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Exotic localized structures based on the symmetrical Lucas function of the (2 + 1)-dimensional modified dispersive water-wave system

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Abstract.

In this paper, with the help of the Lucas Riccati method and a linear variable separation method, new variable separation solutions with arbitrary functions are derived for a (2+1)-dimensional modified dispersive water-wave system. Next, we give a positive answer for the following question: Are there any localized excitations derived by the use of another functions? For this purpose, some attention will be paid to dromion, peakon, dromion lattice, multi dromion-solitoff excitations, regular fractal dromions, lumps with self-similar structures and chaotic dromions patterns based on the golden main and the symmetrical hyperbolic and triangular Lucas functions.

Keywords: Lucas functions, localized excitations, variable separation solutions, modified dispersive water-wave system.

1. Introduction

Modern soliton theory is widely applied in many natural sciences such as chemistry, biology, mathematics, communication and particularly in almost all branches of physics such as fluid dynamics, plasma physics, field theory, nonlinear optics and condensed matter physics, etc. As one of the effective methods in linear physics, the variable separation approach (VSA) has been successfully extended to nonlinear models. The so-called multilinear VSA (MLVSA) has also been established for various (2+1)-dimensional models[1)] since it was applied to the Davey–Stewartsen (DS) equation in 1996[2)]. Recently, along with the variable separation idea, Zheng *et al* realized variable separation for the Broer–Kaup–Kupershmidt (BKK) system[3)] and the dispersive long wave (DLW) equation[4)] by the extended tanh-function method (ETM) based on the mapping method. In ref. 5, they extended the ETM to a selection of (2+1)-dimensional nonlinear equations, including the (2+1)-dimensional generalized Nizhnik–Novikov–Veselov (GNNV) system, the (2+1)-dimensional Burgers equation, the (2+1)-dimensional breaking soliton model, and the (2+1)- dimensional integrable Kortweg-de-Vries (KdV) equation, and so on[6-36)].

There is well-known fact that two mathematical constants of Nature, the π- and e-numbers, play a great role in mathematics and physics. Their importance consists in the fact that they ‘‘generate’’ the main classes of so-called ‘‘elementary functions’’: sin, cosine (the π-number), exponential, logarithmic and hyperbolic functions (the e-number). However, there is the one more mathematical constant playing a great role in modeling of processes in living nature termed the Golden Section, Golden Proportion, Golden Ratio, Golden Mean[37-41]. However, we should certify that a role of this mathematical constant is sometimes undeservedly humiliated in modern mathematics and mathematical education. There is the well-known fact that the basic symbols of esoteric (pentagram, pentagonal star, platonic solids etc.) are connected to the Golden Section closely. Moreover, the ‘‘materialistic’’ science together with it's ‘‘materialistic’’ education had decided to ‘‘throw out’’ the Golden Section. However, in modern science, an attitude towards the Golden Section and connected to its Fibonacci and Lucas numbers is changing very quickly. The outstanding discoveries of modern science based on the Golden Section have a revolutionary importance for development of modern science. These are enough convincing confirmation of the fact that human science approaches to uncovering one of the most complicated scientific notions, namely, the notion of Harmony, which is based on the Golden Section, Harmony was opposed to Chaos and meant the organization of the Universe. In Euclid’s The Elements we find a geometric problem called ‘‘the problem of division of a line segment in the extreme and middle ratio’’. Often this problem is called the golden section problem[38-41]. Solution of the golden section problem reduces to the following algebraic equation $x^2 = x+1$ this equation has two roots. We call the positive root, $\alpha = \dfrac{1+\sqrt{5}}{2}$, the golden proportion, golden mean, or golden ratio. El Naschie’s works[39-43]develop the Golden Mean applications into modern physics. In ref. 43, devoted to the role of the Golden Mean in quantum physics El Naschie concludes the following: "In our opinion it is very worthwhile enterprise to follow the idea of Cantorian space-time with all its mathematical and physical ramifications. The final version may well be a synthesis between the results of quantum topology, quantum geometry and may be also Rossler’s endorphysics, which like Nottale’s latest work makes extensive use of the ideas of Nelson’s stochastic mechanism". Thus, in the Shechtman’s, Butusov’s, Mauldin and Williams’, El- Naschie’s, Vladimirov’s works, the Golden Section occupied a firm place in modern physics and it is impossible to imagine the future progress in physical researches without the Golden Section.

In our present paper, we review symmetrical Lucas functions[33-35] and we find new solutions of the Riccati equation by using these functions. Also, we devise an algorithm called Lucas Riccati method to obtain new exact solutions of NLPDEs.

For a given NPLDE with independent variables $x=(x_0=t, x_1, x_2, x_3, ..., x_n)$ and dependent variable u,

$$P(u, u_t, u_{x_i}, u_{x_i x_j}, ...)=0, \tag{1}$$

where P is in general a polynomial function of its argument, and the subscripts denote the partial derivatives, in order to derive some new solutions with certain arbitrary functions, we assume that its solutions in the form,

$$u(x)=\sum_{i=0}^{n} a_i(x)F^i(x), \tag{2}$$

with

$$F' = A + BF^2 \tag{3}$$

where $x=(x_0=t, x_1, x_2, x_3, ..., x_n)$ and A, B are constants and the prime denotes differentiation with respect to ξ. To determine u explicitly, one may take the following steps: First, similar to the usual mapping approach, determine n by balancing the highest non-linear terms and the highest-order partial terms in the given NLPDE. Second, substituting (2) and (3) into the given NLPDE and collecting coefficients of polynomials of F, then eliminating each coefficient to derive a set of partial differential equations of

a_i $(i=0,1,2,.....,n)$ and ξ. Third, solving the system of partial differential equations to obtain a_i and ξ. Substituting these results into (1), then a general formula of solutions of equation (1) can be obtained. Choose properly A and B in ODE (3) such that the corresponding solution $F(\xi)$ is one of the symmetrical Lucas function given bellow. Some definitions and properties of the symmetrical Lucas function are given in appendix A.

