, \pi\left(\underline{\varphi}|\underline{x}\right) d\underline{\varphi}$$

$$= k \int_{\underline{\varphi}} \varphi_\ell \, \alpha^{a-1}\theta^{a+c-1}e^{-\theta(\alpha+b)}\gamma^{d-1}(1-\gamma)^{q-1}$$

$$\times (e^\alpha - 1)^{-n}\left\{\prod_{i=1}^{r}(w_1 - w_2)\right\}[e^\alpha - w_r]^{n-r} \, d\underline{\varphi}.$$

$$\boldsymbol{\varphi_1, \varphi_2 > 0, \quad 0 < \varphi_3 < 1, \quad \ell = 1,2,3.} \qquad \mathbf{(19)}$$

where $\varphi_1 = \alpha, \ \varphi_2 = \theta \ and \ \varphi_3 = \gamma$.

b. Bayesian estimation for the sf, hrf, and ahrf

The Bayes estimators of the sf, hrf, and ahrf under the SE loss function can be obtained as given below:

$$\tilde{s}_{(SE)} = E(s(x_0)|\underline{x})$$

$$= k \int_0^\infty \alpha^{a-1} (e^\alpha - 1)^{-n-1} \int_0^\infty \theta^{a+c-1} \, e^{-\theta(\alpha+b)} \int_0^1 \gamma^{d-1} \, (1-\gamma)^{q-1}$$

$$\times \left\{\prod_{i=1}^{r}(w_1 - w_2)\right\}[e^\alpha - w_r]^{n-r}(e^\alpha - w_{01}) \, d\gamma d\theta d\alpha, \qquad \mathbf{(20)}$$

$$\tilde{h}_{(SE)} = E(h(x_0)|\underline{x})$$

$$= k \int_0^\infty \alpha^{a-1}(e^\alpha - 1)^{-n} \int_0^\infty \theta^{a+c-1}e^{-\theta(\alpha+b)} \int_0^1 \gamma^{d-1}(1-\gamma)^{q-1}$$

$$\times \left\{\prod_{i=1}^{r}(w_1 - w_2)\right\}[e^\alpha - w_r]^{n-r}\left(\frac{w_1 - w_2}{e^\alpha - w_{01}}\right) d\gamma d\theta d\alpha, \qquad \mathbf{(21)}$$

and

$$\widetilde{ah}_{(SE)} = E\left(h_1(x_0)|\underline{x}\right)$$
$$= k\int_0^{\infty} \alpha^{a-1}(e^{\alpha}-1)^{-n}\int_0^{\infty}\theta^{a+c-1}e^{-\theta(\alpha+b)}\int_0^1 \gamma^{d-1}(1-\gamma)^{q-1}$$
$$\times\left\{\prod_{i=1}^{r}(\boldsymbol{w_1}-\boldsymbol{w_2})\right\}[\boldsymbol{e^{\alpha}}-\boldsymbol{w_r}]^{\boldsymbol{n-r}}\boldsymbol{ln}\left[\left(\frac{\boldsymbol{e^{\alpha}}-\boldsymbol{w_{01}}}{\boldsymbol{e^{\alpha}}-\boldsymbol{w_{02}}}\right)\right]\boldsymbol{d\gamma d\theta d\alpha}, \qquad \textbf{(22)}$$

where w_1, w_2, w_r are given in (8), k is defined by (17),

$w_{01} = e^{\alpha\left(1-\gamma^{(x_0)^{\theta}}\right)^2}$ and $w_{02} = e^{\alpha\left(1-\gamma^{(x_0+1)^{\theta}}\right)^2}$.

2.1.2 Bayesian estimation under linear exponential loss function

Considering the LINEX loss function, the Bayes estimators of the parameters, sf, hrf, and ahrf are derived below.

a. Bayesian estimation for the parameters

The Bayes estimators of the parameters $\underline{\varphi}$ under the LINEX loss function are

$$\varphi^*_{\ell(LINEX)} = \frac{-1}{v}ln\{E(e^{-(v\varphi_\ell)}|\underline{x})\}$$
$$= \frac{-1}{v}ln\left\{\int_{\underline{\varphi}} e^{-(v\varphi_\ell)}\,\pi\left(\underline{\varphi}|\underline{x}\right)d\underline{\varphi}\right\}$$
$$= \frac{-1}{v}ln\left\{\int_{\underline{\varphi}} k\,e^{-(v\varphi_\ell)}\,\alpha^{a-1}\,\theta^{a+c-1}\,e^{-\theta(\alpha+b)}\,\gamma^{d-1}\,(1-\gamma)^{q-1}\times\left\{\prod_{i=1}^{r}(w_1-w_2)\right\}(e^{\alpha}-1)^{-n}\,[e^{\alpha}-w_r]^{n-r}\,d\underline{\varphi}\right\}.$$

$$\varphi_1,\varphi_2 > 0,\quad 0<\varphi_3<1,\quad \ell = 1,2,3. \qquad (23)$$

where v is constant and $v \neq 0$.

b. Bayesian estimation for the sf, hrf, and ahrf

The Bayes estimators of the sf, hrf, and ahrf under the LINEX loss function can be obtained as follows:

$$s^*_{(LINEX)} = \frac{-1}{v} ln\{E(e^{-vs(x_0)}|\underline{x})\}$$

$$= \frac{-1}{v} ln\left\{\int_0^\infty k\, \alpha^{a-1}\, (e^\alpha - 1)^{-n} \int_0^\infty \theta^{a+c-1}\, e^{-\theta(\alpha+b)} \int_0^1 \gamma^{d-1}\, (1-\gamma)^{q-1} \times \{\prod_{i=1}^r (w_1 - w_2)\}\, [e^\alpha - w_r]^{n-r}\, e^{-v\left(\frac{e^\alpha - w_{01}}{e^\alpha - 1}\right)}\, d\gamma d\theta d\alpha\right\}, \quad (24)$$

$$h^*_{(LINEX)} = \frac{-1}{v} ln\{E(e^{-vh(x_0)}|\underline{x})\}$$

$$= \frac{-1}{v} ln\left\{\int_0^\infty k\, \alpha^{a-1}\, (e^\alpha - 1)^{-n} \int_0^\infty \theta^{a+c-1}\, e^{-\theta(\alpha+b)} \int_0^1 \gamma^{d-1}\, (1-\gamma)^{q-1} \times \left\{\prod_{i=1}^r (w_1 - w_2)\right\} [e^\alpha - w_r]^{n-r}\, e^{-v\left(\frac{w_1 - w_2}{e^\alpha - w_{01}}\right)}\, d\gamma d\theta d\alpha\right\}, \quad (25)$$

and

$$ah^*_{(LINEX)} = \frac{-1}{v} ln\{E(e^{-vah(x_0)}|\underline{x})\}$$

$$= \frac{-1}{v} ln\left\{\int_0^\infty k\, \alpha^{a-1}\, (e^\alpha - 1)^{-n} \int_0^\infty \theta^{a+c-1}\, e^{-\theta(\alpha+b)} \int_0^1 \gamma^{d-1}\, (1-\gamma)^{q-1} \times \left\{\prod_{i=1}^r (w_1 - w_2)\right\} [e^\alpha - w_r]^{n-r} \left(\frac{e^\alpha - w_{01}}{e^\alpha - w_{02}}\right)^{-v} d\gamma d\theta d\alpha\right\}. \quad (26)$$

where w_1, w_2, w_r are given in (8), k is defined by (17),

$$w_{01} = e^{\alpha\left(1-\gamma^{(x_0)^\theta}\right)^2} \text{ and } w_{02} = e^{\alpha\left(1-\gamma^{(x_0+1)^\theta}\right)^2}.$$

The Bayes estimates of the parameters, sf, hrf, and ahrf can be obtained by solving (19) - (26) numerically.

