Military Technical College
Kobry Elkobbah,
Cairo, Egypt
May 25-27,2010

5th International Conference on
Mathematics and Engineering
Physics (ICMEP-5)

EM-8

An interactive satisficing approach for solving fuzzy multiobjective Linear optimization problems based on the Attainable Reference Point Method

Taghreed A. Hassan*, A. A. Mousa

Abstract

In a fuzzy environment, Decision Maker (DM) generally gives all objectives their vague targets. DM permits the objectives less than or equal to (or more than or equal to) aspiration levels by predetermining the respective tolerances.
In this research, an approach to the case of the fuzzy multiple objectives linear programming problems with fuzzy goals in objective functions and constraints is described. This approach is associated with modifying the compromise model by reconstructing the membership functions by changing tolerances of the objectives using the principle of the interactive Attainable Reference Point Method to guarantee the optimization problem feasibility.
To demonstrate the effectiveness of the optimization approach proposed here a numerical example is solved.
.

1. Introduction

Several realistic optimization problems require taking into account multiple objectives, on the one hand, and various types of uncertainties, on the other. In the present paper an attempt has been made to consider a type of generalizations with respect to the classical single-objective programming in the framework of multiobjective fuzzy programming (MOFP). fuzzy mathematical programming using fuzzy concepts to represent the ambiguity in systems optimization problems has been also progressed in various ways. In particular, fuzzy linear programming models and fuzzy multiple objective programming problems are designated for such a purpose [1-3]. In fuzzy set theory, a corresponding membership function is usually employed to quantify the fuzzy objectives and constraints. Using the linear membership function, Zimmermann proposed the min operator model to the Fuzzy Multiple Objective Linear Programming Problems (FMOLP) [4,5].

***Correspondence to**
E-Mail : taghreedma2005@yahoo.com

Although the min operator method has been proven to have several nice properties ,the solution generated by min operator does not guarantee compensatory and efficient[6]. Lee and

Li [7] proposed two-phase approach to overcome this difficulty, Yan-Kuen Wu, et al., [8], proposed a compromise model for solving FMOLP, and presented an adjustable compromise index for the Decision-Maker (DM), that he/she only trade off this index; furthermore a fuzzy-efficient solution between non-compensatory and fully compensatory obtained by using their compromise model.
With progressive articulation of preference, there are a few approaches [9–11], they need DM give the reference membership degrees at each step in terms of the current values and trade-off rates between membership functions. However it is difficult to obtain the trade-off rates which are often approximate.
Werners [12] proposed an interactive procedure to ask the DM to modify membership functions of objectives and constraints and applied the min operator to generate a fuzzy-efficient solution.
S.Y. Li, et al.,[13], proposed an interactive method based on the improvements of the objectives by altering their membership functions using the varying-domain method which is only designed for the special preference, where the difference among the objectives is determined by the strict priority order after DM has known the priority preference. Nevertheless, the varying-domain method is not proper when preference is not clear in actual environment. Chaofang Hu, et al.,[14], proposed an interactive satisficing method based on alternative tolerance for MOFP to overcome this difficulty. they obtained the efficient solution of MOFP by solving the lexicographic two-phase programming procedure introduced by E.S. Lee, et al.,[15]. To guarantee the feasibility, the Attainable Reference Point(ARP) Method[16] is referred to improve the objectives by altering their membership functions.
In this paper, an interactive approach for solving FMOLP problems with fuzzy goals in objective functions and constraints is described. This approach is based on the improvement of a compromise model for solving FMOLP[8] by improving the objectives by altering their membership functions using the principle of the ARP Method[16], which guarantee the feasibility. The DM selects objectives attaining his/her preferences, the other objectives are improved. The process is continued until all the objectives are satisficing.
To demonstrate the effectiveness of the optimization approach proposed here two numerical examples are solved.
The rest of this paper is organized in the following manner, section 2, some previous concepts and results are stated. Section 3, is devoted to describe the interactive algorithm. While a numerical example is solved in section 4 . Finally, some conclusions are given in section 5.