Case 1: If $A=\ln\alpha$ and $B=-\ln\alpha$, then (3) possesses solutions

$$\mathrm{tLs}(\xi), \qquad \mathrm{cotLs}(\xi).$$

Case 2: If $A=\frac{\ln\alpha}{2}$ and $B=-\frac{\ln\alpha}{2}$, then (3) possesses a solution

$$\frac{\mathrm{tLs}(\xi)}{1\pm\ \mathrm{secLs}(\xi)}.$$

Case 3: If $A=\ln\alpha$ and $B=-4\ln\alpha$, then (3) possesses a solution

$$\frac{\mathrm{tLs}(\xi)}{1\pm\ \mathrm{tLs}^2(\xi)}.$$

In §2, we apply the Lucas Riccati method to obtain new localized excitations. Also in §3, we pay our attention to dromion, peakon, dromion lattice, multi dromion-solitoff excitations, regular fractal dromions, lumps with self-similar structures and chaotic dromions patterns based on the golden main and the symmetrical hyperbolic and triangular Lucas functions.

2. New variable separation solutions of the (2 + 1)-dimensional modified dispersive water-wave system

We consider here the (2+1)-dimensional modified dispersive water-wave (MDWW) system

$$\begin{aligned} &u_{yt}+u_{xxy}-2v_{xx}+(u^2)_{xy}=0,\\ &v_t-v_{xx}-2\,(vu)_x=0. \end{aligned} \tag{4}$$

The (2+1)-dimensional MDWW system was used to model nonlinear and dispersive long gravity waves travelling in two horizontal directions on shallow waters of uniform depth, and can also be derived from the well-known Kadomtsev-Petviashvili (KP) equation using the symmetry constraint[44,45)]. Abundant propagating localized excitations were derived by Tang et al[9)] with the help of Painlevé-Bäcklund transformation and a multilinear variable separation approach. It is worth mentioning that this system has been widely applied in many branches of physics, such as plasma physics, fluid dynamics, nonlinear optics, etc. So, a good understanding of more solutions of the (2+1)-dimensional MDWW system (4) is very helpful, especially for coastal and civil engineers in applying the nonlinear water model in harbor and coastal design. Meanwhile, finding more types of solutions to system (4) is of fundamental interest in fluid dynamics.

Now we apply the Lucas Riccati method to equations (4). First, let us make a transformation of the system (4): $u=v_y$. Substituting this transformation into system (4), yields

$$u_{ty}-u_{xxy}-(u^2)_{xy}=0. \tag{5}$$

Balancing the highest order derivative term with the nonlinear term in equation (5), gives n = 1, we have the ansätz

$$u(x,y,t)=a_0(x,y,t)+a_1(x,y,t)F(\varphi(x,y,t)), \tag{6}$$

where $a_0(x,y,t)\equiv a_0$, $a_1(x,y,t)\equiv a_1$ and $\varphi(x,y,t)\equiv\varphi$ are arbitrary functions of x, y, t to be determined. Substituting (6) with (3) into (5), and equating each of the coefficients of $F(\varphi)$

to zero, we obtain system of PDEs. Solving this system of PDEs, with the help of Maple, we obtain the following solution:

$$a_0(x,y,t) = -\frac{\varphi_{xx}(x,y,t)-\varphi_t(x,y,t)}{2\varphi_x(x,y,t)} \tag{7}$$

$$a_1(x,y,t) = -B\varphi_x(x,y,t) \tag{8}$$

$$\varphi(x,y,t) = f(x,t)+g(y), \tag{9}$$

where $f(x,t)\equiv f$ and $g(y)=g$ are two arbitrary functions of x, t and y, respectively.

Now. based on the solutions of (3), one can obtain new types of localized excitations of the (2+1)-dimensional MDWW system. We obtain the general formulae of the solutions of the (2+1)-dimensional MDWW system

$$u = -\frac{f_{xx}-f_t}{2f_x} - Bf_x F(f+g), \tag{10}$$

$$v = -ABf_x g_y - B^2 f_x g_y F^2(f+g). \tag{11}$$

By selecting the special values of the A, B and the corresponding function F we have the following solutions of (2+1)-dimensional MDWW system:

$$u_1 = -\frac{f_{xx}-f_t}{2f_x} + f_x \ \mathrm{tLs}(f+g)\ \ln\alpha, \tag{12}$$

$$v_1 = f_x g_y \ln\alpha^2 - f_x g_y \ \mathrm{tLs}^2(f+g)\ln\alpha^2, \tag{13}$$

$$u_2 = -\frac{f_{xx}-f_t}{2f_x} + f_x \ \mathrm{cotLs}(f+g)\ \ln\alpha, \tag{14}$$

$$v_2 = f_x g_y \ln\alpha^2 - f_x g_y \ \mathrm{cotLs}^2(f+g)\ln\alpha^2, \tag{15}$$

$$u_3 = -\frac{f_{xx}-f_t}{2f_x} + \frac{f_x \ \mathrm{tLs}(f+g)\ \ln\alpha}{2[1\pm\mathrm{secLs}(f+g)]}, \tag{16}$$

$$v_3 = \frac{f_x g_y \ln\alpha^2}{4} - \frac{f_x g_y \ \ln\alpha^2}{4}\left[\frac{\mathrm{tLs}(f+g)}{1\pm\mathrm{secLs}(f+g)}\right]^2 \tag{17}$$

