Military Technical College
Kobry Elkobbah,
Cairo, Egypt
May 25-27,2010

5th International Conference on
Mathematics and Engineering
Physics (ICMEP-5)

EM-29

Reliability of Connected $(1,1,2)$-or-$(1,2,1)$–or-$(2,1,1)$ out-of–$(n,2,2)$: F Lattice System

M. Gharib[a], E.M. El-Sayed[b], I. I. H. Nashwan[c]

Abstract

Few papers study the reliability of consecutive-(r_1,r_2,r_3)-out-of-(n_1,n_2,n_3): F system, since the computation of reliability in the 3- dimensional system is more complicated than the other systems; most of researchers study special cases.

In this paper, we study a special case of 3-dimentional systems, it is $(1,1,2)$, $(1,2,1)$ or $(2,1,1)$-out- of- $(n,2,2)$: F or connected 2-out-of-$(n,2,2)$: F system, we compute reliability of the system using three methods; the 1st method, is a direct computation of reliability, we determine the configurations of n parallel 2×2 of the system. Secondly, we find the transition probability matrix of Markov chain to compute system reliability, where the 3rd method, we uses "the recursive algorithm of a connected $(1,2)-\text{or}-(2,1)$-out-of-(m,n): F system to compute the reliability of the system" [1] for computing the reliability of the system.

Introduction

Due to the practical importance of consecutive-k-out-of-n : F systems, the study of this system, has caught the attention of researchers since the early 1980s [2]. This system consists of n components; they might be arranged linearly or circularly, it fails if and only if at least k consecutive components fail. Papastavridis & Lambiris [5-6] gave exact reliability formulas for linear & circular Consecutive-k-out-of-n: F Systems, and a recursive method to compute the reliability for consecutive-k-out-of-n: F system with Markov-dependent components.

Salvia and Lasher [4] extended a consecutive -k-out-of-n: F system to a 2-dimensional version, a consecutive (r, s)-out-of-(m, n): F system, for reliability analysis of a sensor system, an X-ray diagnostic system, a pattern search system.

Few researcher study the 3-dimensional consecutive k-out-of-n: F system; most of them study special cases of the 3-dimensional systems, Tomoaki Akiba, Hisashi Yamamoto and Yasutaka Kainuma studied the 3-dimentional adjacent triangle: F system triangular lattice system this system can be applied as the mathematical model of a scatter water area of a water sprinkler system etc., also Tomoaki Akiba and Hisashi Yammamoto study the lower and upper bound of the k-within- consecutive-(r_1,r_2,r_3)-out-of-(n_1,n_2,n_3):F system[3].

The Connected $(1,1,2)$ -or- $(1,2,1)$ -or- $(2,1,1)$ -out-of- $(n,2,2)$: F system

The Connected $(1,1,2)$ -or- $(1,2,1)$ -or- $(2,1,1)$ -out-of- $(n,2,2)$: F system is a special case of 3-dimensional system, it consists of $4n$ components distributed on the square or rectangle cuboid edges, the cuboid consist n levels, each level could be square or rectangle, 4 knots distributed on the edges of each level, the knots represent the system components which either fail or function, the whole system fails if there exist at least 2 failed connected components (i.e. neighbor knots), Figure 1.a and 1.b are showing some functioning and failed statuses of the system.