2. Preliminaries

2.1 The Attainable Reference Point Method

In general, the multicriteria decision making (MCDM) problem is represented as:

$$\begin{aligned} &F(x)= \{f_i(x)\} \qquad i=1,\dots,k \\ &s.t.\ x \in X \end{aligned} \tag{1}$$

where x is an n-dimential vector of decision variable, $X \subset R^n$ is a feasible set of x, $f_i(x),\ i=1,\dots,k$ are m-distinct real- valued functions of x . These multiple objectives are usually incommensurate and in conflict with one another, Because of this, multiple objective optimisation is not to search for optimal solutions but for efficient (non-inferior, non-dominated or Pareto-optimal) solutions that can best attain the prioritised multiple objectives as greatly as possible. Solving Multiobjective Optimization (MOP) problems usually requires the participation of a human decision maker who is supposed to have better insight into the problem and to express preference relations between alternative solutions. Many researchers have developed various methods for MOP problems are collected in [17,18].
Xiaomin M. Wang [16], proposed an interactive algorithm the ARP method is for finding a satisfactory solution to a general multicriteria decision making problem. The DM is only

required to modify the reference value of the satisfactory objectives to generate a new attainable reference point in each iteration step. The lexicographic weighted Tchebycheff program associated with the attainable reference point is constructed to guarantee the efficiency of all discussed points. The value of the unsatisfactory objective chosen by the decision-maker is improved to be satisfactory. Thus its reference value doesn't need to be modified again in later iterations, and a satisfactory solution can be derived in finite steps.

Definition 2.1 [19] :

x_t is called an efficient (Pareto-optimal) solution of problem (1) if there does not exist any $x \in S$ ($x \neq x_t$), so that $F(x) \leq F(x_t)$ and $F(x) \neq F(x_t\}$, and x_t is called a weakly efficient solution of such problem if there does not exist any $x \in S$ ($x \neq x_t$), so that $F(x) < F(x_t)$, where f_i ($i = 1,..., k$) are assumed for minimisation.

Definition 2.2 [16] :

I). $\bar{f}_i\ , i = 1,...,k$ is called a reference value of objective f_i for (1), if the DM does not wish the value f_i of objective to be more than $\bar{f}_i$, i.e., . Meanwhile, $\bar{f} = \left(\bar{f}_1,\ldots,\bar{f}_k\right)^T$ is called a reference point for (1).

II). $\bar{f} = \left(\bar{f}_1,\ldots,\bar{f}_k\right)^T$ is called an attainable reference point for (1), and $\bar{f}_i\ , i = 1,...,k$ an attainable reference value of objective f_i , if $\bar{f}$ is a reference point and $\bar{f} \in f\left(X\right) + R_+^k$.

2.2 Fuzzy multiobjective optimization problem

In a fuzzy environment, DM generally gives all objectives their vague targets. DM permits the objectives less than or equal to (or more than or equal to) aspiration levels by predetermining the respective tolerances. Such a decision is often defined as follows [5]:

$$\begin{aligned} &Find: \quad x \\ &S.t. \qquad f_i\left(x\right) \tilde{\leq} \left(\tilde{\geq}\right) f_i^*,\ i = 1,\ldots,k \\ &\qquad\qquad g_j\left(x\right) \tilde{\leq} 0,\ j = 1,\ldots,m \end{aligned} \tag{2}$$

Since the pioneer papers [1-5] a great deal of work has been devoted to solve the MOFP problem. In almost all of the cases, and in a parallel way to the classical MOP, the research has been oriented towards the characterization of noninferior solutions in this fuzzy case. In a fuzzy environment, however, we have different possibilities to address an MOFP problem, extending in all the cases and generalizing the conventional MOP.

These extensions and generalizations are as follows [20]:

2.2.1 Fuzziness in the constraints

In particular, when the linear case is considered, that is, when $f_i\left(x\right),\ i = 1,\ldots,k,\quad g_j\left(x\right),\ j = 1,\ldots,m$ are linear, as it will be in this paper, the model becomes a linear multiobjective optimization problem which is typically stated as

$$\begin{aligned} &Min\ \left[c_1x, c_2x,\ldots,c_nx\right] \\ &s.t.\quad Ax \tilde{\leq} b\ , x \geq 0 \end{aligned} \tag{3}$$

Two different models can be considered.