3. Credible interval

In this subsection, the credible intervals for the parameters $\underline{\varphi}$ of the DZW distribution are derived.

In general, $[L_\ell(\underline{x}) < \varphi_\ell < U_\ell(\underline{x})|\underline{x}]$, is a 100 $(1-\tau)$ % credible intervals for $\underline{\varphi}$ if

$$[L_\ell(\underline{x}) < \varphi_\ell < U_\ell(\underline{x})] = \int_{L_\ell(\underline{x})}^{U_\ell(\underline{x})} \pi(\varphi_\ell|\underline{x})\, d\varphi_\ell = 1-\tau. \quad \ell = 1,2,3. \tag{27}$$

The lower and upper bounds $[L_\ell(\underline{x}), U(\underline{x})]$ can be obtained by evaluating

$$P[\varphi_\ell > L_\ell(\underline{x})|\underline{x}] = 1 - \frac{\tau}{2},$$

and

$$P[\varphi_\ell > U_\ell(\underline{x})|\underline{x}] = \frac{\tau}{2}.$$

Then a two sided $100(1-\tau)\%$ CIs for α of the DZW distribution is $[L_1(\underline{x}), U_1(\underline{x})]$,

$$P[\alpha > L_1(\underline{x})|\underline{x}] = k\int_{L_1(\underline{x})}^{\infty} \alpha^{a-1}(e^\alpha - 1)^{-n} \int_0^\infty \theta^{a+c-1} e^{-\theta(\alpha+b)} \int_0^1 \gamma^{d-1}(1-\gamma)^{q-1} \times \left\{\prod_{i=1}^{r}(w_1 - w_2)\right\}[e^\alpha - w_r]^{n-r} d\gamma d\theta d\alpha = 1 - \frac{\tau}{2}, \tag{28}$$

and

$$P[\alpha > U_1(\underline{x})|\underline{x}] = k\int_{U_1(\underline{x})}^{\infty} \alpha^{a-1}(e^\alpha - 1)^{-n} \int_0^\infty \theta^{a+c-1} e^{-\theta(\alpha+b)} \int_0^1 \gamma^{d-1}(1-\gamma)^{q-1} \times \left\{\prod_{i=1}^{r}(w_1 - w_2)\right\}[e^\alpha - w_r]^{n-r} d\gamma d\theta d\alpha = \frac{\tau}{2}. \tag{29}$$

A $100(1-\tau)\%$ CIs for θ of DZW is $[L_2(\underline{x}), U_2(\underline{x})]$,

$$\boldsymbol{P[\theta > L_2(\underline{x})|\underline{x}] = k \int_{L_2(\underline{x})}^{\infty} \theta^{a+c-1} e^{-\theta(\alpha+b)} \int_0^{\infty} \alpha^{a-1}(e^{\alpha}-1)^{-n} \int_0^1 \gamma^{d-1}(1-\gamma)^{q-1}}$$

$$\times \left\{\prod_{i=1}^{r}(w_1 - w_2)\right\}[e^{\alpha} - w_r]^{n-r} d\gamma d\alpha d\theta = 1 - \frac{\tau}{2}, \qquad (30)$$

and

$$\boldsymbol{P[\theta > U_2(\underline{x})|\underline{x}] = k \int_{U_2(\underline{x})}^{\infty} \theta^{a+c-1} e^{-\theta(\alpha+b)} \int_0^{\infty} \alpha^{a-1}(e^{\alpha}-1)^{-n} \int_0^1 \gamma^{d-1}(1-\gamma)^{q-1}}$$

$$\times \left\{\prod_{i=1}^{r}(w_1 - w_2)\right\}[e^{\alpha} - w_r]^{n-r} d\gamma d\alpha d\theta = \frac{\tau}{2}. \qquad (31)$$

A $100(1-\tau)\%$ CIs for θ of DZW is $[L_3(\underline{x}), U_3(\underline{x})]$,

$$P[\gamma > L_3(\underline{x})|\underline{x}]$$
$$= k \int_{L_3(\underline{x})}^{1} \gamma^{d-1}(1-\gamma)^{q-1} \int_0^{\infty} \alpha^{a-1}(e^{\alpha}$$
$$-1)^{-n} \int_0^1 \theta^{a+c-1} e^{-\theta(\alpha+b)}$$

$$\times \left\{\prod_{i=1}^{r}(w_1 - w_2)\right\}[e^{\alpha} - w_r]^{n-r} d\theta d\alpha d\gamma = 1 - \frac{\tau}{2}, \qquad (32)$$

and

$$\boldsymbol{P[\gamma > U_3(\underline{x})|\underline{x}] = k \int_{U_3(\underline{x})}^{1} \gamma^{d-1}(1-\gamma)^{q-1} \int_0^{\infty} \alpha^{a-1}(e^{\alpha}-1)^{-n} \int_0^1 \theta^{a+c-1} e^{-\theta(\alpha+b)}}$$

$$\times \left\{\prod_{i=1}^{r}(w_1 - w_2)\right\}[e^{\alpha} - w_r]^{n-r} d\theta d\alpha d\gamma = \frac{\tau}{2}. \qquad (33)$$

where w_1, w_2, w_r are given in (8) and k is defined by (17).

To obtain the two-sided 100 $(1-\tau)$ % CIs for $\underline{\varphi}$, Equations (28) - (33) can be solved numerically.

4. Bayesian Prediction Based on Two-Sample Prediction

In this section, the Bayes predictor of the observation $Y_{(s)}, s = 1,2,\dots,m$ of DZW(α,θ,γ) are obtained based on Type-II censored sample.

Assume that $X = (X_{(1)} \le X_{(2)} \le \cdots \le X_{(r)})$ represent the informative sample and independent of the future ordered sample of size m, $Y = (Y_1 \le Y_2 \le \cdots \le Y_r)$ from the same distribution. The conditional pmf of s^{th} order statistic is

$$h\left(y_{(s)}|\underline{\varphi}\right) = \frac{m!}{(s-1)!\,(m-s)!}\int_{F(y_{(s)}-1)}^{F(y_{(s)})} v^{s-1}(1-v)^{m-s}dv$$
$$= \frac{m!}{(s-1)!\,(m-s)!}\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\frac{1}{s+j}\left[\left[F\left(y_{(s)}\right)\right]^{s+j} - \left[F\left(y_{(s)}-1\right)\right]^{s+j}\right]$$
$$= \frac{m!}{(s-1)!\,(m-s)!}\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\frac{1}{s+j}\left[\left[\frac{w_{i1}-1}{e^{\alpha}-1}\right]^{s+j} - \left[\frac{w_{i2}-1}{e^{\alpha}-1}\right]^{s+j}\right],$$

$$\boldsymbol{s = 1,2,\dots,m.} \quad (34)$$

where

$$\boldsymbol{w_{i1} = e^{\alpha\left(1-\gamma^{(y_{(s)}+1)^{\theta}}\right)^2} \text{ and } w_{i2} = e^{\alpha\left(1-\gamma^{(y_{(s)})^{\theta}}\right)^2}.} \quad (35)$$

Considering $\underline{\varphi}$ are unknown and independent, then the *Bayesian* predictive *mass* (BPM) function of $Y_{(s)}$ given $\underline{x}$ is given by

$$\boldsymbol{h(y_{(s)}|\underline{x}) = \int_{\underline{\varphi}} h\left(y_{(s)}|\underline{\varphi}\right)\pi\left(\underline{\varphi}|\underline{x}\right)d\underline{\varphi},\quad y_{(s)} = 0,1,\dots,s = 1,2,3,\dots,m.} \quad (36)$$