$$u_4 = -\frac{f_{xx}-f_t}{2f_x} + \frac{4f_x \ \mathrm{tLs}(f+g)\ \ln\alpha}{1+\mathrm{tLs}^2(f+g)}, \tag{18}$$

$$v_4 = 4f_x g_y \ln\alpha^2 - 16 f_x g_y \ \ln\alpha^2\left[\frac{\mathrm{tLs}(f+g)}{1+\mathrm{tLs}^2(f+g)}\right]^2 \tag{19}$$

where $f(x,t)$ and $g(y)$ are two arbitrary variable separation functions. Especially, for the potential $U = u_{1y}$ has the following form

$$U = 4f_x g_y \ \mathrm{secLs}^2(f+g)\ln\alpha^2, \tag{20}$$

3. Novel localized structures of the (2+1)-dimensional MDWW system

All rich localized coherent structures, such as non-propagating solitons, dromions, peakons, compactons, foldons, instantons, ghostons, ring solitons, and the interactions between these solitons[9-34)], can be derived by the quantity U expressed by (20) with the help of the hyperbolic and triangular functions. It is known that for the (2+1)-dimensional integrable models, there are many more abundant localized structures than in (1+1)-dimensional case because some types of arbitrary functions can be included in the explicit solution expression[9)]. Moreover, the periodic waves also have been studied by some authors. In this paper, we try to give an answer for the following question: Are there any localized excitations derived by the use of another functions? Fortunately, the answer is still positive due to some arbitrariness of the functions $f(x,t)$ and $g(y)$ in the potential U given by the equation (20). In order to answer this question, some attention will be paid to dromion, peakon, dromion lattice, multi dromion-solitoff excitations, regular fractal dromions, lumps with self-similar structures and chaotic dromions patterns based on the golden main and the symmetrical hyperbolic and triangular Lucas functions for the potential field U in (2+1) dimensions.

3.1 Dromion, Peakon and Dromion lattice excitations

According to the solution *U*, we first discuss its dromion excitation which is one of the significant localized excitations localized exponentially in all directions are driven by multiple straight-line ghost solitons with some suitable dispersion relation. Also, multiple dromion solutions are driven by curved line and straight line solitons. When the simple selections of the functions $f(x,t)$ and $g(y)$ are given to be

$$f(x,t)=1+\sum_{i=1}^{M}a_i\,\mathrm{tLs}[k_i(x+c_i t)+x_{0i}],\quad g(y)=1+\sum_{j=1}^{N}b_j\,\mathrm{tLs}(K_j y+y_{0j}),\qquad (21)$$

where $a_i, k_i, c_i, b_j, K_j, x_{0i}, y_{0j}$ are arbitrary constants, *M* and *N* are integers, we can obtain a two-dromion excitation for the physical quantity *U*, as shown in Fig.1. and the parameter selections as

$$\begin{aligned}&M=2,\ N=k_1=k_2=-\frac{c_2}{2}=c_1=K_1=1,\ \ x_{01}=y_{01}=x_{02}=0,\\&2a_1=a_2=2b_1=0.2\end{aligned}\qquad (22)$$

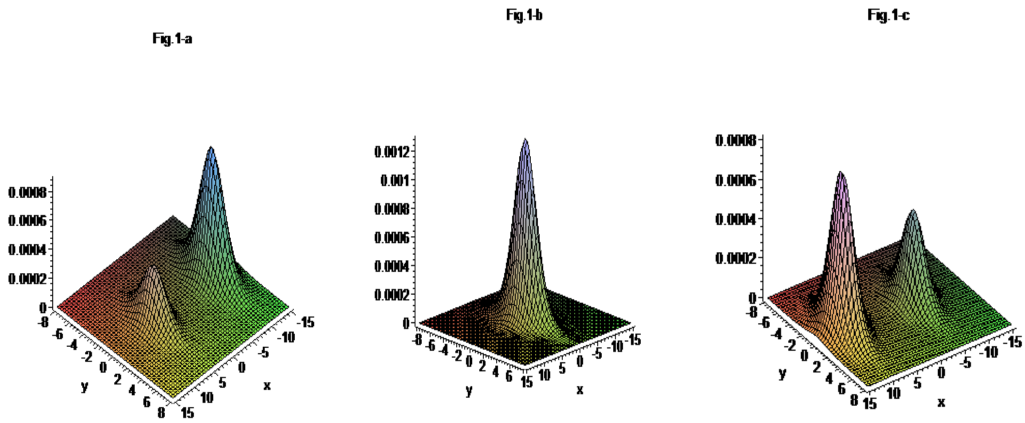

Figure (1): Time evolutional plots of an interaction between two travilling dromions for the potential *U* with selections (21) and (22): (a) when *t=-5* (b) when *t=0* (c) when *t=5*

It is well-known that the interactions of solitons in (1 + 1)-dimensional nonlinear models are usually considered to be elastic. That means there is no exchange of any physical quantities like the energy and the momentum among interacting solitons. That is, the amplitude, velocity and wave shape of a soliton do not undergo any change after the nonlinear interaction. But in higher dimensional non-linear models the interactions between solitary waves may be completely elastic or non-completely elastic. From figure 1, we show that the interaction between a dromion-dromion is non-completely elastic since their shapes and amplitudes are not completely preserved after interaction.

Along with the above lines, if taking

$$f(x,t)=\sum_{n=-N}^{N}0.3\,\mathrm{tLs}(x-ct+4n),\quad g(y)=\sum_{m=-M}^{M}0.3\,\mathrm{tLs}(y+4m),\qquad (23)$$

If *M=N=2* and *c*=1 then we can obtain a 5×5 dromion lattice" excitation for the physical field *U*. The corresponding dromion lattice plot is presented in figure 2.