1. In the first the fuzzification of (2) leads to the following model:

$$\begin{aligned} &Min\ \left[c_1x, c_2x,\ldots,c_nx\right] \\ &s.t.\quad Ax \leq_u b\ , x \geq 0 \end{aligned} \tag{4}$$

where $\leq_u$ indicates, as usual, that there exist membership functions

$\mu_j : R \rightarrow \left[0,1\right],\ j = 1,\ldots,m$

expressing for each $x \in R^N$ the accomplishment degree of the j_{th} constraint.

2.In the second the fuzzification of (2) is translated into both the coefficients of the technological matrix and the right-hand side. Then the model is defined as,

$$\begin{aligned} & Min\ [c_1x, c_2x, ..., c_nx] \\ & s.t.\quad A^f x \le_f b^f\ , x \ge 0 \end{aligned} \tag{5}$$

with A^f an $m \times n$-matrix of fuzzy numbers, b^f an $m-$vector of fuzzy numbers, and the symbol $\le_f$ standing for a fuzzy relation ranking fuzzy numbers

2.2.2 Fuzziness in the objective functions

Two different models can be considered **:**

1. the coefficients in the objective functions are given by fuzzy numbers. Then the corresponding model can be defined as

$$\begin{aligned} & Min\ \left\lfloor c_1^f x, c_2^f x, \ldots, c_n^f x \right\rfloor \\ & s.t.\quad Ax \le b\ , x \ge 0 \end{aligned} \tag{6}$$

where each $c_i^f\ , i = 1, \ldots, n$ is an $N-$vector of fuzzy numbers

2. the existence of fuzzy goals can be assumed. Then the problem is defined as

$$\begin{aligned} & Find:\quad x \in R^N \\ & Such\ that.\qquad c_i x \le_g z_i\ ,\ i = 1, \ldots, n \\ & \qquad\qquad\quad Ax \le_u b\ ,\ x \ge 0 \end{aligned} \tag{7}$$

where z_i are aspirations levels fixed, together with its respective membership functions, by the decision maker.

2.3 The Compromise Model

Yan-Kuen,et al. [8], condidered the FMOLP linear form of problem (2), and determined all membership functions , then converted the FMOLP problem into the following linear programming model by using the min operator method, yielded by Zimmermann.[5].

$$\begin{aligned} & Max\quad \alpha \\ & s.t.\qquad 1 \ge \mu_i(x) \ge \alpha\quad ,\quad \forall i = 1, \ldots, n \\ & \qquad\qquad 1 \ge \mu_j(x) \ge \alpha\quad ,\quad \forall j = 1, \ldots, m \\ & \qquad\qquad x \ge 0\ , \\ & \qquad\qquad \alpha \in [0,1]\,. \end{aligned} \tag{8}$$

where: $\mu_i(x)$ membership function for the i_{th} objective function,

$\mu_j(x)$ membership function for the j_{th} fuzzy constraint function,

Solving the model (8), one optimal value α^* can be yielded. In fact, this α^* denotes that the satisfaction level for all membership functions can simultaneously obtain.

In order to offer any desirable compromise solutions between non-compensatory and fully compensatory to the DM, they associated preceding two-phase approach with the results obtained by min operator and propose following compromise model to solve the FMOLP.

$$
\begin{aligned}
& Max\ \tilde{\alpha} = \frac{1}{n+m}\left[\sum_{i=1}^{n}\mu_i(x) + \sum_{j=1}^{m}\mu_j(x)\right] \\
& s.t. \quad 1 \ge \mu_i(x) \ge \alpha' \ , \quad \forall i = 1,\dots,n \\
& \qquad 1 \ge \mu_j(x) \ge \alpha' \ , \quad \forall j = 1,\dots,m \\
& \qquad x \ge 0
\end{aligned} \tag{9}
$$

$\alpha' \in \left[0, \alpha^*\right]$ may be considered as a compromise index for all membership functions.

As long as the DM determines the compromise degree among 0 and α^* to index α' , the model (9) can be solved and obtained a fuzzy-efficient solution between non-compensatory and fully compensatory[8].

3. Interactive satisficing approach based on the Attainable Reference Point Method.

3.1. Main features.

In this paper, the linear membership function is adopted for decreasing computation.

