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Some Characterizations of The Skew Normal Distribution Using Conditional Moments

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Abstract

Very few researchers have tackled the characterization problem of the skew nor- mal distribution due perhaps to its mathematical tractability. In this article, some new characterization results for the skew normal distribution based on conditional moments have been obtained. The results specialized to the standard normal distri- bution. Some consequences and discussions are, also given in this context.

1- Introduction

The skew-normal distribution was first introduced by O'Hagan and Leonard (1976) as a prior distribution for estimating a normal location parameter.

The skew-normal distribution and its variations have been discussed by several au- thors including Azzalini (1985, 1986), Henze (1986), Azzalini and Dalla Valle (1996), Branco and Dey (2001), Loperfido (2001), Arnold and Beaver (2002), Balakrishnan (2002), and Azzalini and Chiogna (2004) and others for a comprehensive survey of de- velopments on skew-normal distribution and its multivariate form see Azzalini (2005).

A random variable *X* is said to have a standard skew-normal distribution with parameter $\lambda \in \mathbb{R}$, denoted by X ~SN(λ), if its probability density function (pdf) is:

$$f(x,\lambda) = 2\Phi(\lambda x)\,\phi(x) = \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\lambda x} e^{-\frac{t^2}{2}}\, dt. \qquad \lambda \in \mathbb{R}, -\infty < x < \infty \tag{1.1}$$

The cumulative distribution function (cdf) of SN(λ) is given by:

$$F(x,\lambda) = 2\int_{-\infty}^{x}\Phi(\lambda x)\phi(u)du = \int_{-\infty}^{x}\frac{1}{\pi}e^{-\frac{u^2}{2}}\int_{-\infty}^{\lambda u}e^{-\frac{t^2}{2}}\,dt\,du. \tag{1.2}$$

where $\phi(.)$ and $\Phi(.)$ denote the standard normal pdf and cdf, respectively.

Now, we shall give some definitions that are needed in the sequel.

Definition 1 Let $X_1, X_2, \ldots, X_n$ be independent random variables having skew normal pdf $f(x,\lambda)$ and cdf $F(x,\lambda)$. Let $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ denote the corresponding order statistics. Then the pdf of $X_{r:n}$, $1 \le r \le n$ and the joint pdf of $(X_{r:n}, X_{s:n})$ can be written as (Arnold et al. (2008)):

$$f_{r:n}(x,\lambda) = C_{r:n}(F(x,\lambda))^{r-1} f(x,\lambda)(1-F(x,\lambda))^{n-r}, \qquad 1 \le r \le n \tag{1.3}$$

$$f_{r:n}(x,y,\lambda) = C_{r:s:n}(F(x,\lambda))^{r-1} f(x,\lambda)(F(y,\lambda)-F(x,\lambda))^{s-r-1} f(y,\lambda)(1-F(y,\lambda))^{n-s}, \quad 1 \le r < s \le n \tag{1.4}$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$, $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

Definition 2 Let $n \in \mathbb{R}$, $k \ge 1$, $m_1, m_2, \ldots, m_{n-1} \in \mathbb{R}$, $M_r = \sum_{j=r}^{n-1} m_i$, $1 \le r \le n-1$, and let $\bar{m} = (m_1, m_2, \ldots, m_{n-1})$, $m_1 = m_1 = \cdots = m_{n-1} = m$, $m \ne -1$. The pdf of the r<u>th</u> generalized order statistics (gos) (Kamps 1995) is given by:

$$f_{r,n,\bar{m},k}(x,\lambda) = \frac{d_{r-1}}{(r-1)!}(1-F(x,\lambda))^{n-r+k+M_r-1} f(x,\lambda)\left(\frac{1-(1-F(x,\lambda))^{m+1}}{m+1}\right)^{r-1}, \quad 1 \le r \le n \tag{1.5}$$

and the joint pdf of the r<u>th</u> and the s<u>th</u> gos as:

$$f_{r,s,n,\bar{m},k}(x,y,\lambda) = \frac{d_{s-1}}{(r-1)!\,(s-r-1)!}(1-F(x,\lambda))^{m} f(x,\lambda)\left(\frac{1-(1-F(x,\lambda))^{m+1}}{m+1}\right)^{r-1} \times$$

$$(1-F(y,\lambda))^{n-s+k+M_s-1} f(y,\lambda)\left[\frac{(1-F(x,\lambda))^{m+1}-(1-F(y,\lambda))^{m+1}}{m+1}\right]^{s-r-1}, 1 \le r < s \le n \tag{1.6}$$

where $d_{r:n} = \prod_{i=1}^{r} k+n-i+M_i \ge 1$

Let us denote the k$^{\text{th}}$ single moment $E\left(X_{r:n}^{k}\right)$ of $X_{r:n}$ by $\mu_{r:n}^{(k)}$ and the product moment $E\left(X_{r:n}X_{s:n}\right)$ of $X_{r:n}$ and $X_{s:n}$ by $\mu_{r,s:n}$.

2- The Results.

We shall state the main results concerning characterization of the $SN(\lambda)$ distribution.