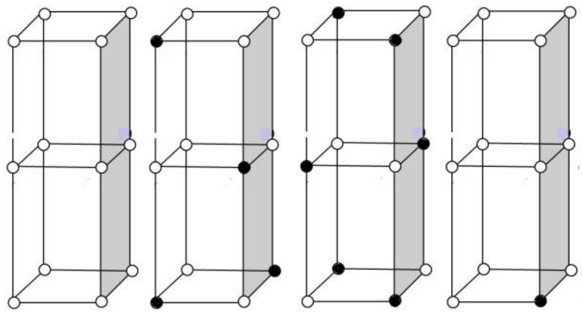


Figure 1.a: Some functioning cases of the system

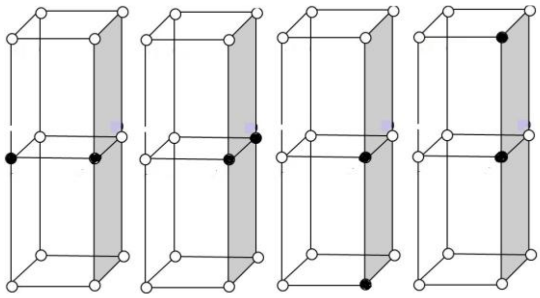


Figure 1.a: Some failed cases of the system

Keywords: consecutive k-out-of-n: F system, consecutive (r,s)-out-of-(m,n): F system, consecutive- (r_1,r_2,r_3) -out-of- (n_1,n_2,n_3) : F system, Markov chain.

Notations

$R(n)$	: Reliability of consecutive $(1,1,2)$, $(1,2,1)$ or $(2,1,1)$ - out- of- $(n,2,2)$: F system
$(0,1)$	: Functioning and failed component represented respectively
$L(n,j)$	: Number of configuration of n parallel 2×2 matrix ($4n$ components) having j total of failures and no two contiguous components fails.
$[x]$	The greatest integer lower bound of x
p,q	: Components Reliability, unreliability respectively
X_k	: Random variable count the number of failed component at the k^th matrix 2×2
$[P_{ij}]$	: Markov matrix including Markov probabilities
$Z_{i,j}$	: Random variable describe the component status at i row and j column, equal 1 if the component fails and 0 otherwise.
χ_u	: Indicator function
$A_{i,j}^c$	: Event of " system will function by the i row and j column component"
$F_j(s)$	: Probability of the status vector $\mathbf{x}$ of the column j have s failed components $P\left\{\bigcap_{u=1}^{5} Z_{u,j}=\chi_u\right\} \quad j=1,2,...,n$ where s number of failed component in the column j.

$R(j,\mathbf{x})$: the reliability connected –(1,2)-or-(2,1) out of $(4,j)$: F lattice system, for $j=1,2,\ldots,n$ given the vector x.

$R(j,X=s)$: is the reliability connected – (1,2)-or-(2,1) out of (n,j): F lattice system given there is s failed components in the column j.

Θ : $\{\mathbf{y}=(y_1,y_2,y_3,y_4): y_i y_{i-1}=0 \wedge y_1 y_4 = 0 : i=1,2,3,4\}$

$\Omega(\mathbf{y})$: $\left\{(x_1,x_2,\ldots,x_n)\in\Theta \mid \left\{\bigcap_{u=1}^{5} A_{u,j}\right\} \cap \left\{\bigcap_{u=1}^{5} Z_{u,j}=\chi_u\right\} \cap \left\{\bigcap_{u=1}^{5} Z_{u,j-1}=\chi_u\right\} \neq \phi\right\}$

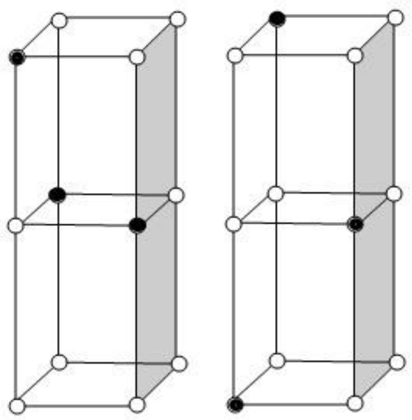


Figure 2.a

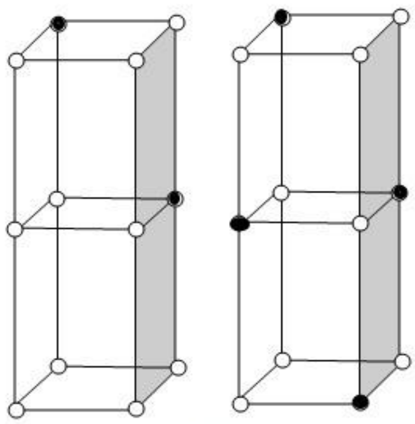


Figure 2.b

1. **Direct Computation of System Reliability**

Assume that the system functioning, since the system consist of n of 2×2 level; we have 2 failed component not connected as a maximum number of failed component at each 2×2 level, we denote the system by a number consist of n digit each digit will be one of $\{0,1,2\}$ i.e. number of failed component, for example, the states in fig 2.a can be denoted by 120,111, and the states in figure 2.b by 110, 121.