Following Werners[21] method, the possible range F_i^o, F_i^1 for the i_{th} objective function can be obtained as follows:

$$
\begin{aligned}
& F_i^o = Max\ F_i(x) \quad , i = 1,\dots,k \\
& s.t. \quad (Ax)_j \le b_j \ , \ j = 1,\dots,m \ , \\
& \qquad x \ge 0
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
& F_i^1 = Max\ F_i(x) \quad , i = 1,\dots,k \\
& s.t. \quad (Ax)_j \le b_j + p_j \ , \ j = 1,\dots,m \ , \\
& \qquad x \ge 0
\end{aligned} \tag{11}
$$

With F_i^o and F_i^1**,** a non-decreasing linear membership function for the i_{th} objective function is defined as following:

$$
\mu_i(x) = \begin{cases} 1 & if \quad F_i(x) > F_i^1 \\ \dfrac{F_i(x) - F_i^o}{F_i^1 - F_i^o} & if \quad F_i^o \le F_i(x) \le F_i^1 \\ 0 & if \quad F_i(x) < F_i^o \end{cases} \tag{12}
$$

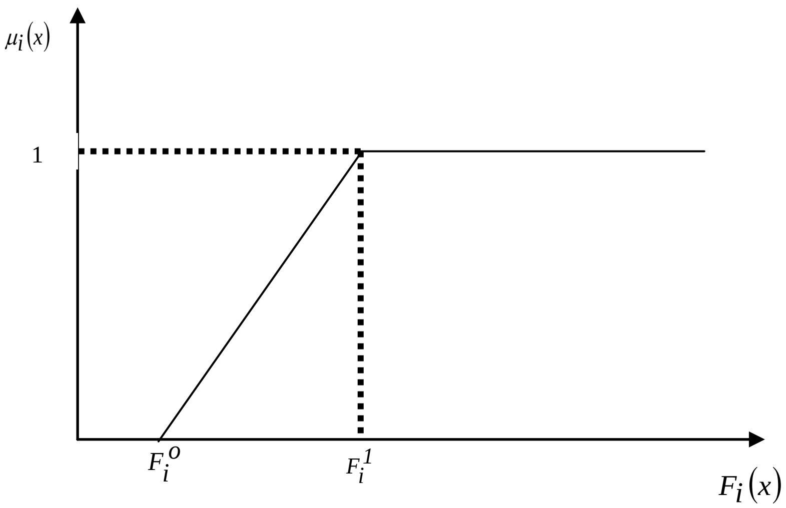

Fig. (1). Linear membership function of objectives for " $\tilde{\ge}$ "

For problem (2), a non-increasing linear membership function for the j_{th} fuzzy constraint is usually formed to as follows:

$$\mu_j = \begin{cases} 1 & \text{if} \quad (Ax)_j < b_j \\ \dfrac{(b_j + p_j) - (Ax)_j}{p_j} & \text{if} \quad b_j \le (Ax)_j \le b_j + p_j \\ 0 & \text{if} \quad (Ax)_j > b_j + p_j \end{cases} \tag{13}$$

For the crisp objective function of problem (2), the original tolerance of the objective may be not legible. Hence, following Chaofang Hu, et al.,[14], method the original tolerance of the i_{th} objective function can be obtained from the payoff table of ideal solution as shown in Table 1. For the minimization problem, the ideal solution is just the optimum of the single objective under the system constraint. That is:

$$F_i\left(x^{i*}\right) = \min_{x \in X} F_i(x), \quad i = 1,\dots,k$$

together with $F_{ij} = F_j\left(x^{i*}\right), \quad i, j = 1,\dots,k$. Then the payoff table of is formed as Table 1.

The original tolerant limit of every objectivecan be obtained from the following equation:

$$F_j^o = \max_{i=1,\dots,k} F_{ij}, \quad j = 1,\dots,k.$$

Table 1. Payoff table of ideal solution

	F_1	$\dots$	F_k
$\min\ F_1(x)$	$F_1\left(x^{1*}\right)$		$F_k\left(x^{1*}\right)$
$\vdots$	$\dots$		$\dots$
$\min\ F_k(x)$	$F_1\left(x^{k*}\right)$		$F_k\left(x^{k*}\right)$

When all membership functions are determined, then the problem (2) can be converted into linear programming model by using the min operator method as described in problem (8) .