Theorem 1

Let X be a continuous random variable with cdf $F(.)$, survival function $\bar{F}(.)$, pdf $f(.)$, failure rate $h(.)$, finite mean μ. Then X has a skew normal distribution$SN(\lambda)$ with mean μ if and only if.

$$E\left(X^{2k} \,\middle|\, X > y\right) = (2k-1)!! +$$

$$\frac{1}{F(y,\lambda)}\sum_{i=1}^{k}\frac{(2k-1)!!}{(2i-1)!!}\left(y^{2i-1}f(y,\lambda)+\frac{2\lambda}{\sqrt{2\pi}}\sum_{j=1}^{i}\frac{(2i-1)!!}{(2j-1)!!}\frac{y^{2j-2}\phi\left(y\sqrt{1+\lambda^2}\right)}{(1+\lambda)^{2i-j+1}}\right)$$

with k = 1.2.3…, (2.1)

or

$$E\left(X^{2k} \,\middle|\, X > y\right) =$$

$$\left(\frac{1}{\bar{F}(y,\lambda)}\sum_{i=o}^{k}\frac{(2k)!!}{(2i)!!}\right)\left(y^{2i}f(y,\lambda)+\frac{(2i-1)!!\,\Phi\left(-y\sqrt{1+\lambda^2}\right)}{(1+\lambda)^{i+1}}\right)+$$

$$\left(\frac{1}{F(y,\lambda)}\sum_{i=0}^{k}\frac{(2k)!!}{(2i)!!}\right)\frac{2\lambda}{\sqrt{2\pi}}\sum_{j=1}^{i}\frac{(2i-1)!!}{(2j-1)!!}\frac{y^{2j-1}\phi\left(y\sqrt{1+\lambda^2}\right)}{(1+\lambda)^{i-j+1}}$$

with k =0,1.2.3…, (2.2)

where $(n)!! = n(n-2)!!$

Remarks

(1) From Theorem 1, for k=0, (2.2) reduces to:

$$E(X|X>y)=h(y)+\mu\frac{\Phi\left(-y\sqrt{1+\lambda^2}\right)}{\bar{F}(y,\lambda)}.$$

(2.3)

The " only if " part of (2.3) in this case can be proved simply as follows:

If (2.4) is true, then:

$$E(X|X>y)=\frac{1}{\bar{F}(y,\lambda)}\int_y^\infty x\ f(x,\lambda)dx=h(y,\lambda)+\mu\frac{\Phi\left(-y\sqrt{1+\lambda^2}\right)}{\bar{F}(y,\lambda)}.$$

Then we have:

$$\int_y^\infty x\ f(x,\lambda)dx=f(y,\lambda)+\mu\Phi\left(-y\sqrt{1+\lambda^2}\right)$$

Differentiating both sides with respect to , we obtain:

$$-y\ f(x,\lambda)=2\lambda\phi(\lambda y)\phi(y)+2\Phi(\lambda y)\phi(y)(-y)-\mu\phi\left(y\sqrt{1+\lambda^2}\right)\sqrt{1+\lambda^2}$$

Or, $-y\ f(x,\lambda)=\frac{2\lambda}{\sqrt{2\pi}}\phi\left(y\sqrt{1+\lambda^2}\right)+2\Phi(\lambda y)\phi(y)(-y)-\mu\phi\left(y\sqrt{1+\lambda^2}\right)\sqrt{1+\lambda^2}$

Or, $-y\ f(x,\lambda)=2\Phi(\lambda y)\phi(y)(-y)$

Or, $f(x,\lambda)=2\Phi(\lambda y)\phi(y)$

which is the pdf of the skew normal distribution $SN(\lambda)$.

(2) Theorem 1 reducedto the case of the standard normal distribution by putting $\lambda=0$.

Theorem 2

Let X be a random variable having pdf $f(x)$ and cdf $F(x)$. Then X have a skew normal distribution $SN(\lambda)$ if and only if

$$E\left(X_{s:n}^k|X_{r:n}=x\right)=x^k+\frac{(n-r)!}{(1-F(x,\lambda))^{n-r}}\eta_s(x,\lambda),|1\le r<s\le n, \qquad (2.4)$$

where:

$\eta_s(x,\lambda) =$

$$\sum_{j=r+1}^{s} k \int_{x}^{\infty} y^{k-1} \frac{\left[\left[\left(1-\int_{-\infty}^{y} \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\lambda x} e^{-\frac{t^2}{2}}\, dt\, dx\right)^{(n-j=1)}\right] \psi_1(x,y,\lambda)\right]}{(n-j+1)!\,(j-r-1)!} dy$$

and

$\psi_1(x,y,\lambda) =$

$$\left[\left(\int_{-\infty}^{y} \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\lambda x} e^{-\frac{t^2}{2}} dt\, dx - \int_{-\infty}^{x} \frac{1}{\pi} e^{-\frac{u^2}{2}} \int_{-\infty}^{\lambda u} e^{-\frac{t^2}{2}} dt\, du\right)^{(j-r-1)}\right]$$

Remarks:

(1) From Theorem 2, at $s = r + 1$, we have:

$$E\left(X_{r+1:n}^{k} \middle| X_{r:n} = x\right) = x^k + \frac{1}{\left(1-F(x,\lambda)\right)^{n-r}} \eta(x,\lambda)$$

k = 1, 2, 3, …. (2.5)

where $\eta(x,\lambda) = k \int_{x}^{\infty} y^{k-1} \left(1 - \int_{-\infty}^{y} \frac{1}{\pi|} e^{-\frac{x^2}{2}} \int_{-\infty}^{\lambda x} e^{-\frac{t^2}{2}}\, dt\, dx\right)^{(n-r)} dy.$

(2) The result (2.4) reduces to the standard normal distribution case if λ = 0.

i.e.

$$E\left(X_{s:n}^{k} \middle| X_{r:n} = x\right) = x^k + \frac{(n-r)!}{\left(1-F(x,\lambda)\right)^{n-r}} \eta_s(x), \qquad 1 \le r < s \le n$$

(2.6)

where

$$\eta_s(x) = \sum_{j=r+1}^{s} k \int_x^{\infty} y^{k-1} \frac{\left|\left[\left(1 - \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx\right)^{(n-j+1)}\right] \psi_1(x,y)\right|}{(n-j+1)!\,(j-r-1)!} dy,$$

and

$$\psi_1(x,y) = \left[\left(\int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx - \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du\right)^{(j-r-1)}\right].$$

Theorem 3

Let X be a random variable having pdf $f(x)$ and cdf $F(x)$. Then X have a skew normal distribution SN(λ) if and only if

$$E\left(x_{s:n,\tilde{m},k}^{k} \middle| X_{r:n,\tilde{m},k} = x\right) =$$

$$x^k + \left(\frac{[(m+1)(n-s-1)+k]}{(m+1)}\right)! \frac{\prod_{i=r+1}^{s}[(m+1)(n-r-1)+k](m+1)^{s-r}}{(1-F(x,\lambda))^{(m+1)(n-r-1)+k}} \gamma_s(x,\lambda),$$

$$1 \le r < s \le n \tag{2.7}$$

where

$$\gamma_s(x,\lambda) = \sum_{j=r+1}^{s} k \int_x^{\infty} y^{k-1} \frac{\left|\left[\left(1 - \int_{-\infty}^{y} \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\lambda x} e^{-\frac{t^2}{2}} dt\, dx\right)^{(m+1)(n-j)+k}\right] \psi_1(x,y,\lambda)\right|}{\left(\frac{[(m+1)(n-j)+k]}{(m+1)}\right)!\,(j-r-1)!} dy.$$

and

$$\psi_m(x,y,\lambda)$$

$$= \left[\left(1 - \int_{-\infty}^{x} \frac{1}{\pi} e^{-\frac{u^2}{2}} \int_{-\infty}^{\lambda u} e^{-\frac{t^2}{2}} dt\, du\right)^{m+1} - \left(1 - \int_{-\infty}^{y} \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\lambda x} e^{-\frac{t^2}{2}} dt\, dx\right)^{m+1}\right]^{(j-r-1)}$$

Remarks.

(1) At $s = r + 1$, (2.7), reduces to

$$E\left(X^k_{r+1:n,\bar{m},k}\middle|X_{r:n,\bar{m},k}=x\right)=x^k+\frac{1}{\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}}\gamma(x,\lambda),\qquad k=1,2,3,\ldots$$

where $\gamma(x,\lambda)=k\int_x^{\infty} y^{k-1}\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx\right)^{(m+1)(n-r-1)}dy.$

(2) Also (2.7) reduces to the standard normal distribution case if $\lambda = 0$.

i.e.

$$E\left(X^k_{s:n,\bar{m},k}\middle|X_{r:n,\bar{m},k}=x\right)=$$

$$x^k+\left(\frac{[(m+1)(n-s-1)+k]}{(m+1)}\right)!\,\frac{\prod_{i=r+1}^{s}[(m+1)(n-r-1)+k](m+1)^{s-r}}{\left(1-F(x)\right)^{(m+1)(n-r-1)+k}}\gamma_s(x),$$

$1 \le r < s \le n$ (2.9)

where

$$\gamma_s(x)=\sum_{j=r+1}^{s}k\int_x^{\infty}y^{k-1}\frac{\left[\left[\left(1-\int_{-\infty}^{y}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right)^{(m+1)(n-j)+k}\right]\psi_1(x,y)\right]}{\left(\frac{[(m+1)(n-j)+k]}{(m+1)}\right)!\,(j-r-1)!}dy$$

and

$$\psi_m(x,y)=\left[\left(1-\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{u^2}{2}}du\right)^{m+1}-\left(1-\int_{-\infty}^{y}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx\right)^{m+1}\right]^{(j-r-1)}$$

3- The proofs

proof of theorem 1

Necessity.