Fig.2-a

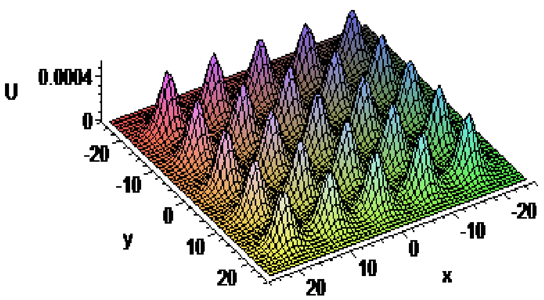

Figure 2: A dromion-lattice plot of the potential U with (23) when $t=0$

Similarly, based on the field U we can obtain some important weak localized excitations such as peakons (weak continuous solution) which is discontinuous at its crest $[u(x,t)=-k+c\,e^{(-|x-ct|)},\ k\to 0]$, was first found in the celebrated (1+1)-dimensional Camassa-Holm equation[46)]

$$u_t + 2ku_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

We find many (2+1)-dimensional nonlinear models also possess these soliton excitations[15-33)]. When selecting $f(x,t)$ and $g(y)$ to be some piecewise smooth functions

$$f(x,t)=1+\sum_{i=1}^{M}\begin{cases} f_i(x+c_i t) & x+c_i t\le 0,\\ -f_i(-x-c_i t) & x+c_i t>0,\end{cases}\qquad g(y)=1+\sum_{i=1}^{N}\begin{cases} g_i(y) & y\le 0,\\ -g_i(-y) & y>0,\end{cases}\tag{24}$$

where the functions $f_i(x+c_i t)$ and $g_i(y)$ are differentiable functions of the indicated arguments and possess boundary condition $f_i(\pm\infty)=C_{\pm i}$,$i=1,2,\dots,M$ and $g_i(\pm\infty)=D_{\pm i}$,$i=1,2,\dots,N$ with $C_{\pm i}$ and $D_{\pm i}$ being constants and /or even infinity. For instance, when choosing

$$f_1=0.1\alpha^{x+t},\quad f_2=0.05\alpha^{x-2t}\quad g(y)=0.1\alpha^{y},\quad M=2N=2,\tag{25}$$

We can derive a propagating two-peakon excitation for the potential field U. The corresponding two-peakon excitation profile is depicted in figure 3. Also, a simple example of peaked solitary waves with periodic behavior is depicted in figure 4. with the selection function reads

$$f_1=0.1\alpha^{x+t},\quad f_2=0.05\alpha^{x-2t}\quad g(y)=\alpha^{\mathrm{sTLs}(y)},\tag{26}$$

where $\mathrm{sTLs}(y)=\dfrac{1}{i}(\alpha^{iy}-\alpha^{-iy})$, $(i=\sqrt{-1})$ is the symmetrical triangular Luas sine function[33)].

Fig.3-a Fig.3-b Fig.3-c

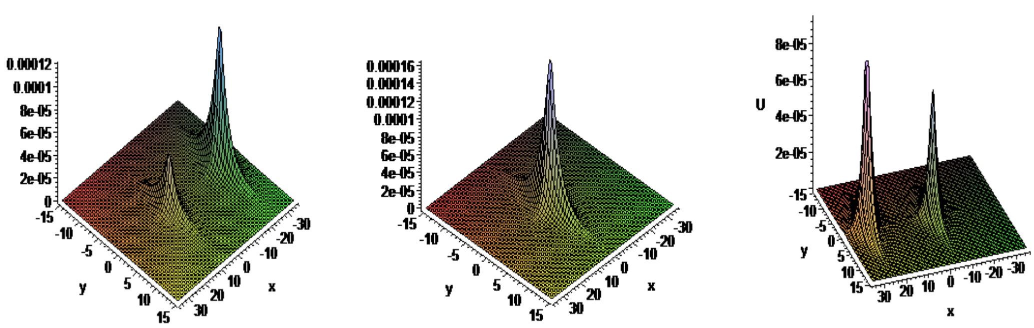

Figure (3): Time evolutional plots of the two peaked solitary waves for the potential U with the function selections (24) and (25): (a) when *t=-5* (b) when *t=0* (c) when *t=5*

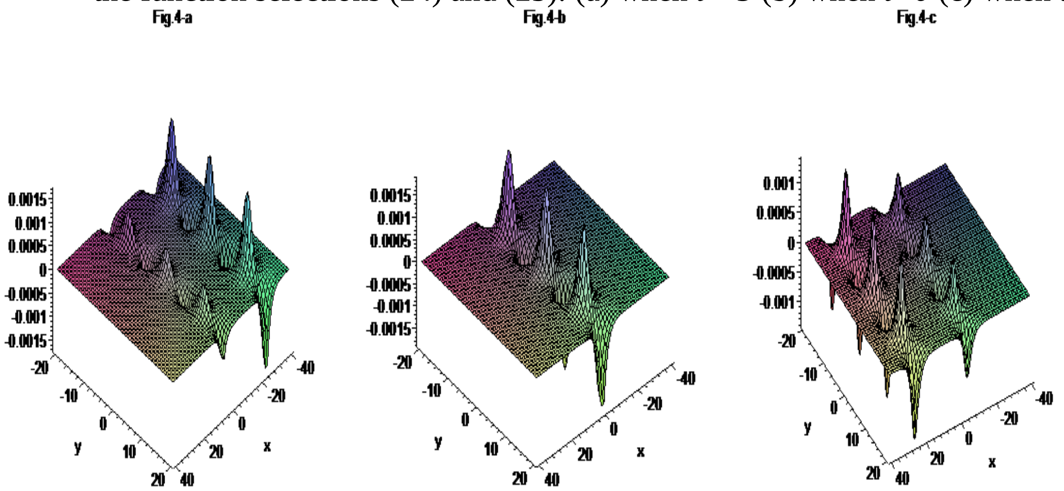

Figure (4): Time evolutional plots of the two peaked solitary waves with periodic behavior for the

potential *U* with the function selections (24) and (26): (a) when *t=-5* (b) when *t=0* (c) when *t=5*

From figure 3, we show that the interaction between a peakon-peakon is non-completely elastic since their shapes and amplitudes are not completely preserved after interaction. Also with the help of figure 4, one can easily say that the multi-peakon excitation possess periodic behavior.