After solving problem (8), one optimal value α^* can be yielded. In fact, this α^* denotes that the satisfaction level for all membership functions can simultaneously obtain.

As long as the DM determines the compromise degree among 0 and α^* to index α' , the problem (2) can be solved by the compromise model (9).

Let the optimal solution obtained from the compromise model (9) is $\bar{x}$

Yan et.al,[8] have proved that the optimal solution yielded by the compromise model (9), is a fuzzy-efficient solution of (FMO) problem.

Then the membership functions are improved by means of changing the tolerances of the objectives. The alternative membership functions during a solution process reflect the progressive preference, the determination of tolerance is key to the interactive method. Then the algorithm solve the ARP method [16], the new tolerance is acquired by means of attainable reference point.

The attainable reference value can be regarded as the tolerant limit of the objective . Suppose the solution is $\bar{x}$ in certain iteration and the tolerant limits of the objectives are respectively $\bar{F}_i^o$, $i = 1,\ldots,k$.

For the dissatisficing objective $F_d(x)$, its future value is expected to lie between goal value F_d^1 and the optimization value $F_d(\bar{x})$ in this iteration.

In order to improve the value of the objective $F_d(x)$, DM is required to give the information about how to modify its tolerance.

The other objectives that are satisficing $F_s(x)$ can relaxed to $\bar{F}_s^o < \hat{F}_s^o \le F_s(\bar{x})$.

Then the following auxiliary programming is used to find the new tolerance and guarantee its attainability .

$$\begin{aligned} &Min \quad F_d(x) \\ &s.t. \quad F_s(x) \le \hat{F}_s^0 \ , \ s = 1,\ldots,k, \ s \ne d \\ &\qquad (Ax)_j \le b_j + p_j \ , \ j = 1,\ldots,m \ , \\ &\qquad x \ge 0 \end{aligned} \tag{14}$$

The optimal value $F_d(\hat{x})$ of the d_{th} objective is the desirable level. Its new tolerant limit $\hat{F}_d^o$ can be chosen as : $F_d(\bar{x}) < \hat{F}_d^o < F_d(\hat{x})$. This assures that its next optimization result is able to locate the new tolerance.

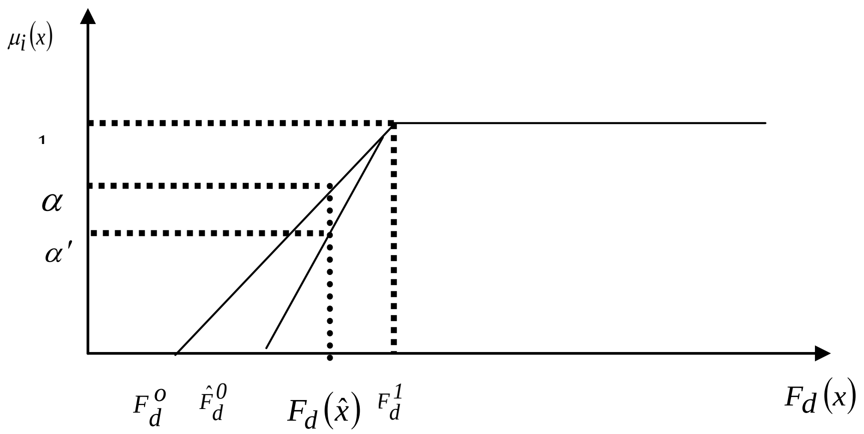

Fig. (2) . New membership function for dissatisficing objective

$\hat{F}_d^o$ is the new tolerant limit of the objective $F_d(x)$ whose new membership degree α' for the solution $\hat{x}$. The increase of the tolerance reduces the feasible region.This principle dwindles the distance between the value of the objective and its aspiration level.

According to the determined tolerances of all objectives, the new membership functions are constructed. Then the solution is obtained by solving the compromise model (9) again.The process goes on until a solution accepted by the DM is found.