We have

$$E\left(X^k\middle|X>y\right)=\frac{1}{\bar{F}(y,\lambda)}\int_y^{\infty}x^k f(x,\lambda)dx$$

$$= \frac{2}{\bar{F}(y,\lambda)} \int_{y}^{\infty} x^{k}\, \Phi(\lambda x)\phi(x)d$$

$$= \frac{-2}{\bar{F}(y,\lambda)} \int_{y}^{\infty} x^{k-1}\, \Phi(\lambda x)d\phi(x)$$

(3.1)

Integrating by parts we get

$$E\left(X^{k}\middle|X > y\right) = \frac{2y^{k-1}\Phi(\lambda y)\phi(y)}{\bar{F}(y,\lambda)} + \frac{2(k-1)\int_{y}^{\infty} x^{k-2}\Phi(Xx)\phi(x)dx}{\bar{F}(y,\lambda)} +$$

$$\frac{2\lambda\int_{y}^{\infty} x^{k-1}\phi(\lambda x)\phi(x)dx}{\bar{F}(y,\lambda)}$$

(3.2)

$$= y^{k-1}h(y,\lambda) + (k-1)E\left(X^{k-2}\middle|X > y\right) + \frac{2\lambda}{\bar{F}(y,\lambda)} \int_{y}^{\infty} x^{k-1}\phi(\lambda x)\phi(x)dx$$

Let

$$\alpha(y,\lambda) = 2\lambda \int_{y}^{\infty} x^{k-1}\emptyset(\lambda x)\phi(x)dx = \frac{\lambda}{\pi}\int_{y}^{\infty} x^{k-1}e^{-\frac{1}{2}(1+\lambda^{2})x^{2}}\, dx$$

Put $z = x^{2}$ we get

$$\alpha(y,\lambda) = \frac{\lambda}{2\pi} \int_{y^{2}}^{\infty} z^{\frac{k-2}{2}}\, e^{-\frac{1}{2}(1+\lambda^{2})z} dz$$

For $= 2r$, we have

$$\alpha_{1}(y,\lambda) = \frac{\lambda}{2\pi} \int_{y^{2}}^{\infty} z^{r-1}e^{-\frac{1}{2}(1+\lambda^{2})z}\, dz$$

Integrating by parts we get

$$\alpha_1(y,\lambda) = \left(\frac{k-2}{2}\right)!\,\mu\phi\left(y\sqrt{1+\lambda^2}\right)\sum_{i=0}^{\frac{k-2}{2}} \frac{2^i y^{k-2-2i}}{\left(\frac{k-2-2i}{2}\right)!\,(1+\lambda^2)^{i+\frac{1}{2}}}$$

Hence

$$E\left(X^k \middle| X > y\right) = y^{k-1}h(y,\lambda) + (k-1)E\left(X^{k-2}\middle|X > y\right) + \frac{1}{\bar{F}(y,\lambda)}\alpha_1(y,\lambda).$$

Now for $k = 2r - 1$, we have

$$\alpha_2(y,\lambda) = \frac{\lambda}{2\pi}\int_{y^2}^{\infty} z^{r-1-\frac{1}{2}} e^{-\frac{1}{2}(1+y^2)z}\,dz$$

Integrating by parts we get

$$\alpha_2(y,\lambda)$$

$$= \left(\frac{k-2}{2}\right)!\,\mu\phi\left(y\sqrt{1+\lambda^2}\right)\sum_{i=0}^{\frac{k-3}{2}} \frac{2^i y^{k-2-2i}}{\left(\frac{k-2-2i}{2}\right)!\,(1+\lambda^2)^{i+\frac{1}{2}}} + \mu\frac{k-2}{2}!\,\frac{\Phi\left(-y\sqrt{1+\lambda^2}\right)2^{\frac{k-1}{2}}}{\left(\sqrt{1+\lambda^2}\right)^{k-1}\left(\frac{-1}{2}\right)!}$$

Therefore,

$$E\left(X^k \middle| X > y\right) = y^{k-1}h(y,\lambda) + (k-1)E\left(X^{k-2}\middle|X > y\right) + \frac{1}{\bar{F}(y,\lambda)}\alpha_2(y,\lambda).$$

Hence the result..

Sufficiently:

First when $= 2r$,

we have

$$\frac{1}{\bar{F}(y,\lambda)}\int_{y}^{\infty} x^k\, f(x,\lambda)dx = y^{k-1}h(y,\lambda) + (k-1)E\left(X^{k-2}\middle|X > y\right) + \frac{1}{\bar{F}(y,\lambda)}\alpha_1(y,\lambda).$$

Or, after some maniposition

$$\int_y^\infty x^k f(x,\lambda)dx = y^{k-1} f(y,\lambda) + \int_y^\infty x^{k-2} f(x,\lambda)dx + \alpha_1(y,\lambda)$$