Theorem: Reliability of connected $(1,1,2)$, $(1,2,1)$ or $(2,1,1)$ - out- of- $(n,2,2)$: F system is

$$R(n)=\sum_{j=0}^{2n} L(n,j)\, p^{4n-j} q^{j}$$

Where $L(n,j)$ is the number of non-zero configuration of $4n$ components having j total of failed and no two connected failures. The integer of $L(n,j)$ for $n\leq 6$ are given in table 1.

Table 1: $L(n,j)$ for consecutive $(1,1,2),(1,2,1),(1,1,2)$ -out-of- $(n,2,2)$: F system

n \ j	1	2	3	4	5	6
0	1	1	1	1	1	1
1	4	8	12	16	20	24
2	2	16	46	92	154	232
3		8	68	240	588	1176
4		2	40	302	1208	3430
5			12	192	1384	6000
6			2	72	926	6472

7				16	396	4480
8				2	112	2110
9					20	696
10					2	160
11						24
12						2

Example: consider the case of $n=2$; all possibilities operating state are listed in table 2

Number of configuration	00	10	01	20	02	11	12	21	22
Probability	p^8	$4p^7q$	$4p^7q$	$2p^6q^2$	$2p^6q^2$	$12p^6q^2$	$4p^5q^3$	$4p^5q^3$	$2p^4q^4$

$R(2)=p^8+8p^7q+16p^6q^2+8p^5q^3+2p^4q^4$

2. Computing Reliability using Markov chain

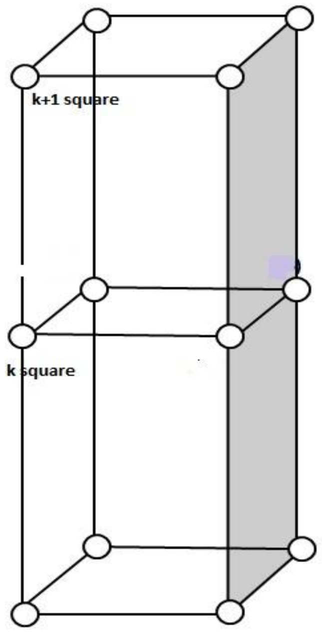
Assume that the system function, and define the variable X_k counting the number of failed components at the level k in, the maximum number of failed components which allows the system to be functioning is 2.

X_k	0	1	2
Number of Configurations	1	4	2

Assume the system functions when we have k of (2×2) level, add the new level $k+1$, since the system failed if 2 connected component vertically, we must take in consideration that the failed components in the level k, and whom in the level k+1 must not connected, hence the configuration of failed components in the level $k+1$ depend only on location of the failed components on the k level, which meet the postulate of Markov chain (does not depend on the history of the $X_{k-1}, X_{k-2}, \dots X_1$ failures). Now, represent the knots of level k by a 2×2 matrix, take the level k for example of the form $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ the following configurations for the level k+1 will be:

$k+1$ level $\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$ $\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$

k level $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$



$$P\left(X_{k+1}=x_{k+1} \mid X_k = x_k\right)=$$
$$P\left(X_{k+1}=x_{k+1} \mid X_k = x_k, X_{k-1}=x_{k-1},\ldots.., X_1 = x_1\right)$$
$$x_i = 0,1,2 \ \forall 1\le i\le k\le n$$
$$P=\begin{pmatrix} p^4 & 4p^3q & 2p^2q^2 \\ p^4 & 3p^3q & p^2q^2 \\ p^4 & 2p^3q & p^2q^2 \end{pmatrix}$$

See appendix for proof.