Let us now describe the proposed interactive algorithm step by step**.**

3.2. Step-by-step description of the proposed algorithm

Step 0: Ask the DM to specify the satisficing fuzzy resources $\tilde{b}_j, [b_j, b_j + p_j]$ with given p_j, $0 \le p_j < \infty$, $j = 1,\ldots,m$ for each of the constraints in (2). Then the algorithm will solve problems (10), and (11), obtaining the range: $\left[F_i^o, F_i^1\right]$ for i_{th} objective function.
Step 0' : When the original tolerance of the objective may be not legible, the algorithm constructs the pay off table as shown in: Table.1 to obtain F_i^o for i_{th} objective function.
Step 1: With the previous results, the algorithm costructs a linear membership function for the i_{th} objective function , and j_{th} constraint, let $h = 1$.
Step 2: The problem (2) will be converted into the min operator model as described in problem (8), obtaining one optimal value α^*
Step 3: Ask the DM to determine the compromise degree α' with $\alpha' \in \left[0, \alpha^*\right]$.
Step 4: The compromise model (9) will be solved, with optimal solution $\bar{x}^h$, with this solution the algorithm calculates the obective values $F_i\left(\bar{x}^h\right)$ and the corresponding membership degree $\mu_i\left(\bar{x}^h\right)$, the DM is shown these results.
Step 5: Ask the DM if he/she satisfied with the these results stop, Else, ask him to fix a satisficing objective $F_s(x)$, and dissatisficing objective $F_d(x)$, at certain compromise degree α' .
Step 6: The algorithm relaxed $F_s\left(x^h\right)$ in the intervale $\left[F_s^o\left(\bar{x}^h\right), F_s\left(\bar{x}^h\right)\right]$ by increasing its value where $F_s^o\left(\bar{x}^h\right)$ is the tolerant limit of the objective, and define $\hat{F}_s^o$ satisficing $\bar{F}_s^o \le \hat{F}_s^o \le F_s(\bar{x})$ as the new tolerant limit for $F_s(x)$
Step 7: With the previous data, the auxiliary problem (14) will be solved .
With optimal solution $\left(\hat{x}^h\right)$, with this solution $F_s\left(\hat{x}^h\right)$ will be calculated .
Step 8: If the value $F_s\left(\hat{x}^h\right)$ of the d_{th} objective does not satisfy DM, go back to the step 6 to continue relaxing; if $F_s\left(\hat{x}^h\right)$ is still dissatisficing to DM when $F_s(x) = F_s^o\left(\bar{x}^h\right)$, there is not satisficing efficient and weak efficient solution; otherwise, go to the next step.
Step 9: According to the determined tolerances of all objectives, the new membership functions are reconstructed. Then go to step 2. let $h = h + 1$.
Step 10: The process goes on until a solution accepted by the DM is found.

4. Numerical examples

To show that the results obtained from the compromise model is improved using the proposed approach, numerical examples will be solved.

Example.1

$$
\begin{aligned}
& F_1(x) = -8x_1 - 7x_2 - 4x_3 - 4x_4 - 6x_5 \;\tilde{\lesssim}\; -80 \\
& F_2(x) = 10x_1 + 15x_2 + 15x_3 + 14x_4 + 7x_5 \;\tilde{\lesssim}\; 10 \\
s.t. \quad & 5x_1 + 11x_2 + 8x_3 + 15x_4 + 3x_5 \le 106.6 \\
& 9x_1 + 6x_2 + 8x_3 + 3x_4 + 12x_5 \le 109.796 \\
& 3x_1 + 7x_2 + 13x_3 + 5x_4 + 15x_5 \le 107.248 \\
& 12x_1 + 10x_2 + 3x_3 + 10x_4 + 5x_5 \le 109.736 \\
& 8x_1 + 3x_2 + 15x_3 + 8x_4 + 10x_5 \le 113.312 \\
& x_1, x_2, x_3, x_4, x_5 \ge 0
\end{aligned}
\tag{15}
$$

Their tolerant limits are not described, so the payoff table is obtained as follow:

Table 2. Payoff table of ideal solution of example 1.