Differentiating both sides with respect to y, we obtain

$$-y^k f(y,\lambda) = (k-1)y^{k-2} f(y,\lambda) + y^{k-1}2\lambda\phi(\lambda y)\phi(y) + y^{k-1}2\Phi(\lambda y)\phi(y)(-y)$$

$$+\left(\frac{k-2}{2}\right)!\,\mu\phi\left(y\sqrt{1+\lambda^2}\right)\sum_{i=0}^{\frac{k-2}{2}}\frac{2^i\, y^{k-2-2i}(-y)(1+\lambda^2)}{\left(\frac{k-2-2i}{2}\right)!\,(1+\lambda^2)^{i+\frac{1}{2}}}$$

$$+\left(\frac{k-2}{2}\right)!\,\mu\phi\left(y\sqrt{1+\lambda^2}\right)\sum_{i=0}^{\frac{k-2}{2}}\frac{2^i\, y^{k-3-2i}(k-2-2i)}{\left(\frac{k-2-2i}{2}\right)!\,(1+\lambda^2)^{i+\frac{1}{2}}}$$

which reduces to

$$-y^k f(y,\lambda) = y^{k-1}2\Phi(\lambda y)\phi(y)(-y)$$

Second when $k = 2r-1$,

we have

$$\frac{1}{\bar{F}(y,\lambda)}\int_y^\infty x^k\; f(x,\lambda)\,dx = y^{k-1}h(y,\lambda) + (k-1)E\left(X^{k-2}\middle|X>y\right) + \frac{1}{\bar{F}(y,\lambda)}\alpha_2(y,\lambda)$$

Or,

$$\int_y^\infty x^k\; f(x,\lambda)\,dx = y^{k-1}h(y,\lambda) + \int_y^\infty x^{k-2}\; f(x,\lambda)\,dx + \alpha_2(y,\lambda).$$

Differentiating both sides with respect to y, we obtain

$$-y^k f(y,\lambda) = (k-1)y^{k-2} f(y,\lambda) + y^{k-1}2\lambda\phi(\lambda y)\phi(y) + y^{k-1}2\Phi(\lambda y)\phi(y)(-y)$$

$$+\left(\frac{k-2}{2}\right)!\,\mu\phi\left(y\sqrt{1+\lambda^2}\right)\sum_{i=0}^{\frac{k-2}{2}}\frac{2^i y^{k-2-2i}(-y)(1+\lambda^2)}{\left(\frac{k-2-2i}{2}\right)!\,(1+\lambda^2)^{i+\frac{1}{2}}}$$

$$+\left(\frac{k-2}{2}\right)!\,\mu\phi\left(y\sqrt{1+\lambda^2}\right)\sum_{i=0}^{\frac{k-2}{2}}\frac{2^i y^{k-3-2i}(k-2-2i)}{\left(\frac{k-2-2i}{2}\right)!\,(1+\lambda^2)^{i+\frac{1}{2}}}$$

$$+\mu\left(\frac{k-2}{2}\right)!\,\frac{\phi\left(y\sqrt{1+\lambda^2}\right)2^{\frac{k-1}{2}}\left(-\sqrt{1+\lambda^2}\right)}{\left(\sqrt{1+\lambda^2}\right)^{k-1}\left(\frac{-1}{2}\right)!}$$

Which reduces to

$$-y^k f(y,\lambda) = y^{k-1} 2\Phi(\lambda y)\phi(y)(-y).$$

Or, $f(y,\lambda) = 2\Phi(\lambda y)\phi(y)$.

Proof of theorem 2

Necessity

we have

$$E\left(X_{s:n}^k \middle| X_{r:n} = x\right) = \int_x^\infty y^k \frac{f_{r,s:n}(x,y,\lambda)}{f_{r:n}(x,\lambda)}\,dy$$

$$= \frac{C_{s:n}}{C_{r:n}}\left(\frac{\int_x^\infty y^k\, F(x,\lambda)^{r-1} f(x,\lambda)\left(F(y,\lambda)-F(x,\lambda)\right)^{s-r-1} f(y,\lambda)\left(1-F(y,\lambda)\right)^{n-s} dy}{F(x,\lambda)^{r-1} f(x,\lambda)\left(1-F(x,\lambda)\right)^{n-r}}\right)$$

$$= \frac{\frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}}{\frac{n!}{(r-1)!\,(n-r)!}}\left(\frac{\int_x^\infty y^k\left(F(y,\lambda)-F(x,\lambda)\right)^{s-r-1} f(y,\lambda)\left(1-F(y,\lambda)\right)^{n-s} dy}{\left(1-F(x,\lambda)\right)^{n-r}}\right)$$

$$= \frac{(n-r)!\int_x^\infty y^k\left(F(y,\lambda)-F(x,\lambda)\right)^{s-r-1} f(y,\lambda)\left(1-F(y,\lambda)\right)^{n-s} dy}{\left(1-F(x,\lambda)\right)^{n-r}} \qquad (3.4)$$