The $R(n)=\sum_{j=0}^{2} P_{0j}^{n} = \sum_{j=0}^{2} P^{n}\left(X_{k+1}=j \mid X_k = 0\right)$

Example: The reliability of connected ( ) , ( ) or ( ) -out-of-( ) : F system

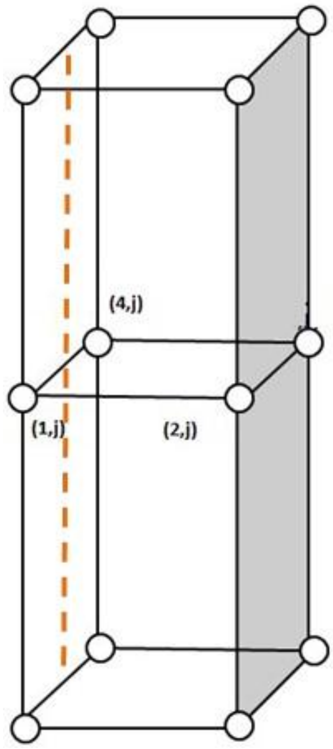
$$P^2=\begin{pmatrix} p^8+4p^7q+2p^6q^2 & 4p^7q+12p^6q^2+4p^5q^3 & 2p^6q^2+4p^5q^3+2p^4q^4 \\ p^8+3p^7q+p^6q^2 & 4p^7q+9p^6q^2+2p^5q^3 & 2p^6q^2+3p^5q^3+p^4q^4 \\ p^8+2p^7q+p^6q^2 & 4p^7q+6p^6q^2+2p^5q^3 & 2p^6q^2+2p^5q^3+p^4q^4 \end{pmatrix}$$

$$R(2)=\sum_{j=0}^{2} P_{0j}^{2} = p^8+8p^7q+16p^6q^2+8p^5q^3+2p^4q^4$$

3. Using Recursive Algorithm for the Reliability of Connected $(1,2)-\text{or}-(2,1)$ -out- of-(m,n): F Lattice system to find the System Reliability

In this section we transform the system to be in an applicable form that helps us to apply the recursive algorithm of a connected $(1,2)-\text{or}-(2,1)$-out-of-(m,n): F system for computing the reliability of the connected $(1,1,2)$, $(1,2,1)$ or $(2,1,1)$ - out- of- $(n,2,2)$: F system [1].

Make a vertical cutting of the connected $(1,1,2)$, $(1,2,1)$ or $(2,1,1)$ - out- of- $(n,2,2)$: F system between the 4th knot and the 1st one and rotate the graph 90 degree with clockwise direction as shown at figure 2, the shape will be of two dimension as a linear connected $(1,2)-or-(2,1)$ out of (m,n): F lattice system, taking in consideration the resulted fail from the 1st and 4th see figure (2)



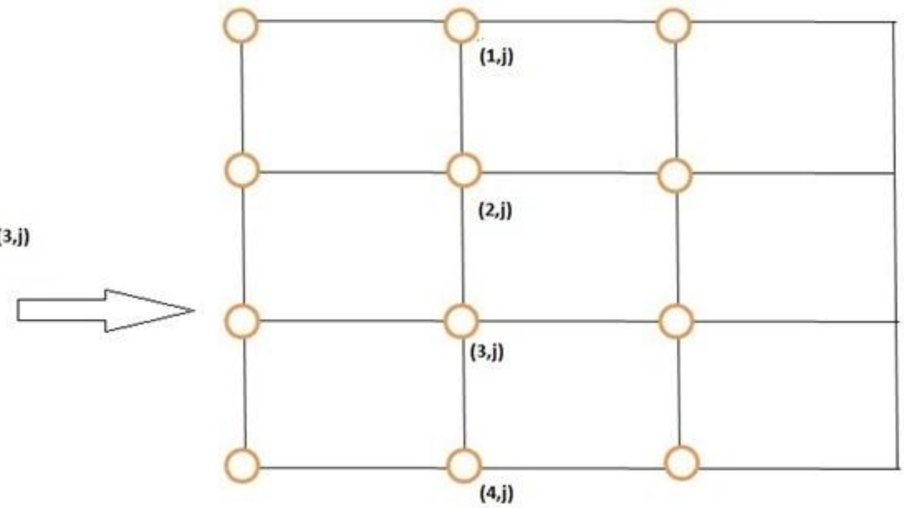


Figure (2)