The BPM function of $Y_{(s)}$ given $\underline{x}$ can be obtained by substituting (16) and (34) in (36) as follows:

$$h\left(y_{(s)} \middle| \underline{x}\right)$$

$$= k\frac{m!}{(s-1)!\,(m-s)!}\int_0^\infty\int_0^\infty\int_0^1\left\{\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\frac{1}{s+j}\right.$$

$$\times\left[\left[\frac{w_{i1}-1}{e^\alpha-1}\right]^{s+j}\right.$$

$$\left.\left.-\left[\frac{w_{i2}-1}{e^\alpha-1}\right]^{s+j}\right]\right\}\alpha^{a-1}\,\theta^{a+c-1}e^{-\theta(b+\alpha)}\,\gamma^{d-1}\,(1-\gamma)^{q-1}$$

$$\times\ (e^\alpha-1)^{-n}\left\{\prod_{i=1}^{r}(w_1-w_2)\right\}\,[e^\alpha-w_r]^{n-r}\ d\gamma d\theta d\alpha,$$

$$y_{(s)}=0,1,2,\ldots,\ \ s$$

$$=1,2,3,\ldots,m, \qquad (37)$$

where w_1, w_2 and w_r are given in (8).

5. Point predictor

The Bayes point predictor is derived under two types of loss functions SE and LINEX loss functions.

a. Squared error loss function

The *Bayes predictor* (BP) for the future observation $Y_{(s)}$, under the SE loss function, can be derived as given below

$$\tilde{y}_{(s)(SE)}=E\left(Y_{(s)}\middle|\underline{x}\right)=\sum_{y_{(s)}=0}^{\infty}y_{(s)}\,h\left(y_{(s)}\middle|\underline{x}\right)$$

$$= k\frac{m!}{(s-1)!\,(m-s)!}\sum_{y_{(s)}=0}^{\infty}y_{(s)}\int_0^\infty\int_0^\infty\int_0^1\left\{\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\frac{1}{s+j}\right.$$

$$\times\left[\left[\frac{w_{i1}-1}{e^{\alpha}-1}\right]^{s+j}-\left[\frac{w_{i2}-1}{e^{\alpha}-1}\right]^{s+j}\right]\Bigg\}\alpha^{a-1}\theta^{a+c-1}e^{-\theta(b+\alpha)}\gamma^{d-1}(1-\gamma)^{q-1}$$

$$\times(e^{\alpha}-1)^{-n}\left\{\prod_{i=1}^{r}(w_1-w_2)\right\}[e^{\alpha}-w_r]^{n-r}d\gamma d\theta d\alpha,$$

$$y_{(s)}=0,1,\dots,s=1,2,3,\dots,m. \quad (38)$$

b. Linear exponential loss function

The BP for the future observation $Y_{(s)}$, under the LINEX loss function can be obtained as follows:

$$y^*_{(s)(\text{LINEX})}=\frac{-1}{v}\ln\{E(e^{-vy_{(s)}}|\underline{x})\}=\sum_{y_{(s)}=0}^{\infty}e^{-vy_{(s)}}h(y_{(s)}|\underline{x})$$

$$=k\frac{-1}{v}\frac{m!}{(s-1)!\,(m-s)!}\ln\left\{\sum_{y_{(s)}=0}^{\infty}e^{-vy_{(s)}}\int_0^{\infty}\int_0^{\infty}\int_0^{1}\left\{\sum_{j=0}^{m-s}\binom{m-s}{j}(-1)^j\frac{1}{s+j}\right.\right.$$

$$\times\left[\left[\frac{w_{i1}-1}{e^{\alpha}-1}\right]^{s+j}\left[\frac{w_{i2}-1}{e^{\alpha}-1}\right]^{s+j}\right]\Bigg\}\alpha^{a-1}\ \theta^{a+c-1}\ e^{-\theta(b+\alpha)}\ \gamma^{d-1}\ (1-\gamma)^{q-1}$$

$$\times(e^{\alpha}-1)^{-n}\left\{\prod_{i=1}^{r}(w_1-w_2)\right\}[e^{\alpha}-w_r]^{n-r}d\gamma d\theta d\alpha\Bigg\},$$

$$y_{(s)}=0,1,\dots,s=1,2,3,\dots,m. \quad (39)$$

6. Bayesian predictive bounds

A 100(1-τ)% *Bayesian predictive bounds* (BPB) for the future observation $Y_{(s)}$, is

$$P\left[L_{(s)}(\underline{x}) < Y_{(s)} < U_{(s)}(\underline{x}) | \underline{x}\right] = \sum_{L_{(s)}(\underline{x})}^{U_{(s)}(\underline{x})} h\left(y_{(s)} | \underline{x}\right) = 1 - \tau.$$

The lower and upper bounds $\left[L_{(s)}(\underline{x}), U_{(s)}(\underline{x})\right]$ can be obtained by evaluating,

$$P\left[Y_{(s)} > L_{(s)}(\underline{x}) | \underline{x}\right] = \sum_{L_{(s)}(\underline{x})}^{\infty} h\left(y_{(s)} | \underline{x}\right) = 1 - \frac{\tau}{2}$$

$$= k \frac{m!}{(s-1)!\,(m-s)!} \sum_{L_{(s)}(\underline{x})}^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{1} \left\{ \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j \frac{1}{s+j} \right.$$
$$\times \left[\left[\frac{w_{i1}-1}{e^{\alpha}-1}\right]^{s+j} \left[\frac{w_{i2}-1}{e^{\alpha}-1}\right]^{s+j} \right] \Bigg\} \alpha^{a-1}\, \theta^{a+c-1}\, e^{-\theta(b+\alpha)}\, \gamma^{d-1}\, (1-\gamma)^{q-1}$$
$$\times (e^{\alpha}-1)^{-n} \left\{ \prod_{i=1}^{r} (w_1 - w_2) \right\} [e^{\alpha} - w_r]^{n-r}\; d\gamma d\theta d\alpha = 1 - \frac{\tau}{2},$$
$$s = 1,2, \ldots, m, \tag{40}$$

and

$$P\left[Y_{(s)} > U_{(s)}(\underline{x}) | \underline{x}\right] = \sum_{U_{(s)}(\underline{x})}^{\infty} h\left(y_{(s)} | \underline{x}\right) = \frac{\tau}{2}$$

$$= k \frac{m!}{(s-1)!\,(m-s)!} \sum_{0}^{U_{(s)}(\underline{x})} \int_0^{\infty} \int_0^{\infty} \int_0^{1} \left\{ \sum_{j=0}^{m-s} \binom{m-s}{j} (-1)^j \frac{1}{s+j} \right.$$
$$\times \left[\left[\frac{w_{i1}-1}{e^{\alpha}-1}\right]^{s+j} \left[\frac{w_{i2}-1}{e^{\alpha}-1}\right]^{s+j} \right] \Bigg\} \alpha^{a-1} \theta^{a+c-1} e^{-\theta(b+\alpha)} \gamma^{d-1} (1-\gamma)^{q-1}$$
$$\times (e^{\alpha}-1)^{-n} \left\{ \prod_{i=1}^{r} (w_1 - w_2) \right\} [e^{\alpha} - w_r]^{n-r} d\gamma d\theta d\alpha = \frac{\tau}{2},$$
$$s = 1,2, \ldots, m. \tag{41}$$

BPB can be obtained by solving the previous equations numerically.