3.2 Multi Dromion-Solitoff Excitations

According to potential *U*, we discuss its multi dromion-solitoff excitations. That can be expressed by means of Lucas functions in the form

$$f(x,t)=\sum_{n=-N}^{N} 0.2\,\mathrm{secLs}(x+5n+t), \quad g(y)=\sum_{m=-M}^{M} 0.2\,\mathrm{secLs}(y+5m), \tag{27}$$

If $m=n=2$, we can obtain a multi dromion-solitoff excitation for the physical quantity *U* depicted in figure 5 with *t=0*.

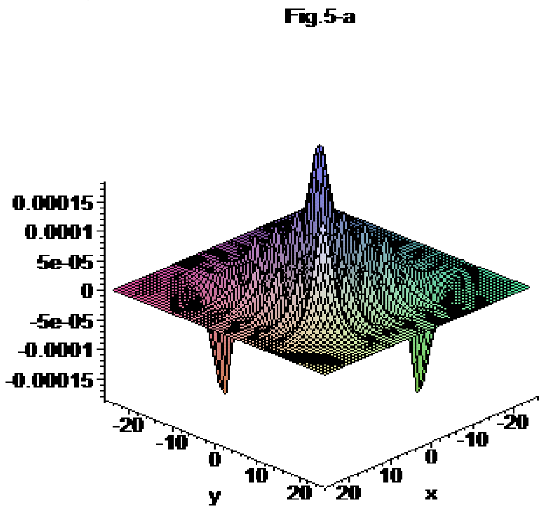

Figure 5: A plot of a special type of multi dromion-solitoff structure for the physical quantity *U* with the choice (27) and *t = 0*

For instance, when $f(x,t)$ and $g(y)$ are considered to be

$$
\begin{aligned}
f(x,t) &= \sum_{n=-N}^{N} [0.2\,\mathrm{secLs}(x+5n+t) + 0.2\,\mathrm{secLs}(x+5n-t)], \\
g(y) &= \sum_{m=-M}^{M} 0.2\,\mathrm{secLs}(y+5m),
\end{aligned}
\tag{28}
$$

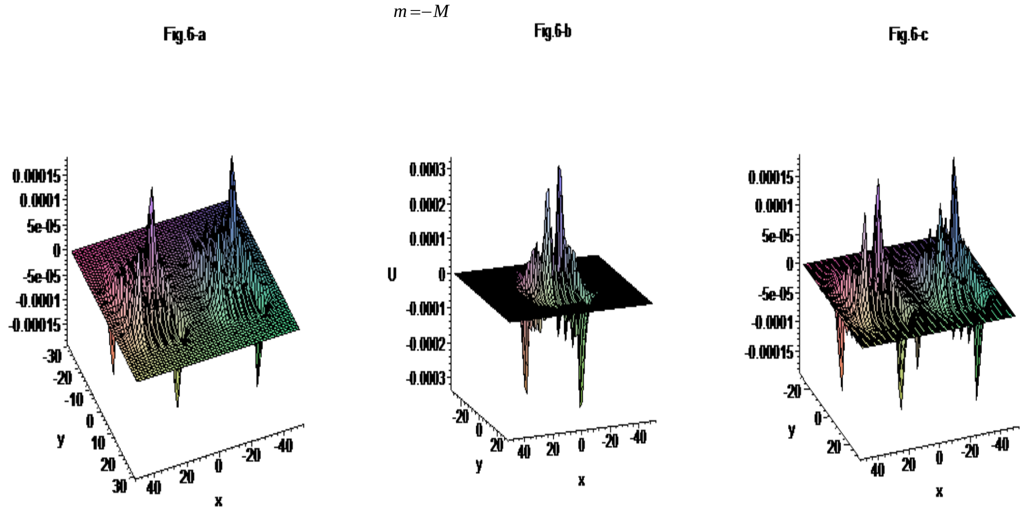

Figure (6): Time evolutional plots of an interaction between two multi dromion-solitoffs

for the potential *U* with the choice (28): (a) when *t=-25* (b) when *t=0* (c) when *t=25*

We can obtain the interactions between two multi dromion-solitoffs. Figure 6 shows an evolutional profile corresponding to the physical quantity *U*. From figure 6 and through detailed analysis, we find that the shapes, amplitudes, and velocities of the two multi dromion-solitoffs are completely conserved after their interactions. Consequently, the interaction between two multi dromion-solitoffs are completely elastic. Now we focus our attention on the intriguing evolution of a dromion in a background wave for the potential field *U*. For instance, if we choose $f(x,t)$ and $g(y)$ as

$$
\begin{aligned}
f(x,t) &= 3 + 0.12\ \mathrm{tLs}(0.5x - t) + 0.02\ \mathrm{sn}(0.4x, 0.3), \\
g(y) &= 3 + 0.12\ \mathrm{tLs}(0.5y) + 0.02\ \mathrm{sn}(0.4y, 0.3),
\end{aligned}
\tag{29}
$$

where, sn is the Jacobi sine function, we can obtain an evolutional profile of single-dromion in the background wave for the physical quantity *U* presented in figure 7 at different times (a) when *t* = -10 (b) *t=0* (c) *t=10.*