Assume the system is functioning, and let (i,j) denote the location of the component in the connected $(1,1,2)$, $(1,2,1)$ or $(2,1,1)$ - out- of- $(n,2,2)$: F system, where $i=1,2,3,4$, define the location variable $Z_{i,j}$ denote the status of the component in the column j and row i in the new figure of the system, such that:

$$Z_{i,j}=\begin{cases}1 & \text{component fail}\\ 0 & \text{componant function}\end{cases} \qquad i=1,2,3,4,5 \quad j=1,2,..,n \text{ where } Z_{1,j}=Z_{5,j}$$

The failure event resulted by the component (i,j) is $A_{i,j}=\{Z_{i,j}=1\}\cap[\{Z_{i,j-1}=1\}\cup\{Z_{i-1,j}=1\}]$, where $A_{5,v}=A_{1,v}$, hence the reliability for the component (i,j) will be $A^c_{i,j}$, which implies that the reliability of the system is $R(n)=P\left\{\bigcap_{u=1}^{5}\bigcap_{v=1}^{n}A^c_{u,v}\right\}$.

For any column j, define the set Θ consist of 4-dimensional vector consist of 0,1 (functioning and failed status respectively) i.e. $\Theta=\{\mathbf{y}=(y_1,y_2,y_3,y_4):y_iy_{i-1}=0\wedge y_1y_4=0:i=1,2,3,4\}$, according [1] the $R(j)$ is the reliability connected $(1,2)-or-(2,1)$-out-of-$(4,j)$: F lattice system, for $j=1,2,...,n$,and $R(j,\mathbf{y})$ is the reliability connected –(1,2)-or-(2,1) out of $(4,j)$: F lattice system, for $j=1,2,...,n$ with given components state $\mathbf{y}$ on column j, and $R(j)=\sum_{\mathbf{y}\in\Theta}R(j,\mathbf{y})$, where

$$R(j,\mathbf{y}) = \begin{Bmatrix} F_j(\mathbf{y}) \sum_{\mathbf{x}\in\Omega(\mathbf{y})} R(j-1,\mathbf{x}) & j \ge 2 \\ F_j(\mathbf{y}) & j = 1 \end{Bmatrix}, \quad F_j(\mathbf{y}) = P\left\{ \bigcap_{u=1}^{5} Z_{u,j} = \chi_u \right\} \quad j = 1,2,\dots,n$$

and

$$\Omega(\mathbf{y}) = \left\{ (x_1, x_2, \dots, x_n) \in \Theta \mid \left\{ \bigcap_{u=1}^{5} A_{u,j} \right\} \cap \left\{ \bigcap_{u=1}^{5} Z_{u,j} = \chi_u \right\} \cap \left\{ \bigcap_{u=1}^{5} Z_{u,j-1} = \chi_u \right\} \neq \phi \right\}.$$

Since the system is $i.i.d$, for any column j, define a random variable X counts the number of failed components, using the recursive algorithm [1]:

$$\Theta = \Omega(0000) = \{(0101)(1010)(1000)(0100)(0010)(0001)(0000)\} \Rightarrow$$

$$\Omega(1000) = \{(0101)(0100)(0010)(0001)(0000)\} \qquad \Omega(0100) = \{(1010)(1000)(0010)(0001)(0000)\}$$

$$\Omega(0010) = \{(0101)(1000)(0100)(0001)(0000)\} \qquad \Omega(0001) = \{(1010)(1000)(0100)(0010)(0000)\}$$

$$\Omega(1010) = \{(0100)(0100)(0001)(0000)\} \qquad \Omega(0101) = \{(1010)(0100)(0010)(0000)\}$$

Note that

1. $R(1,(0000)) = F_1(0000) = F_1(0) = p^4$
2. $R(1,(0100)) = R(1,(0010)) = R(1,(0001)) = F_1(0100) = F_1(1) = p^3 q$
3. $R(1,(1010)) = R(1,(0101)) = F_1(1010) = F_1(2) = p^2 q^2$

Example: Compute the reliability of connected $(1,1,2)$, $(1,2,1)$ or $(2,1,1)$-out-of-$(2,2,2)$: F system:

The reliability of connected $(1,1,2)$, $(1,2,1)$ or $(2,1,1)$-out-of-$(2,2,2)$: F system is the reliability of $(1,2)$ or $(2,1)$ –out-of- $(2,n)$: F circular system, then apply the recursive algorithm

$$\left.\begin{aligned} &R(1, X=0) = F_1(0000) = p^4 \\ &R(1, X=1) = \left[R(1,(0100)) + R(1,(0010)) + R(1,(0001)) \right] = 3p^3 q \\ &R(1, X=2) = \left[R(1,(1010)) + R(1,(0101)) \right] = 2F_1(2) = 2p^2 q^2 \end{aligned}\right\} \Rightarrow R(1) = p^4 + 4p^3 q + 2p^2 q^2$$

$$\left.\begin{aligned} &R(2, X=0) = F_2(0) \sum_{\mathbf{x}\in\Omega(0000)} R(1,\mathbf{x}) = p^4\left[p^4 + 4p^3 q + 2p^2 q^2 \right] \\ &R(2, X=1) = 4F_2(1) \sum_{\mathbf{x}\in\Omega(0001)} R(1,\mathbf{x}) = 4p^3 q\left[p^4 + 3p^3 q + p^2 q^2 \right] \\ &R(2, X=2) = 2F_2(1) \sum_{\mathbf{x}\in\Omega(0101)} R(1,\mathbf{x}) = 2p^2 q^2\left[p^4 + 2p^3 q + p^2 q^2 \right] \end{aligned}\right\} \Rightarrow \begin{aligned} R(2) = {} & p^8 + 8p^7 q + 16p^6 q^2 \\ & + 8p^5 q^3 + 2p^4 q^4 \end{aligned}$$

Appendix

$$R(n)=\sum_{j=0}^{2n} L(n,j)\,p^{4n-j}q^{j}$$

Theorem 1:

Let n be given positive integer, and suppose we have a system whose components are ordered like the elements of $(n,2,2)$-cuboids, the components are either in an operating status or in failed status. The system which fails if and only if two connected failed components occurs. That is, associated with any fixed n is an $1\times 2n$ array L, where $L=L(n,j);0\le j\le 2n$, it's clear that

$$L(n,j)=\begin{cases} 1 & j=0 \\ 4n & j=1 \\ 2 & j=2n \end{cases}$$

If $j=2$ then $L(n,2)=L(n-1,2)+2L(n-2,2)+4L(n-1,1)+4\times 3L(n-3,0)$ where the 1st term is the number of all states which end in 0, the 2nd term is the number of all states which end in 02, the 3rd term is the number of all states which end in 01, and the last term is the number of all states end in 011.

$$L(n,3)=L(n-1,3)+2L(n-2,1)+4L(n-2,2)+4\times 3L(n-3,1)+4\times 3^{2}L(n-4,1)+4\times 2L(n-3,0)$$

$$=L(n-1,3)+2L(n-2,1)+4\times\sum_{i=1}^{2}3^{i}L(n-i-2,\,j-i-1)+4\times 2L(n-3,0)$$

Where the 1st term is the number of all states which end in 0, the 2nd term is the number of all states which end in 02, the 3rd term is the number of all states which end in 01, 011,011, and the last term is the number of all states end in 012,021.In general

$$L(n,j)=L(n-1,j)+2\sum_{i=1}^{\left[\frac{j}{2}\right]}L(n-i-1,\,j-2i)+4\sum_{i=0}^{j-1}3^{i}L(n-i-2,\,j-i-1)$$

$$+4\times\sum_{i=1}^{j-3}\left[\binom{2}{1}3^{i}+2\binom{2}{2}3^{i-1}\right]L(n-i-3,\,j-i-3)+\ldots+$$

$$+4\sum_{i=1}^{j-2s+1}\left[3^{i}\binom{s}{1}+2\times 3^{i-1}\binom{s}{2}+\ldots+2^{s-1}\times 3^{i-s+1}\binom{s}{s}\binom{i}{s-1}\right]L(n-i-s-1,\,j-i-2s+1)$$