Integrating by parts we get

$$E\left(X_{s:n}^{k}\middle|X_{r:n}=x\right)=x^{k}+\left[\frac{(n-r)!}{\left(1-F(x,\lambda)\right)^{n-r}}\right]\times$$

$$\left[\sum_{j=r+1}^{s}k\int_{x}^{\infty}y^{k-1}\frac{\left[\left[\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^{2}}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t2}{2}}\,dt\,dx\right)^{(n-j+1)}\right]\beta_{1}(x,\lambda)\right]}{(j-r-1)!\,(n-j+1)!}dy\right].$$

$$E\left(X_{s:n}^{k}\middle|X_{r:n}=x\right)=x^{k}+\frac{(n-r)!}{\left(1-F(x,\lambda)\right)^{n-r}}\eta_{s}(x,\lambda),$$

where

$$\eta_{s}(x,\lambda)=\sum_{j=r+1}^{s}k\int_{x}^{\infty}y^{k-1}\frac{\left[\left[\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^{2}}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^{2}}{2}}\,dt\,dx\right)^{(n-j+1)}\right]\psi_{1}(x,y,\lambda)\right]}{(j-r-1)!\,(n-j+1)!}dy,$$

and

$$\psi_{1}(x,y,\lambda)=\left[\left(\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^{2}}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^{2}}{2}}\,dt\,dx-\int_{-\infty}^{x}\frac{1}{\pi}e^{-\frac{u^{2}}{2}}\int_{-\infty}^{\lambda u}e^{-\frac{t^{2}}{2}}\,dt\,du\right)^{(j-r-1)}\right].$$

which complete the proof.

Sufficiently:

Assume that $E\left(X_{s:n}^{k}\middle|X_{r:n}=x\right)=x^{k}+\frac{1}{\left(1-F(x,\lambda)\right)^{n-r}}\eta_{s}(x,\lambda)$.

$$\frac{(n-r)!}{(s-r-1)!\,(n-s)!}\left(\frac{\int_{x}^{\infty}y^{k}\left(F(y,\lambda)-F(x,\lambda)\right)^{s-r-1}f(y,\lambda)\left(1-F(y,\lambda)\right)^{n-s}dy}{\left(1-F(x,\lambda)\right)^{n-r}}\right)=$$

$$x^{k}+\frac{1}{\left(1-F(x,\lambda)\right)^{n-r}}\eta_{s}(x,\lambda)$$

Multiplying both sides by $\left(1-F(x,\lambda)\right)^{(n-r)}$, we obtain

$$\frac{(n-r)!}{(s-r-1)!(n-s)!}\left(\int_x^\infty y^k \left(F(y,\lambda)-F(x,\lambda)\right)^{s-r-1} f(y,\lambda)\left(1-F(y,\lambda)\right)^{n-s} dy\right)=$$

$$x^k\left(1-F(x,\lambda)\right)^{n-r}+\eta_s(x,\lambda).$$

Differentiating both sides with respect to x, we obtain

$$\frac{-(n-r)!}{(s-r-1)!(n-s)!}\left[x^k\left(F(x,\lambda)-F(x,\lambda)\right)^{s-r-1} f(x,\lambda)\left(1-F(x,\lambda)\right)^{n-s}\right]$$

$$-\frac{(n-r)!\,(s-r-1)}{(s-r-1)!\,(n-s)!}\left(\int_x^\infty y^k \left(F(y,\lambda)-F(x,\lambda)\right)^{s-r-2} f(x,\lambda) f(y,\lambda)\left(1-F(y,\lambda)\right)^{n-s} dy\right)$$

$$=kx^{k-1}\left(1-F(x,\lambda)\right)^{n-r}-(n-r)x^k\left(1-F(x,\lambda)\right)^{n-r-1} f(x,\lambda)$$

$$-(n-r)!\sum_{j=r+1}^{s} kx^{k-1}\frac{\left[\left[\left(1-\int_{-\infty}^{x}\frac{1}{\pi}e^{-\frac{u^2}{2}}\int_{-\infty}^{\lambda u}e^{-\frac{t^2}{2}}\,dt\,du\right)^{(n+j+1)}\right]\beta_1(x,\lambda)\right]}{(j-r-1)!\,(n-j+1)!}$$

$$-(n-r)!\sum_{j=r+2}^{s} k\int_x^\infty y^{k-1}\frac{\left[\left[\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx\right)^{(n-j+1)}\right]f(x,\lambda)\psi_2(x,y,\lambda)\right]}{(j-r-2)!\,(n-j+1)!}dy.$$

Where $\psi_2(x,y,\lambda)=\left[\left(\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx-\int_{-\infty}^{x}\frac{1}{\pi}e^{-\frac{u^2}{2}}\int_{-\infty}^{\lambda u}e^{-\frac{t^2}{2}}\,dt\,du\right)^{(j-r-2)}\right]$

$$-\frac{(n-r)!}{(8-r-2)!\,(n-s)!}\left(\int_x^\infty y^k \left(F(y,\lambda)-F(x,\lambda)\right)^{s-r-2} f(y,\lambda)\left(1-F(y,\lambda)\right)^{n-s} dy\right)$$

$$=-(n-r)x^k\left(1-F(x,\lambda)\right)^{n-r-1}$$

$$-(n-r)!\sum_{j=r+2}^{s} k\int_{x}^{\infty} y^{k-1}\frac{\left|\left[\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx\right)^{(n-j+1)}\right]\psi_2(x,y,\lambda)\right|}{(j-r-2)!\,(n-j+1)!}dy.$$

Now, let $I_4(x,\lambda)=\int_x^{\infty} y^k\left(F(y,\lambda)-F(x,\lambda)\right)^{s-r-2} f(y,\lambda)\left(1-F(y,\lambda)\right)^{ns} dy.$