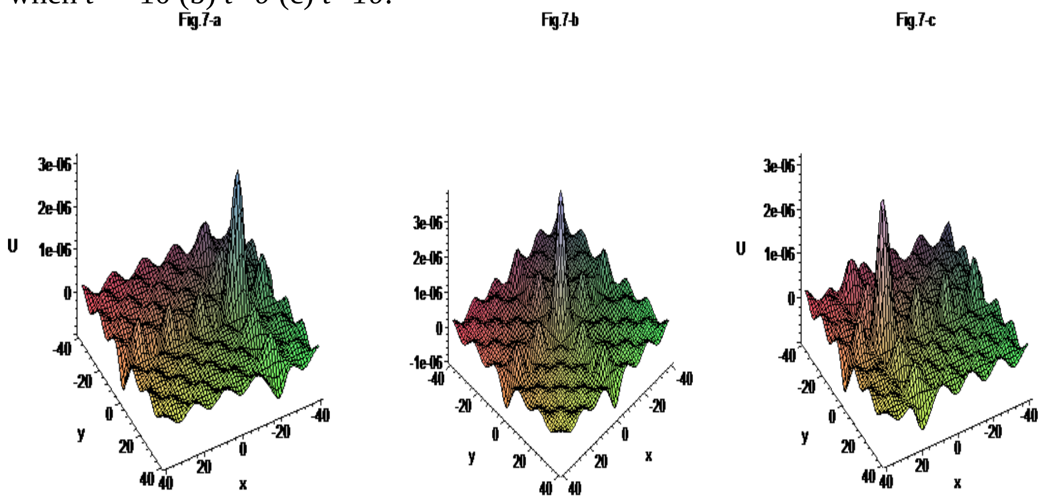

Figure (7): Time evolutional plots of the single dromion in the background wave for the potential *U* with selection (29): (a) when *t=-10* (b) when *t=0* (c) when *t=10*

From figure 7 and through detailed analysis, this figure shows the corresponding profile of the complex wave excitation presenting the propagation of a dromion moving on the determined double periodic wave background. However, its wave shape and wave velocity do not suffer any change, which is very close to many actual physical processes in the natural world.

3.3 Regular Fractal Dromions

Recently, it has been found that many lower-dimensional piecewise smooth functions with fractal structure can be used to construct exact localized solutions of the higher-dimensional soliton system which also possesses fractal structures. If we appropriately select the arbitrary functions $f(x,t)$ and $g(y)$, we find that some special types of fractal dromions for the potential U can be revealed. For example, if we take

$$
\begin{aligned}
f(x,t) &= 1+\alpha^{|x-t|[x-t+\mathrm{cTLs}(\ln(x-t)^2)+\mathrm{sTLs}(\ln(x-t)^2)]}, \\
g(y) &= 1+\alpha^{|y|[y+\mathrm{cTLs}(\ln(y)^2)+\mathrm{sTLs}(\ln(y)^2)]},
\end{aligned}
\tag{30}
$$

where $\mathrm{cTLs}(y)=\alpha^{iy}+\alpha^{-iy}$, $(i=\sqrt{-1})$ is the symmetrical triangular Lucas cosine function[33)], then we can obtain a simple fractal dromion.

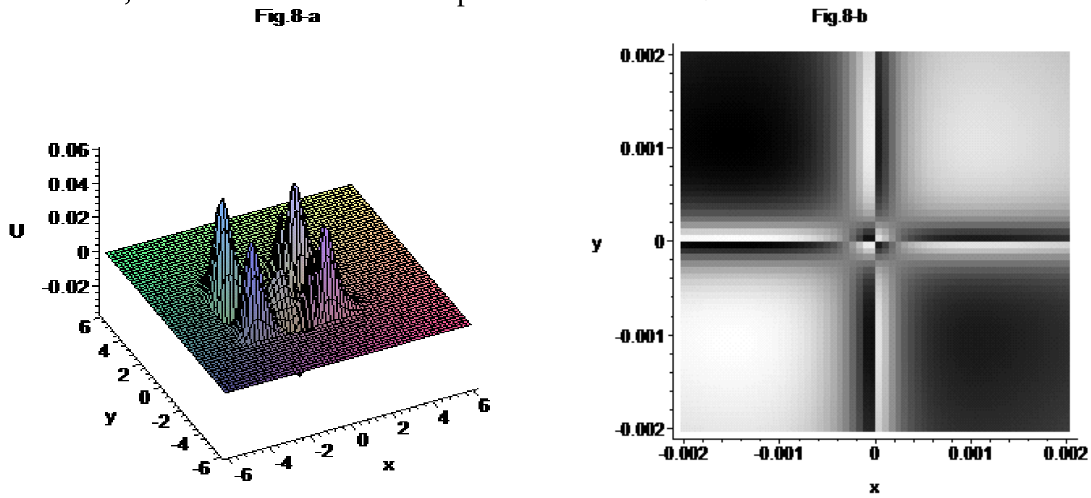

Figure 8: (a) Fractal dromion structure for the potential U with the choice (30): at $t=0$, (b) Density of the fractal structure of the dromion in the region {x=[-0.002,0.002],y=[-0.002,0.002]}

From figure 8 and through detailed analysis, figure (8-a) shows a plot of this special type of fractal dromion structure for the potential U given by equation (20) with the selection functions (30) when t=0. Figure (8-b) shows the density of the fractal structure of the dromion in the region {x=[-0.002,0.002], y=[-0.002,0.002]}. To observe the self-similar structure of the fractal dromion clearly, one may enlarge a small region near the centre of figure (8-b). For instance, if we reduce the region of figure (8-b) to {x=[-0.0002,0.0002], y=[-0.0002,0.0002]}, {x=[-0.00001,0.00001], y=[-0.00001,0.00001]}, and so on, we find a totally similar structure to that presented in figure (8-b).