7. Numerical Study

This section aims to investigate the precision of the theoretical results of the Bayes estimates and predictors through the *Markov chain Monte Carlo* (MCMC) simulation study and some applications.

MCMC method: The multiple levels of integration are necessary to obtain the normalizing constant k and the marginal posterior densities. For complicated models, these integrations are often analytically intractable, and sometimes even a numerical integration cannot be directly obtained. In these cases, MCMC simulation is the easiest way to get reliable results without evaluating integrals [see Gelman *et al.* (2003)]. The MCMC methods are used to do simulations based on constructing Markov Chain and are widely applied for physics, statistics, biology, genetics, cryptography, and others. A MCMC algorithm that is particularly useful in high dimensional problems is the alternating conditional sampling called Gibbs sampling. Each iteration of the Gibbs sampling cycles through the unknown parameters, drawing a sample of one parameter conditioned on the latest values of all the other parameters. When the number of iterations is large enough, the samples drawn on one parameter can be regarded as simulated observations from its marginal posterior distribution. Functions of the model parameters, such as the sf, hrf, and ahrf of the lifetime distribution can also be conveniently sampled. Posterior inference can be computed using sample statistics.

8. Simulation study

In this subsection, a simulation study is conducted to illustrate the performance of the Bayes estimators and two-sample predictors (point and interval) for a future observation using the MCMC method based on the generated data from the DZW(α, θ, γ) distribution. The Bayes averages, *relative absolute biases* (RABs), *estimated risks* (ERs), and 95% CIs for the parameters, sf, hrf, and ahrf based on Type-II censoring are calculated**.** All the computations are performed using the R programming language.

The following steps are used to generate Type-II censored samples from DZW (α, θ, γ) distribution as follows:

- The transformation between the uniform distribution and DZW distribution is

$$x_i = \left[\left(\frac{ln\left[1 - \left(\frac{ln[u(e^{\alpha} - 1) + 1]}{\alpha}\right)^{\frac{1}{2}}\right]}{ln\gamma} \right)^{\frac{1}{\theta}} - 1 \right], \qquad i = 1,2,..,n.$$

- Several data sets are generated from the DZW distribution for a combination of the population parameter values $(\alpha = 3, \theta = 0.5, \gamma = 0.3)$ and $(\alpha = 2, \theta = 0.5, \gamma = 0.9)$ where the samples of size (30, 60 and 100) are used. For each sample size, the uncensored level is 70% and 100%. and *number of replications* (N) = 10000 for each sample size.
- Calculate the Bayes estimates of the parameters, sf, hrf, and ahrf under the SE and LINEX loss functions.
- The Bayes averages, RABs, and ERs of the Bayes estimates for the parameters, sf, hrf, and ahrf are computed as follows:

$$\text{Average} = \frac{\sum_{i=1}^{N} estimated\ value}{N},$$

$$\text{RAB} = \frac{|\text{bias}\ (estimated\ value)|}{\text{true value}},$$

$$\text{Estimated risk} = \frac{\sum_{i=1}^{N} (estimated\ value - true\ value)^2}{N}.$$

- The Bayes predictors (point and interval) for a future observation from the DZW distribution based on Type-II censored data are computed for the two-sample case.

Table 1

Bayes averages, RABs, ERs, and 95% CIs for the parameters α, θ and γ under SE and LINEX loss functions based on Type-II censoring for different samples of size n, ($N = 10000,\ \alpha = 2, \theta = 0.5$ and $\gamma = 0.9$)

n	r	φ	SE						LINEX(ν= 0.5)					
			Average	RAB	ER	UL	LL	Length	Average	RAB	ER	UL	LL	Length
30	21	α	2.00242	0.00121	0.00235	2.00400	1.99955	0.00445	1.99825	0.00087	0.00122	1.99999	1.99599	0.00400
		θ	0.49910	0.00180	0.00033	0.50179	0.49633	0.00546	0.50104	0.00209	0.00043	0.50323	0.49902	0.00421
		γ	0.89845	0.00172	0.00096	0.90023	0.89591	0.00432	0.90136	0.00151	0.00074	0.90272	0.89992	0.00280
	30	α	1.99791	0.00105	0.00175	1.99968	1.99606	0.00362	2.00120	0.00060	0.00058	2.00279	1.99929	0.00350
		θ	0.50058	0.00116	0.00013	0.50155	0.49935	0.00220	0.49915	0.00171	0.00029	0.50022	0.49705	0.00318
		γ	0.89874	0.00140	0.00063	0.89985	0.89687	0.00298	0.90084	0.00094	0.00028	0.90207	0.89972	0.00235
60	42	α	1.99791	0.00105	0.00175	1.99991	1.99608	0.00382	1.99841	0.00079	0.00101	1.99945	1.99673	0.00272
		θ	0.49928	0.00144	0.00021	0.50139	0.49741	0.00398	0.49944	0.00111	0.00012	0.50089	0.49802	0.00287
		γ	0.89896	0.00115	0.00043	0.90051	0.89679	0.00372	0.89930	0.00078	0.00020	0.90035	0.89776	0.00259
	60	α	1.99803	0.00098	0.00155	1.99977	1.99649	0.00328	1.99918	0.00041	0.00027	2.00012	1.99766	0.00247
		θ	0.49943	0.00113	0.00012	0.50022	0.49805	0.00216	0.49960	0.00080	0.00006	0.50025	0.49825	0.00200
		γ	0.90095	0.00106	0.00036	0.90176	0.89985	0.00191	0.90060	0.00066	0.00014	0.90184	0.89952	0.00232
100	70	α	2.00107	0.00054	0.00046	2.00205	1.99971	0.00234	2.00074	0.00037	0.00022	2.00206	1.99945	0.00261
		θ	0.49951	0.00099	0.00010	0.50032	0.49808	0.00224	0.50020	0.00040	0.00002	0.50146	0.49874	0.00272
		γ	0.89906	0.00105	0.00036	0.90015	0.89721	0.00294	0.89975	0.00024	0.00002	0.90086	0.89843	0.00244
	100	α	1.99925	0.00038	0.00023	1.99998	1.99827	0.00227	1.99932	0.00034	0.00019	2.00024	1.99803	0.00221
		θ	0.49973	0.00054	0.00005	0.50058	0.49870	0.00184	0.49988	0.00025	0.00001	0.50042	0.49862	0.00180
		γ	0.89990	0.00011	0.00001	0.90041	0.89889	0.00218	0.89956	0.00049	0.00008	0.90027	0.89870	0.00157

Table 2

Bayes averages, RABs, ERs, and 95% CIs for the parameters α, θ and γ under the SE and LINEX loss functions based on Type-II censoring for different samples of size n, ($N = 10000,\ \alpha = 3, \theta = 0.5$ and $\gamma = 0.3$)