Integrating by parts, gives

$$I_4(x,\lambda)=x^k(n-s)!\,(s-r-2)!\left(1-F(x,\lambda)\right)^{n-r-1}$$

$$+\int_{x}^{\infty} ky^{k-1}(n-s)!\,(s-r-2)!\left[\sum_{j=r+2}^{s}\frac{\left(F(y,\lambda)-F(x,\lambda)\right)^{j-r-2}\left(1-F(y,\lambda)\right)^{n-j+1}}{(j-r-2)!\,(n-j+1)1}\right]dy.$$

$$(n-r)!\,x^k\left(1-F(x,\lambda)\right)^{n-r-1}+$$

$$\int_{x}^{\infty} ky^{k-1}(n-r)!\left[\sum_{j=r+2}^{s}\frac{\left(F(y,\lambda)-F(x,\lambda)\right)^{j-r-2}\left(1-F(y,h)\right)^{n-j+1}}{(j-r-2)!\,(n-j+1)!}\right]dy$$

$$=(n-r)x^k\left(1-F(x,\lambda)\right)^{n-r-1}$$

$$+(n-r)!\sum_{j=r+2}^{s} k\int_{x}^{\infty} y^{k-1}\frac{\left|\left[\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx\right)^{(n-j+1)}\right]\psi_2(x,y,\lambda)\right|}{(j-r-2)!\,(n-j+1)!}dy.$$

Where $\psi_2(x,y,\lambda)=\left[\left(\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx-\int_{-\infty}^{x}\frac{1}{\pi}e^{-\frac{u^2}{2}}\int_{-\infty}^{\lambda u}e^{-\frac{t^2}{2}}\,dt\,du\right)^{(j-r-2)}\right],$

Or, $\int_x^{\infty} ky^{k-1}(n-r)!\left|\sum_{j=r+2}^{s}\frac{\left(F(y,\lambda)-F(x,\lambda)\right)^{j-r-2}\left(1-F(y,\lambda)\right)^{n-j+1}}{(j-r-2)!(n-j+1)!}\right|dy=$

$$(n-r)!\sum_{j=r+2}^{s} k\int_{x}^{\infty} y^{k-1}\frac{\left|\left[\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx\right)^{(n-j+1)}\right]\psi_2(x,y,\lambda)\right|}{(j-r-2)!\,(n-j+1)!}dy.$$

Where $\psi_2(x,y,\lambda) = \left[\left(\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx - \int_{-\infty}^{x}\frac{1}{\pi}e^{-\frac{u^2}{2}}\int_{-\infty}^{\lambda u}e^{-\frac{t^2}{2}}\,dt\,du\right)^{(j-r-2)}\right].$

Finally, this implies that

$$F(x,\lambda) = \int_{-\infty}^{x}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx.$$

Which is the cdf of the skew normal distribution with parameter λ. ■

Proof. Theorem 3

Necessity.

We have

$$E\left(X_{s:n,\bar{m},k}^{k}\middle|X_{r:n,\bar{m},k}=x\right) = \int_{x}^{\infty} y^k \frac{f_{r,s:n,\bar{m},k}(x,y,\lambda)}{f_{r:n,\bar{m},k}(x,\lambda)}\,dy$$

$$= \frac{d_{s-1}}{d_{r-1}}\frac{\int_x^\infty y^k\left[\left(1-F(x,\lambda)\right)^{m+1}-\left(1-F(y,\lambda)\right)^{m+1}\right]^{s-r-1} f(y,\lambda)\left(1-F(y,\lambda)\right)^{(m+1)(n-s)+k-1}dy}{(m+1)^{s-r-1}(s-r-1)!\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}}, \quad (3.5)$$

Integrating by parts we get

$$E\left(X_{s:n,\bar{m},k}^{k}\middle|X_{r:n,\bar{m},k}=x\right) = x^k$$

$$+\left(\frac{(m+1)(n-s-1)+k}{m+1}\right)!\left[\frac{(m+1)^{s-r}\prod_{j=r+1}^{s}[(m+1)(n-j)+k]}{\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}}\right]\times$$

$$\left[\sum_{j=r+1}^{s} k\int_{x}^{\infty} y^{k-1}\frac{\left[\left[\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx\right)^{(n-j)(m+1)+k}\right]\psi_m(x,y,\lambda)\right]}{(j-r-1)!\,(n-j+1)!}dy\right].$$

$$E\left(X_{s:n}^{k}\middle|X_{r:n}=x\right) =$$

$$x^k + \left(\frac{(m+1)(n-s-1)+k}{m+1}\right)!\,\frac{(m+1)^{s-r}\prod_{j=r+1}^{s}[(m+1)(n-j)+k]}{\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}}\gamma_s(x,\lambda),$$

where

$$\gamma_s(x,\lambda)=\sum_{j=r+1}^{s} k\int_{x}^{\infty} y^{k-1}\frac{\left[\left[\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx\right)^{(m+1)(n-j)+k}\right]\psi_1(x,y,\lambda)\right]}{\left(\frac{[(m+1)(n-j)+k]}{(m+1)}\right)!\,(j-r-1)!}$$

and

$$\psi_m(x,y,\lambda)=$$

$$\left[\left(1-\int_{-\infty}^{x}\frac{1}{\pi}e^{-\frac{u^2}{2}}\int_{-\infty}^{\lambda u}e^{-\frac{t^2}{2}}\,dt\,du\right)^{m+1}-\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx\right)^{m+1}\right]^{(j-r-1)}$$

which proves necessity.