3.4 Lumps with self-similar structures

It is also known that in high dimensions, such as the KP equations and the (2+1)-dimensional Korteweg-de Vries (KdV) equations, a special type of localized structure, which is called the lump solution (algebraically localized in all directions), has been formed by rational functions. This localized coherent soliton structure is another type of significant localized excitation. If we select the functions $f(x,t)$ and $g(y)$ of the potential U appropriately, we can find some types of lump solutions with fractal behavior. Figure (9-a) shows a fractal lump structure for the quantity U, where $f(x,t)$ and $g(y)$ are selected to be

$$f(x,t)=1+\frac{|x-t|}{1+(x-t)^4}[\mathrm{cTLs}(\ln(x-t)^2)+\mathrm{sTLs}(\ln(x-t)^2)]^2,$$
$$g(y)=1+\frac{|y|}{1+y^4}[\mathrm{cTLs}(\ln(y^2))+\mathrm{sTLs}(\ln(y^2))]^2, \tag{31}$$

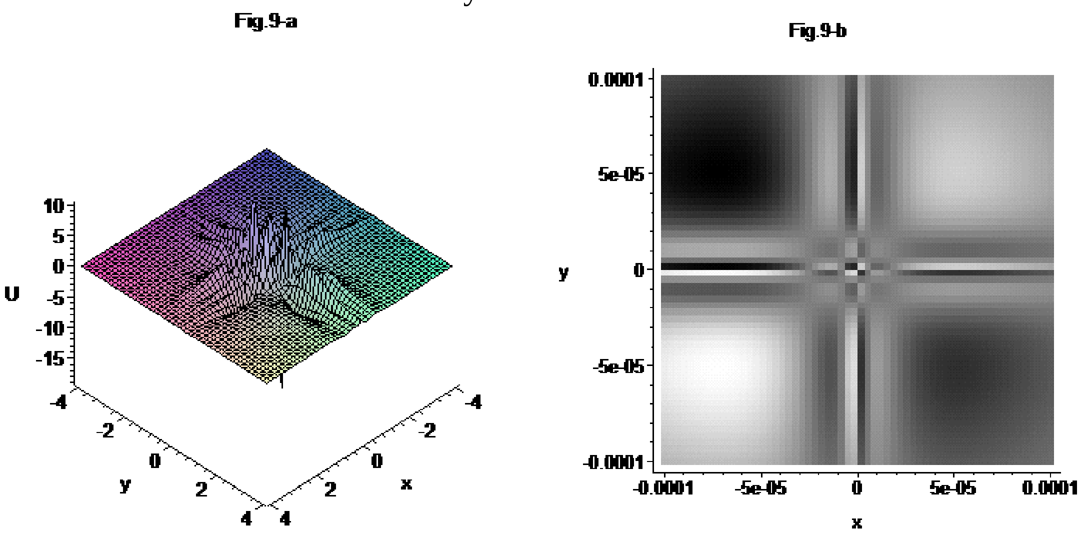

Figure 9: (a) Fractal lump structure for the potential U with conditions (31), (b) Density of the fractal lump related to (a) in the region {x = [-0.0001, 0.0001], y = [-0.0001, 0.0001]}.

From figure (9-a), we can see that the solution is localized in all directions. Near the center there are infinitely many peaks which are distributed in a fractal manner. In order to investigate the fractal structure of the lump, we should look at the structure more carefully. Figure (9-b) presents a density plot of the structure of the fractal lump at the region {x = [-0.0001, 0.0001], y = [-0.0001, 0.0001]}. More detailed studies will show us the self-similar structure of the lump. For example, if we reduce the region of figure (9-b) to {x = [-0.00066, 0.00066], y = [-0.00066, 0.00066]}, {x = [-0.000028, 0.000028], y = [-0.000028, 0.000028]} and so on, we can find a totally similar structure to that plotted in figure (9-b).

3.5 Chaotic Dromions patterns

Now, if we select the functions $f(x,t)$ and $g(y)$ as

$$f(x,t)=1+(100+s(t))\alpha^x, \quad g(y)=1+\alpha^y, \tag{32}$$

where *s(t)* is arbitrary function of time t. From the potential field U with the selections (32), one knows that the amplitude of the dromion is determined by the function *s(t)*. If we select the function *s(t)* as a solution of a chaotic system, then we can obtain a type of chaotic dromion solution. In figure (10-a), we exhibit the shape of the dromion for the physical quantity U shown by equation (20) at a fixed time (for *s(t)* = 0) with the function selection (32). The amplitude A of the dromion is changed chaotically with *s(t)*, where *s(t)* is a chaotic solution of the following nuclear spin generator system depicted in figure (10-b) [47]:

$$s_t=-bs+g, \quad g_t=-s-bg(1-ch), \quad h_t=b[a(1-h)-cg^2], \tag{33}$$

where *s, g* and h are functions of *t, a, b, c* are model parameters. The nuclear spin generator system is a high-frequency oscillator which generates and controls the oscillations of a nuclear magnetization vector in a magnetic field. The nuclear spin generator system exhibits a large variety of chaotic attractors and displays rich structures. One of typical chaotic attractors for the nuclear spin generator system (33) is depicted in figure (10-b) when the model parameters and initial values set