Where the 1st term is the number of all states which end in 0, the 2nd term is the number of all states which end in 02, 022,0222,…, 222…22, the 3rd term is the number of all states which end in 01, 011,011,…., 111…11, and the last term is the number of all states end in 012,021, 0112, 0211, …, 2111…1, the last term is the sum of all states which end in 012…2, 0212…2, ….,222…21, and hence

$$L(n,j)=L(n-1,j)+2\sum_{i=1}^{\left[\frac{j}{2}\right]}L(n-i-1,\,j-2i)$$

$$+4\sum_{l=1}^{\left[\frac{j+1}{2}\right]}\sum_{i=0}^{j-2l+1}\sum_{k=1}^{l}3^{i-k+1}\times 2^{k-1}\binom{l}{k}\binom{i}{k-1}L(n-i-l-1,\,j-i-2l+1)$$

Where $L(-1,0)=1,\ \ L(0,0)=1,\ \ L(-i,0)=0,\ \ i\succ 1,\ \ L(n,j)=0 \text{ if } j\succ 2n$

Markov Chain

$$P_{ij} = P(X_{k+1} = j \mid X_k = i) = \frac{P(X_{k+1} = j, X_k = i)}{P(X_k = i)}$$

$$P_{00} = \frac{P(X_{k+1} = 0, X_k = 0)}{P(X_k = 0)} = \frac{p^8}{p^4} = p^4$$

$$P_{01} = \frac{P(X_{k+1} = 1, X_k = 0)}{P(X_k = 0)} = \frac{4p^7 q}{p^4} = 4p^3 q$$

$$P_{02} = \frac{P(X_{k+1} = 2, X_k = 0)}{P(X_k = 0)} = \frac{2p^6 q^2}{p^4} = 2p^2 q^2$$

$$P_{10} = \frac{P(X_{k+1} = 0, X_k = 1)}{P(X_k = 1)} = \frac{4p^7 q}{4p^3 q} = p^4$$

$$P_{11} = \frac{P(X_{k+1} = 1, X_k = 1)}{P(X_k = 1)} = \frac{12p^6 q^2}{4p^3 q} = 3p^3 q$$

$$P_{12} = \frac{P(X_{k+1} = 2, X_k = 1)}{P(X_k = 1)} = \frac{4p^5 q^3}{4p^3 q} = p^2 q^2$$

$$P_{20} = \frac{P(X_{k+1} = 0, X_k = 2)}{P(X_k = 2)} = \frac{2p^6 q^2}{2p^2 q^2} = p^4$$

$$P_{21} = \frac{P(X_{k+1} = 1, X_k = 2)}{P(X_k = 2)} = \frac{4p^5 q^3}{2p^2 q^2} = 2p^3 q$$

$$P_{22} = \frac{P(X_{k+1} = 2, X_k = 2)}{P(X_k = 2)} = \frac{2p^4 q^4}{2p^2 q^2} = p^2 q^2$$

References:

[1] Hisashi Yamamoto, Tomoaki Akiba, Hiddeki Nagatsuka, Yurie Moriyama, Recursive algorithm for the reliability of connected $(1,2)-or-(2,1)$ out of (m,n): F lattice system, European Journal of Operational Research 188 (2008) 854-864.

[2] Thomas Cluzeau, Jörg Keller, Winfrid Schneeweiss, “an Efficient Algorithm for Computing the Reliability of Consecutive-k-Out-Of-n: F System”, IEEE TRANSACTIONS ON RELIABILITY, VOL. 57, NO. 1, MARCH 2008, pp 84-87.

[3] Tomoaki Akiba, Hisashi Yamamoto, and Yasutaka Kainuma, “Reliability of a 3-dimensional adjacent triangle: F triangular Lattice system”, Proceedings of the Fifth Asia Pacific Industrial Engineering and Management Systems Conference, 2004.

[4] Salvia, A. A. and W.C. Lasher “2-dimensional consecutive k-out-of-n: F system models” & IEEE, Trans. Reliability, Vol 39- 382-385, 1992. (1990).

[5] Menelaos Lambiris, Stavros Papastavridis, “Exact Reliability Formulas for Linear & Circular Consecutive-k-out-of-n: F Systems”, IEEE TRANSACTIONS ON RELIABILITY, VOL. R-34, NO. 2, 1985 JUNE, pp124-126.

[6] Papastavridis, M. Lambiris, “Reliability of a consecutive-k-out-of-n: F system for Markov-dependent components”, IEEE TRANSACTIONS ON RELIABILITY, VOL. R-36, 1987 Apr, pp 78-79.