Sufficiently:

Assume that

$$E\left(X_{s:n}^{k}\middle|X_{r:n}=x\right)=x^k+$$

$$\left(\frac{(m+1)(n-s-1)+k}{m+1}\right)!\,\frac{(m+1)^{s-r}\prod_{j=r+1}^{s}[(m+1)(n-j)+k]}{\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}}\gamma_s(x,\lambda).$$

$$\frac{d_{s-1}}{d_{r-1}}\frac{\int_x^\infty y^k\left[\left(1-F(x,\lambda)\right)^{m+1}-\left(1-F(y,\lambda)\right)^{m+1}\right]^{s-r-1} f(y,\lambda)\left(1-F(y,\lambda)\right)^{(m+1)(n-s)+k-1}dy}{(m+1)^{s-r-1}(s-r-1)!\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}}$$

$$=x^k+\left(\frac{(m+1)(n-s-1)+1}{m+1}\right)!\,\frac{(m+1)^{s-r}\prod_{j=r+1}^{s}[(m+1)(n-j)+k]}{\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}}\gamma_s(x,\lambda).$$

Multiplying both sides by $\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}$, we obtain

$$\frac{d_{s-1}}{d_{r-1}}\frac{\int_x^\infty y^k\left[\left(1-F(x,\lambda)\right)^{m+1}-\left(1-F(y,\lambda)\right)^{m+1}\right]^{s-r-1}f(y,\lambda)\left(1-F(y,\lambda)\right)^{(m+1)(n-s)+k-1}dy}{(m+1)^{s-r-1}(s-r-1)!}$$

$$=x^k\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}+$$

$$\left(\frac{(m+1)(n-s-1)+k}{m+1}\right)!\,(m+1)^{s-r}\prod_{j=r+1}^{s}[(m+1)(n-j)+k]\gamma_s(x,\lambda).$$

Now, differentiating both sides with respect to x, we get

$$\frac{d_{s-1}}{d_{r-1}}\frac{\int_x^\infty y^k\left[\left(1-F(x,\lambda)\right)^{m+1}-\left(1-F(y,\lambda)\right)^{m+1}\right]^{s-r-1}f(y,\lambda)\left(1-F(y,\lambda)\right)^{(m+1)(n-s)+k-1}dy}{(m+1)^{s-r-1}(s-r-1)!}$$

$$\times(m+1)\left(1-F(x,\lambda)\right)^{m}\left(-f(x,\lambda)\right)=kx^{k-1}\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}+$$

$$x^k[(m+1)(n-r-1)+k]\left(-f(x,\lambda)\right)\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}+$$

$$\sum_{j=r+1}^{s}k\int_x^\infty y^{k-1}\frac{\left[\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\,dx\right)^{(m+1)(n-j)+k}\right]}{\left(\frac{[(m+1)(n-j)+k]}{(m+1)}\right)!\,(j-r-1)!}\times$$

$$(j-r-1)(m+1)\left(\frac{-1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda x}e^{-\frac{t^2}{2}}\,dt\right)\left(1-\int_{-\infty}^{x}\frac{1}{\pi}e^{-\frac{u^2}{2}}\int_{-\infty}^{\lambda u}e^{-\frac{t^2}{2}}\,dt\,du\right)^{m}\times$$

$$\left[\left(1-\int_{-\infty}^{x}\frac{1}{\pi}e^{-\frac{u^2}{2}}\int_{-\infty}^{\lambda u}e^{-\frac{t^2}{2}}\,dt\,du\right)^{m+1}-\left(1-\int_{-\infty}^{y}\frac{1}{\pi}e^{-\frac{x^2}{2}}\int_{-\infty}^{\lambda u}e^{-\frac{t^2}{2}}\,dt\,du\right)^{m+1}\right]^{(j-r)}-$$

$$kx^{k-1}\left(1-F(x,\lambda)\right)^{(m+1)(n-r-1)+k}$$

Now let

$$I_5=\int_x^\infty y^k\left[\left(1-F(x,\lambda)\right)^{m+1}-1-F(y,\lambda)^{m+1}\right]^{s-r-1}f(y,\lambda)\left(1-F(y,\lambda)\right)^{(m+1)(n-s)+k-1}dy$$

Integrating by parts, we get

$$F(x,\lambda) = \int_{-\infty}^{x} \frac{1}{\pi} e^{-\frac{x^2}{2}} \int_{-\infty}^{\lambda x} e^{-\frac{t^2}{2}} \, dt \, dx.$$

Which is the cdf of the skew normal distribution with parameter λ.

This completes the proof theorem 3. ■

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