$$a=0.2, \quad b=1.3, \quad c=3, \quad f(0)=2, \; g(0)=1, \; h(0)=0, \tag{34}$$

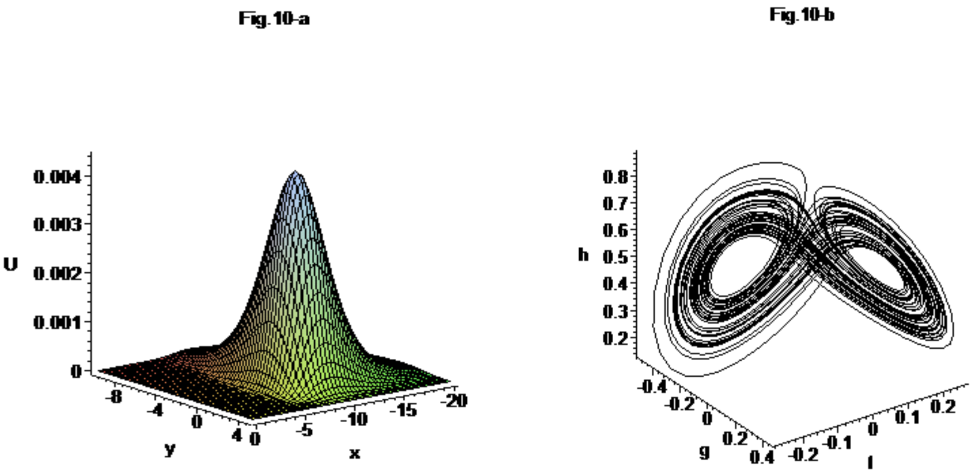

Figure 10: (a) Single dromion structure for the physical quantity U with the selections (32) and
$s(t) = 0$. (b) A typical attractor plot of the nuclear spin generator system (33) with the condition (34).

4. Summary and Discussion

In conclusion, the Lucas Riccati method is applied to obtain variable separation solutions of (2+1)-dimensional MDWW equation. With the help of the quantity (20), some localized excitations, such as dromion, peakon, dromion lattice, multi dromion-solitoff excitations, regular fractal dromions, lumps with self-similar structures and chaotic dromions patterns, are obtianed based on the golden main and the symmetrical hyperbolic and triangular Lucas functions. We hope that in future experimental studies these localized excitations obtained here can be realized in some fields. Actually, our present short paper is merely a beginning work; more application to other nonlinear physical systems should be conducted and deserve further investigation. In our future work, on the one hand, we devote to generalizing this method to other (2+1)-dimensional nonlinear systems such as the ANNV system and BKK system, Boiti-Leon-Pempinelle system etc. On the other hand, we will look for more interesting localized excitations.

5. Appendix A

Stakhov and Rozin in[37)] introduced a new class of hyperbolic functions that unite the characteristics of the classical hyperbolic functions and the recurring Fibonacci and Lucas series. The hyperbolic Fibonacci and Lucas functions, which are the being extension of Binet's formulas for the Fibonacci and Lucas numbers in continuous domain, transform the Fibonacci numbers theory into ''continuous'' theory because every identity for the hyperbolic Fibonacci and Lucas functions has its discrete analogy in the framework of the Fibonacci and Lucas numbers. Taking into consideration a great role played by the hyperbolic functions in geometry and physics, (''Lobatchevski's hyperbolic geometry'', ''Four-dimensional Minkowski's world'', etc.), it is possible to expect that the new theory of the hyperbolic functions will bring to new results and interpretations on mathematics, biology, physics, and cosmology. In particular, the result is vital for understanding the relation between transfinitness i.e. fractal geometry and the hyperbolic symmetrical character of the disintegration of the neural vacuum, as pointed out by El Naschie.

The definition and properties of the symmetrical Lucas functions The symmetrical Lucas sine function (sLs), the symmetrical Lucas cosine function (cLs) and the symmetrical Lucas tangent function (tLs) are defined[33-35, 37-40)] as

$$\mathrm{sLs}(x)=\alpha^{x}-\alpha^{-x}, \quad \mathrm{cLs}(x)=\alpha^{x}+\alpha^{-x}, \quad \mathrm{tLs}(x)=\frac{\alpha^{x}-\alpha^{-x}}{\alpha^{x}+\alpha^{-x}}, \tag{A.1}$$

They are introduced to consider so-called symmetrical representation of the hyperbolic Lucas functions and they may present a certain interest for modern theoretical physics taking into consideration a great role played by the Golden Section, Golden Proportion, Golden ratio, Golden Mean in modern physical researches[37-39]. The symmetrical Lucas cotangent function (cotLs) is $\mathrm{cotLs}(x)=\frac{1}{\mathrm{tLs}(x)}$, the symmetrical Lucas secant function (secLs) is $\mathrm{secLs}(x)=\frac{1}{\mathrm{cLs}(x)}$, the symmetrical Lucas cosecant function (cscLs) is $\mathrm{cscLs}(x)=\frac{1}{\mathrm{sLs}(x)}$.

These functions satisfy the following relations[37-39]

$$\begin{gathered}\mathrm{cLs}^2(x)-\mathrm{sLs}^2(x)=4, \qquad 1-\mathrm{tLs}^2(x)=4\mathrm{secLs}^2(x)\\ \mathrm{cotLs}^2(x)-1=4\mathrm{cscLs}^2(x)\end{gathered} \tag{A.2}$$

Also, from the above definition, we give the derivative formulas of the symmetrical Lucas functions as follows:

$$\frac{d\ \mathrm{sLs}(x)}{dx}=\mathrm{cLs}(x)\ln\alpha, \quad \frac{d\ \mathrm{cLs}(x)}{dx}=\mathrm{sLs}(x)\ln\alpha, \quad \frac{d\ \mathrm{tLs}(x)}{dx}=4\mathrm{secLs}^2(x)\ln\alpha, \tag{A.3}$$

The above symmetrical hyperbolic Lucas functions are connected with the classical hyperbolic functions by the following simple correlations:

$$\mathrm{sLs}(x)=2\sinh(x\ln\alpha), \quad \mathrm{cLs}(x)=2\cosh(x\ln\alpha), \quad \mathrm{tLs}(x)=\tanh(x\ln\alpha). \tag{A.4}$$

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