n	r	φ	SE						LINEX(ν= 0.5)					
			Average	RAB	ER	UL	LL	Length	Average	RAB	ER	UL	LL	Length
30	21	α	2.99758	0.00081	0.00235	2.99913	2.99572	0.00341	3.00225	0.00075	0.00203	3.00322	2.99991	0.00331
		θ	0.50162	0.00324	0.00105	0.50236	0.50003	0.00233	0.49879	0.00243	0.00059	0.50005	0.49688	0.00317
		γ	0.30161	0.00538	0.00104	0.30302	0.29991	0.00311	0.29823	0.00590	0.00125	0.29952	0.29700	0.00252
	30	α	3.00108	0.00036	0.00046	3.00261	2.99938	0.00323	2.99815	0.00062	0.00137	2.99986	2.99655	0.00330
		θ	0.50069	0.00138	0.00019	0.50164	0.49959	0.00205	0.50075	0.00150	0.00023	0.50199	0.49930	0.00270
		γ	0.29905	0.00317	0.00036	0.30031	0.29737	0.00294	0.30139	0.00466	0.00078	0.30220	0.29988	0.00233
60	42	α	2.99813	0.00062	0.00140	2.99990	2.99674	0.00317	2.99868	0.00044	0.00070	2.99999	2.99738	0.00261
		θ	0.50051	0.00102	0.00010	0.50139	0.49942	0.00197	0.50063	0.00125	0.00016	0.50188	0.49939	0.00249
		γ	0.30092	0.00306	0.00034	0.30225	0.29929	0.00296	0.30094	0.00314	0.00036	0.30166	0.29949	0.00217
	60	α	3.00092	0.00031	0.00034	3.00222	2.99955	0.00267	2.99921	0.00026	0.00025	3.00005	2.99805	0.00199
		θ	0.50042	0.00084	0.00007	0.50114	0.49949	0.00165	0.49948	0.00105	0.00011	0.50044	0.49812	0.00232
		γ	0.30022	0.00074	0.00002	0.30098	0.29902	0.00196	0.30073	0.00242	0.00021	0.30140	0.29958	0.00181
100	70	α	2.99902	0.00033	0.00039	3.00042	2.99770	0.00272	3.00076	0.00025	0.00023	3.00189	2.99930	0.00259
		θ	0.49969	0.00061	0.00004	0.50041	0.49853	0.00188	0.50044	0.00089	0.00008	0.50126	0.49911	0.00215
		γ	0.30067	0.00222	0.00018	0.30169	0.29919	0.00250	0.30034	0.00114	0.00005	0.30126	0.29941	0.00184
	100	α	3.00079	0.00026	0.00025	3.00212	2.99960	0.00252	2.99979	0.00007	0.00002	3.00056	2.99879	0.00177
		θ	0.49971	0.00057	0.00003	0.50026	0.49875	0.00151	0.50046	0.00092	0.00008	0.50102	0.49976	0.00126
		γ	0.30006	0.00019	0.00000	0.30056	0.29864	0.00192	0.29976	0.00080	0.00002	0.30054	0.29879	0.00175

Table 3

Bayes averages, RABs, ERs and 95% CIs for the sf, hrf, and ahrf under the SE and LINEX loss functions based on Type-II censoring for different samples of size n, ($N = 10000, x_0 = 1,\ \alpha = 2, \theta = 0.5\ and\ \gamma = 0.9$)

n	r	sf, hrf and ahrf	SE						LINEX(ν= 0.5)					
			Average	RAB	ER	UL	LL	Length	Average	RAB	ER	UL	LL	Length
30	21	$S^*(x_0)$	0.99666	0.00251	0.00252	0.99920	0.99371	0.00549	0.99697	0.00221	0.00195	0.99905	0.99526	0.00380
		$h^*(x_0)$	0.00017	0.92525	0.00187	0.00119	-0.00074	0.00194	0.00059	0.74939	0.00123	0.00198	-0.00027	0.00224
		$ah^*(x_0)$	0.00453	0.93463	0.00191	0.00586	0.00217	0.00369	0.00364	0.55384	0.00067	0.00508	0.00157	0.00351
	30	$S^*(x_0)$	0.99985	0.00208	0.00173	1,00225	0.99958	0.00268	0,99810	0.00108	0.00046	0.99967	0.99675	0.00262
		$h^*(x_0)$	0.00379	0.62196	0.00085	0.00480	0.00295	0.00185	0.00335	0.43161	0.00041	0.00433	0.00210	0.00223
		$ah^*(x_0)$	0.00430	0.83770	0.00154	0.00569	0.00214	0.00356	0.00362	0.54906	0.00066	0.00540	0.00190	0.00350
60	42	$S^*(x_0)$	0.99688	0.00230	0.00211	0,99931	0.99537	0.00394	0,99727	0.00191	0.00146	0,99916	0.99620	0.00296
		$h^*(x_0)$	0.00284	0.21372	0.00010	0.00381	0.00216	0.00165	0.00146	0.37385	0.00031	0.00232	0.00046	0.00186
		$ah^*(x_0)$	0.00395	0.68832	0.00104	0.00499	0.00256	0.00243	0.00330	0.40888	0.00037	0.00502	0.00208	0.00295
	60	$S^*(x_0)$	0,99788	0.00129	0.00067	0.99900	0.99662	0.00237	0,9987	0.00060	0.00014	0,99956	0.99662	0.00211
		$h^*(x_0)$	0.00283	0.21118	0.00009	0.00377	0.00215	0.00162	0.00268	0.14729	0.00005	0.00356	0.00215	0.00176
		$ah^*(x_0)$	0.00362	0.54675	0.00055	0.00485	0.00259	0.00226	0.00325	0.38836	0.00033	0.00430	0.00259	0.00252
100	70	$S^*(x_0)$	0,99860	0.00058	0.00013	0,99974	0.99699	0.00275	0,9987	0.00060	0.00015	0,9951	0.99769	0.00182
		$h^*(x_0)$	0.00278	0.18820	0.00008	0.00337	0.00220	0.00117	0.00312	0.33469	0.00024	0.00390	0.00228	0.00161
		$ah^*(x_0)$	0.00336	0.43702	0.00042	0.00445	0.00250	0.00195	0.00285	0.21892	0.00010	0.00390	0.00192	0.00198
	100	$S^*(x_0)$	0,99934	0.00017	0.00001	1,00027	0.99819	0.00208	0,99975	0.00057	0.00013	1,00067	0.99892	0.00175
		$h^*(x_0)$	0.00252	0.08186	0.00001	0.00311	0.00186	0.00125	0.00256	0.09663	0.00002	0.00331	0.00203	0.00128
		$ah^*(x_0)$	0.00287	0.22808	0.00011	0.00364	0.00217	0.00147	0.00333	0.42317	0.00039	0.00414	0.00252	0.00162

Table 4

Bayes averages, RABs, ERs and 95% CIs for the sf, hrf, and ahrf under the SE and LINEX loss functions based on Type-II censoring for different samples of size n, ($N = 10000, x_0 = 1,\ \alpha = 3, \theta = 0.5\ and\ \gamma = 0.3$)

n	r	sf, hrf and ahrf	SE						LINEX(ν= 0.5)					
			Average	RAB	ER	UL	LL	Length	Average	RAB	ER	UL	LL	Length
30	21	$S^*(x_0)$	٠,٩٥٧٨٩	٠,٠٠٢٣٩	٠,٠٠٢٠٩	٠,٩٥٩٢٦	٠,٩٥٥٦٦	٠,٠٠٣٦٠	٠,٩٥٦٩٧	٠,٠٠١٤٣	٠,٠٠٠٧٤	٠,٩٥٨٦٩	٠,٩٥٥٤٣	٠,٠٠٣٢٦
		$h^*(x_0)$	٠,١٣٥٢١	٠,٠١٤٤١	٠,٠٠١٥٦	٠,١٣٧٤٥	٠,١٣٢٦٩	٠,٠٠٤٧٧	٠,١٣٩٦٣	٠,٠١٧٨١	٠,٠٠٢٣٩	٠,١٤١٥٣	٠,١٣٧٦٠	٠,٠٠٣٩٣
		$ah^*(x_0)$	٠,١٤٩٩٩	٠,٠١٦٥٥	٠,٠٠٢٣٩	٠,١٥١٢٥	٠,١٤٧٤٢	٠,٠٠٣٨٣	٠,١٤٩١٩	٠,٠١١٠٦	٠,٠٠١٠٧	٠,١٥٠٩١	٠,١٤٧٤٨	٠,٠٠٣٤٣
	30	$S^*(x_0)$	٠,٩٥٤٩٩	٠,٠٠١٣٣	٠,٠٠٠٦٤	٠,٩٥٥٨٢	٠,٩٥٢٢٨	٠,٠٠٣٥٤	٠,٩٥٤٧٦	٠,٠٠١٣٣	٠,٠٠٠٢٩	٠,٩٥٥٧٢	٠,٩٥٣١٥	٠,٠٠٢٥٧
		$h^*(x_0)$	٠,١٣٥١٤	٠,٠١٠٣٠	٠,٠٠٠٨٠	٠,١٣٧٩٤	٠,١٣٣٥٨	٠,٠٠٤٣٦	٠,١٣٩٠٠	٠,٠١٣٢٧	٠,٠٠١٣٢	٠,١٤٠٨٧	٠,١٣٧٣٩	٠,٠٠٣٤٨
		$ah^*(x_0)$	٠,١٤٨٥٩	٠,٠٠٩٥٦	٠,٠٠٠٨٠	٠,١٤٩٩١	٠,١٤٧٥٨	٠,٠٠٢٣٣	٠,١٤٨١٢	٠,٠٠٣٨١	٠,٠٠٠١٣	٠,١٤٩٣٦	٠,١٤٦٩٧	٠,٠٠٢٣٩
60	42	$S^*(x_0)$	٠,٩٥٤٠٦	٠,٠٠١٦٢	٠,٠٠٠٩٦	٠,٩٥٥٧٠	٠,٩٥٢٠٢	٠,٠٠٣٦٨	٠,٩٥٤٧٦	٠,٠٠٠٨٩	٠,٠٠٠٢٩	٠,٩٥٦٠٨	٠,٩٥٣٥٦	٠,٠٠٢٥٢
		$h^*(x_0)$	٠,١٣٨٧٣	٠,٠١١٢٣	٠,٠٠٠٩٥	٠,١٤٠٥٤	٠,١٣٦٥١	٠,٠٠٤٠٣	٠,١٣٥٥٥	٠,٠١١٩٥	٠,٠٠١٠٨	٠,١٣٧٣٠	٠,١٣٤١١	٠,٠٠٣١٩
		$ah^*(x_0)$	٠,١٤٥٢٤	٠,٠١٥٧٣	٠,٠٠٢١٥	٠,١٤٧١٩	٠,١٤٣٦٤	٠,٠٠٣٥٥	٠,١٤٨٢٤	٠,٠٠٤٦٣	٠,٠٠٠١٩	٠,١٤٩٣٩	٠,١٤٧١٤	٠,٠٠٢٢٥
	60	$S^*(x_0)$	٠,٩٥٥٠٧	٠,٠٠٠٥٦	٠,٠٠٠١٢	٠,٩٥٦٥٣	٠,٩٥٣٦٩	٠,٠٠٢٨٣	٠,٩٥٥١٩	٠,٠٠٠٤٥	٠,٠٠٠٠٧	٠,٩٥٦٠٨	٠,٩٥٤٤٧	٠,٠٠١٦٠
		$h^*(x_0)$	٠,١٣٥٨٦	٠,٠٠٩٦٦	٠,٠٠٠٧٠	٠,١٣٧٢٢	٠,١٣٤٠٦	٠,٠٠٣١٧	٠,١٣٦٧٢	٠,٠٠٣٤٢	٠,٠٠٠٠٩	٠,١٣٧٥٢	٠,١٣٥٦٨	٠,٠٠١٨٤
		$ah^*(x_0)$	٠,١٤٦٤٩	٠,٠٠٧٢٥	٠,٠٠٠٤٦	٠,١٤٧٥١	٠,١٤٥٦٢	٠,٠٠١٨٩	٠,١٤٧٩٣	٠,٠٠٢٥٦	٠,٠٠٠٠٦	٠,١٤٩١٠	٠,١٤٦٩٧	٠,٠٠٢١٣
100	70	$S^*(x_0)$	٠,٩٥٦٥٦	٠,٠٠٠٩٩	٠,٠٠٠٣٦	٠,٩٥٧٣٩	٠,٩٥٥٥٢	٠,٠٠١٨٧	٠,٩٥٤٩٨	٠,٠٠٠٦٦	٠,٠٠٠١٦	٠,٩٥٦٠٥	٠,٩٥٣٨٩	٠,٠٠٢١٦
		$h^*(x_0)$	٠,١٣٦٤٧	٠,٠٠٥٢٦	٠,٠٠٠٢١	٠,١٣٨٠١	٠,١٣٥٠٦	٠,٠٠٢٩٥	٠,١٣٨١٢	٠,٠٠٦٨٣	٠,٠٠٠٣٥	٠,١٣٩٢٨	٠,١٣٦٧٨	٠,٠٠٢٥٠
		$ah^*(x_0)$	٠,١٤٦٩٥	٠,٠٠٤١٠	٠,٠٠٠١٥	٠,١٤٨٠٨	٠,١٤٥٨٨	٠,٠٠٢٢١	٠,١٤٨١٥	٠,٠٠٤٠٤	٠,٠٠٠١٤	٠,١٤٩٠٦	٠,١٤٧٢٣	٠,٠٠١٨٣
	100	$S^*(x_0)$	٠,٩٥٦٠٧	٠,٠٠٠٤٨	٠,٠٠٠٠٨	٠,٩٥٧٠١	٠,٩٥٥١٨	٠,٠٠١٨٣	٠,٩٥٥٧٢	٠,٠٠٠١١	٠,٠٠٠٠٠٥	٠,٩٥٦٦٣	٠,٩٥٥٢٠	٠,٠٠١٤٢
		$h^*(x_0)$	٠,١٣٦٥١	٠,٠٠٤٩٥	٠,٠٠٠١٨	٠,١٣٧٤٨	٠,١٣٥٤٦	٠,٠٠٢٠٢	٠,١٣٦٨٢	٠,٠٠١٣٩	٠,٠٠٠٠١٤	٠,١٣٧٨٩	٠,١٣٦٣٤	٠,٠٠١٥٦
		$ah^*(x_0)$	٠,١٤٧٣٣	٠,٠٠١٥٤	٠,٠٠٠٠٢	٠,١٤٨١٤	٠,١٤٦٣٦	٠,٠٠١٧٨	٠,١٤٧٣٩	٠,٠٠١٥٤	٠,٠٠٠٠١١	٠,١٤٨٠١	٠,١٤٦٦٠	٠,٠٠١٤١

Table 5
Bayes point predictors and 95% credible intervals for the future observation from the DZW distribution under two-sample prediction
$(N = 10000, n = 100,\ r = 70, m = 25)$

s	SE				LINEX($v = 0.5$)			
	$\tilde{y}_{(s)}$	LL	UL	Length	$y^*_{(s)}$	LL	UL	Length
1	0.1986	0.1974	0.19982	0.0024	0.2003	0.1996	0.2012	0.0016
13	5.3979	5.3968	5.4002	0.0034	5.3996	5.3986	5.4005	0.0019
25	14.9994	14.9977	15.0012	0.0036	15.0006	14.9994	15.0016	0.0022

9. Applications

This subsection aims to demonstrate how the proposed method can be used in practice. Three real data sets are used for this purpose. The DZW(α, θ, γ) distribution is fitted to this data using the Kolmogorov-Smirnov goodness of fit test through Mathematica 11.

Application I

The first data set consists of the 2003 final examination marks of 48 slow space students in mathematics at the Indian Institute of Technology at Kanpur. These data are given by Gupta and Kundu (2009).

The data are**: 29, 25, 50, 15, 13, 27, 15, 18, 7, 7, 8, 19, 12, 18, 5, 21, 15, 86, 21, 15, 14, 39, 15, 14, 70, 44, 6, 23, 58, 19, 50, 23, 11, 6, 34, 18, 28, 34, 12, 37, 4, 60, 20, 23, 40, 65, 19, and 31.**

Application II

The second data set refers to the survival times of 44 patients suffering from head and neck cancer who retreated using a combination of radiotherapy. These data are proposed by Afify *et al.* (2021).
The data are: **12, 32, 37, 24, 24, 74, 81, 26, 41, 58, 63, 68, 78, 47, 55, 84, 155, 159, 92, 94, 110, 127, 130, 133, 140, 112, 119, 146, 173, 179, 194, 195, 339, 432, 209, 249, 281, 319, 469, 725, 817, 519, 633, and 1776**.

Application III

The third data represents 40 observations of time-to-failure (10^3h) of a turbocharger of one type of engine. These data are provided by Xu *et al.* (2003).

The data are: **2, 2, 3, 3, 3, 4, 4, 5, 5, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 8, 8, 8, 8, 9, 9, 9, and 9.**

Kolmogorov–Smirnov and the χ^2 goodness of fit test are applied to check the validity of the fitted model. The p-values are given, respectively, by 0.8375, 0.9430, and 0.3650. The p-value given in each case showed that the model fits the data very well.

Table 6

Bayes estimates and standard errors of the parameters for the real data sets based on Type-II censoring

Applications	n	r	$\underline{\varphi}^*$	SE		LINEX$(v = 0.5)$	
				Estimates	standard errors	Estimates	standard errors
I	٤٨	٣٤	α^*	٥,٠٠١٤	٠,٠٠٤٧	٥,٠٠١١	٠,٠٠٣٥
			θ^*	١,٠١٨٧	٠,٠٠٤١	١,٠١٩٧	٠,٠٠٣٩
			γ^*	٠,٨٩٩٦	٠,٠٠٥٠	٠,٩٠٠٢	٠,٠٠٤١
		٤٨	α^*	٤,٩٩٩٧	٠,٠٠٣٥	٤,٩٩٨٩	٠,٠٠٢٦
			θ^*	١,٠١٩٤	٠,٠٠٣٩	١,٠١٩٦	٠,٠٠٣٦
			γ^*	٠,٩٠١٣	٠,٠٠٤٨	٠,٨٩٩٥	٠,٠٠٢٤
II	٤٤	٣١	α^*	٢,٠٠١٢	٠,٠٠٤٣	٢,٠٠١٤	٠,٠٠٣٧
			θ^*	٠,٤٩٩٣	٠,٠٠٣٦	٠,٤٩٩٢	٠,٠٠٣١
			γ^*	٠,٨٩٨٨	٠,٠٠٤٨	٠,٨٩٩٦	٠,٠٠٣٦
		٤٤	α^*	١,٩٩٩٠	٠,٠٠٣٤	٢,٠٠١٢	٠,٠٠٣١
			θ^*	٠,٥٠١٥	٠,٠٠٣٥	٠,٤٩٩٠	٠,٠٠٢٩
			γ^*	٠,٩٠١٤	٠,٠٠٤٥	٠,٩٠٠٣	٠,٠٠٣٤
III	٤٠	٢٨	α^*	٢,٩٩٩٩	٠,٠٠٥٣	٣,٠٠٠٥	٠,٠٠٤٩
			θ^*	١,٩٩٩٩	٠,٠٠٣٨	١,٩٩٧٩	٠,٠٠٣٥
			γ^*	٠,٨٠٠١	٠,٠٠٤٣	٠,٨٠٠٦٠	٠,٠٠٣٨
		٤٠	α^*	٣,٠٠٠٤	٠,٠٠٤٣	٢,٩٩٩٣	٠,٠٠٤٢
			θ^*	١,٩٩٩٨	٠,٠٠٣٥	١,٩٩٩٥	٠,٠٠٣٢
			γ^*	٠,٧٩٨٨	٠,٠٠٤١	٠,٧٩٩٢	٠,٠٠٣٨

Table 7

Bayes estimates and standard errors of the sf, hrf, and ahrf for the real data sets based on Type-II censoring

Applications	n	r	sf, hrf and ahrf	SE		LINEX($v = 0.5$)	
				Estimates	standard errors	Estimates	standard errors
I	٤٨	٣٤	$S^*(x_0)$	٠,٩٩٩٤	٠,٠٠٤٣	٠,٩٩٨٤	٠,٠٠٤٠
			$h^*(x_0)$	٠,٠٠١٠	٠,٠٠٤٠	٠,٠٠١١	٠,٠٠٣٤
			$ah^*(x_0)$	٠,٠٠٣٣	٠,٠٠٤٤	٠,٠٠٢٨	٠,٠٠٤٠
		٤٨	$S^*(x_0)$	٠,٩٩٨٥	٠,٠٠٣٦	٠,٩٩٩٨	٠,٠٠٣٣
			$h^*(x_0)$	٠,٠٠١٦	٠,٠٠٣١	٠,٠٠٠٤	٠,٠٠٢٩
			$ah^*(x_0)$	٠,٠٠٢٦	٠,٠٠٣٧	٠,٠٠٢٤	٠,٠٠٣٦
II	٤٤	٣١	$S^*(x_0)$	٠,٩٩٧٦	٠,٠٠٤٨	٠,٩٩٩٧	٠,٠٠٤٢
			$h^*(x_0)$	٠,٠٠٥٩	٠,٠٠٥٨	٠,٠٠٢٨	٠,٠٠٤٧
			$ah^*(x_0)$	٠,٠٠٤٨	٠,٠٠٤٢	٠,٠٠٤٧	٠,٠٠٤١
		٤٤	$S^*(x_0)$	٠,٩٩٨٤	٠,٠٠٤٩	٠,٩٩٩٠	٠,٠٠٤١
			$h^*(x_0)$	٠,٠٠٣٩	٠,٠٠٤٥	٠,٠٠٢٨	٠,٠٠٤٣
			$ah^*(x_0)$	٠,٠٠٤٣	٠,٠٠٤٠	٠,٠٠٤٦	٠,٠٠٣٩
III	٤٠	٢٨	$S^*(x_0)$	٠,٩٧٤٤	٠,٠٠٤٦	٠,٩٧٣٧	٠,٠٠٣٩
			$h^*(x_0)$	٠,٠٧٣٥	٠,٠٠٣٣	٠,٠٧٣٦	٠,٠٠٣٢
			$ah^*(x_0)$	٠,٠٧٧٣	٠,٠٠٥١	٠,٠٧٦١	٠,٠٠٤٦
		٤٠	$S^*(x_0)$	٠,٩٧٦٠	٠,٠٠٣٨	٠,٩٧٦٣	٠,٠٠٣٧
			$h^*(x_0)$	٠,٠٧٤٣	٠,٠٠٣١	٠,٠٧٣١	٠,٠٠٣٠
			$ah^*(x_0)$	٠,٠٧٥٤	٠,٠٠٤٤	٠,٠٧٦٨	٠,٠٠٣٢

Table 8

Bayes point predictors and 95% credible intervals for the future observation from the DZW distribution under two-sample prediction for three applications

Applications	s	SE				LINEX($v = 0.5$)			
		$\tilde{y}_{(S)}$	LL	UL	Length	$y^{*}_{(s)}$	LL	UL	Length
I	1	0.2002	0.1994	0.2013	0.0019	0.1997	0.1990	0.2005	0.0015
	23	7.4990	7.4978	7.5003	0.0024	7.4989	7.4979	7.5001	0.0022
	45	20.0005	19.9985	20.0017	0.0032	19.9995	19.9981	20.0010	0.0029
II	1	0.3009	0.2996	0.3021	0.0025	0.2993	0.2982	0.3004	0.0022
	18	4.8005	4.7990	4.8024	0.0034	4.7991	4.7974	4.8005	0.0031
	35	14.902	14.8989	14.9041	0.0042	14.9001	14.8981	14.9016	0.0035
III	1	0.5013	0.5004	0.5026	0.0022	0.4989	0.4982	0.5001	0.0019
	13	2.3994	2.3981	2.4005	0.0024	2.4004	2.3994	2.4014	0.0020
	25	9.0004	8.9989	9.0019	0.0027	9.0009	9.0000	9.0024	0.0023

10. Concluding remarks

1) It is observed from Tables 1, and 2 that the RABs and ERs of the Bayes averages for the parameters $\alpha, \theta, and\ \gamma$ perform better when the sample size increases and the level of censoring decreases. The lengths of the credible intervals get shorter when the sample size increases and the level of censoring decreases.

2) It is noticed from Tables 3, and 4 that the RABs and ERs of the Bayes averages; for the sf, hrf, and ahrf perform better when the sample size increases and the level of censoring decreases. The lengths of the credible intervals become narrower when the sample size increases and the level of censoring decreases.

3) From Tables 5, and 8, one can observe that the length of the interval of the first order statistic is smaller than the length of the interval of

the last order statistic. Also, the BPB include the predictive values between the (LL and UL).

4) The lengths of the intervals of the Bayes predictors under the LINEX loss function are less than the lengths of the intervals of the Bayes predictors under the SE loss function.

11. Conclusion

In this paper, Bayesian estimation and prediction (point and interval) for discrete Zubair Weibull distribution under Type-II censored data are proposed. Informative priors (gamma and beta) were used to estimate the unknown parameters, survival function, hazard rate function, and alternative hazard rate function under SE loss function as a symmetric loss function and the LINEX loss function as an asymmetric loss function. The performance of the Bayes estimates was examined through some measurements of accuracy. From the numerical results, it is concluded that the Bayes estimates perform better when the sample size increases. Moreover, the Bayes estimates under the LINEX loss function behave quite close to the corresponding estimates under the SE loss function. The Bayes predictors (point and interval) for a future observation from the DZW distribution based on Type-II censored sample are derived. Also, three real data sets were applied to show the applicability and flexibility of the distribution in practice.

References

AL-Kashlan A. G., EL-Helbawy, A. A., Hegazy, M. A. and AL-Dayian, G. R. (2022). A discrete analogue of complementary exponentiated-G Poison family of distributions: properties and estimation. *Asian Journal of Probability and Statistics*, 21(4): 34-63. https://doi.org/10.9734/ajpas/2023/v21i4471.

Afify, A. Z., El-Morshedy, M., Eliwa, M. S. (2021). A New skewed discrete model: properties, inference, and applications. *Pakistan Journal of statistics and operation research*, 17(4): 799-816. https://pjsor.com/index.php/index.

Alamatsaz, M. H., Dey, S., Dey, T. and Shams Harandi, S. (2016). Discrete generalized Rayleigh distribution. *Pakistan Journal of Statistics*, 32(1): 1-20.

AL-Huniti, A. A. and AL-Dayian, G. R. (2012). Discrete Burr Type III distribution. *American Journal of Mathematics and Statistics*, 2(5): 145-152.

AL-Hussaini, E. K. and Jaheen, Z. F. (1992). Bayesian estimation of the parameters, reliability and failure rate functions of the Burr Type XII failure model. *Journal of Statistical Computation and Simulation*, 41: 31-40.

Almetwally, E. M. (2020). Discrete alpha power inverse Lomax distribution with application of COVID-19 Data. *International Journal of Applied Mathematics and Statistical Sciences*, 9(6): 2319-3972.

Ashour, S. K. and Muiftah, M. S. A. (2019). Bayesian estimation of the parameters of discrete Weibull Type (I) distribution. *Journal of Modern Applied Statistical Methods,* 18(2): 2740.

Bhattacharya, S. K. (1967). Bayesian approach to life testing and reliability estimation. *Journal of the American Statistical Association,* 62: 48-62.

El-Morshedy, M., Eliwa, M. S. and Abhishek Tyagi. (2021). Discrete analogue of odd Weibull-G family of distributions: properties, classical and Bayesian estimation with applications to count data. *Journal of Applied Statistics*, 49(11):2928-2952. https://doi.org/10.1080/02664763.2021.1928018.

Gelman, A., Carlin, J. B., Stern, H. S. and Rubin, D. B. (2003). *Bayesian data analysis*, the 2nd Edition. New York, Chapman and Hall/CRC.

Gomez-Deniz, E. and Calderin-Ojeda, E. (2011). The discrete Lindley distribution: properties and application. *Journal of Statistics Computation and Simulation,* 81(11): 1405-1416.

Gupta, R. D. and Kundu., D. (2009). A new class of weighted exponential distributions. *Statistics,* 43(6): 621-634.

Hegazy, M. A., EL-Helbawy, A. A., Abd EL-Kader, R. E. and AL-Dayian, G. R. (2021). Bayesian estimation and prediction of discrete Gompertz distribution. *Journal of Advances in Mathematics and Computer Science,* 36(2): 1-21.

Hegazy, M. A., EL-Helbawy, A. A., Abd EL-Kader, R. E. and AL-Dayian, G. R. (2018). Discrete Gompertz distribution: properties and estimation. *Journal of Biostatistics and Biometric Applications,* 3(3): 307.

Jazi, M. A., Lai, C. D. and Alamatsaz, M. H. (2010). A discrete inverse Weibull distribution and estimation of its parameters. *Elsevier Statistical Methodology,* 7(2): 121- 132.

Kamari, H., Bevrani, H. and Kamary, K. (2016). Bayesian estimation of discrete Burr distribution with two parameters. arXiv:1501.05876v3 [stat.CO]. https://doi.org/10.48550/arXiv.1501.05876

Kemp, A.W. (1997). Characterization of a discrete normal distribution**.** *Journal of Statistical Planning and Inference*, 63(2): 223-229.

Khan, M. S. A., Khalique, A. and Abouammoh, A. M. (1989). On estimating parameters in a discrete Weibull distribution. *IEEE Transactions on Reliability,* 383: 348-350.

Krishna, H. and Pundir, P. S. (2009). Discrete Burr and discrete Pareto distributions. *Statistical Methodology,* 6(2): 177-188.

Migdadi, H. S. (2015). Bayesian inference for the scale parameter of the discrete Rayleigh distribution. *MAGNT Research Report,* 3(2):1073-1080.

Mudholkar, G. S. and Srivastava, D. K. (1993). Exponentiated Weibull family for analyzing bathtub failure-rate data. *Reliability, IEEE Transactions on*, 42(2): 299-302.

Nakagawa, T. and Osaki, S. (1975). The discrete Weibull distribution. *IEEE Transactions on Reliability,* 24(5): 300-301.

Para, B. A. and Jan, T. R. (2018). On three parameter discrete generalized inverse Weibull distribution: properties and applications. *Annals of Data Science,* 6(3):549–570.

Roy, D. (2003). Discrete normal distribution. *Communication in Statistics-Theory and Methods*, 32(10): 1871-1883.

Xu, K., Xie, M., Tang, L. C. and Ho, S. L. (2003) Application of neural networks in forecasting engine systems reliability, *Applied Soft Computing*, 2(4): 255-268.