	F_1	F_2
$\min\ F_1(x)$	-86.021	135.769
$\min\ F_2(x)$	0	0

From the payoff table $F_1^o = 0$ **,** and F_2^o =135.769,

Then the membership function of the two objective functions can be defined as follows:

$$
\mu_1(x) = \begin{cases} 1 & \text{if} \quad F_1(x) \le -80 \\ \dfrac{0 - F_1(x)}{0 + 80} & \text{if} \quad -80 \le F_1(x) \le 0 \\ 0 & \text{if} \quad F_1(x) \ge 0 \end{cases}
$$

$$
\mu_2(x) = \begin{cases} 1 & \text{if} \quad F_2(x) \le 10 \\ \dfrac{135.769 - F_2(x)}{125.769} & \text{if} \quad 10 \le F_2(x) \le 135.769 \\ 0 & \text{if} \quad F_2(x) \ge 135.769 \end{cases}
$$

- Solve the min operator model as described in problem (8) with this optimal solution .

$x^o = [0.9475, 0, 0, 0, 6.96]^T$, $\alpha^* = 0.616$

$F_1(x^o) =$ -49.36, $\quad \mu_1(x^o) =$ 0.617

$F_2(x^o) =$ 58.22, $\quad \mu_2(x^o) =$ 0.617

Let h=1.

According to step 3. The DM is ask to determine the compromise degree α' with $\alpha' \in [0, 0.617]$. The following compromise model will be solved.

$$Max\ \tilde{\alpha} = \frac{1}{2}[0.0205x_1 - 0.0318x_2 - 0.0693x_3 - 0.06132x_4 + 0.01934x_5 + 1.079511]$$

$$\begin{aligned} s.t.\quad & 8x_1 + 7x_2 + 4x_3 + 4x_4 + 6x_5 \le 80 \\ & 8x_1 + 7x_2 + 4x_3 + 4x_4 + 6x_5 \ge 80\alpha' \\ & -10x_1 - 15x_2 - 15x_3 - 14x_4 - 7x_5 \le -10 \\ & -10x_1 - 15x_2 - 15x_3 - 14x_4 - 7x_5 \ge 125.769\alpha' - 135.769 \\ & x \in S \end{aligned} \qquad (16)$$

Where S is the feasible region described in problem (15).

Let the optimal solution of problem (16) be $\bar{x}$.

With this solution the algorithm calculates the obective values $F_i(\bar{x})$ and the corresponding membership degree $\mu_i(\bar{x})$, and the DM is shown these results.

Table 3. The results of the compromise solution of iteration 1,for example 1.

α'	$F_1(\bar{x})$	$\mu_1(\bar{x})$	$F_2(\bar{x})$	$\mu_2(\bar{x})$
0.000	-80.002	1	98.117	0.299
0.300	- 80.400	1	98.600	0.296
0.500	-61.000	0.763	72.940	0.500
0.600	-51.000	0.638	60.300	0.600
0.616	-49.440	0.618	58.320	0.616

- **At** $\alpha' = 0$

The result of $F_2(x)$ is not satisficind to DM. Thus, the algorithm improve it by relaxing $F_1(x)$.According to the original tolerance of $F_1(x)$ and the optimization result, the new tolerant limit $\hat{F}_1^o$ is determined, as follow: $-80.002 < \hat{F}_1^o \le 0$

Assume that the new tolerant limit $\hat{F}_1^o = -60$

The following auxiliary problem will be solved:

$$\begin{aligned} Min\quad & F_2'(x) \\ s.t\quad & F_1'(x) \le -60 \\ & x \in S \end{aligned}$$

The result of $F_2(\hat{x}) = 71.679$, is satisficing such that its new tolerant limit is taken as $71.679 < \hat{F}_2^o < 98.117$, let $\hat{F}_2^o$=80.Then the corresponding membership function of the two objective functions are rewritten as:

$$\mu_{1new}(x) = \begin{cases} 1 & if\quad F_1(x) \le -80 \\ \dfrac{-60 - F_1(x)}{20} & if\quad -80 \le F_1(x) \le -60 \\ 0 & if\quad F_1(x) \ge -60 \end{cases}$$

$$\mu_{2new}(x)=\begin{cases}1 & \text{if} \quad F_2(x)\le 10\\ \dfrac{80-F_2(x)}{70} & \text{if} \quad 10\le F_2(x)\le 80\\ 0 & \text{if} \quad F_2(x)\ge 80\end{cases}$$

Let h=h+1.

with the above new membership function of the two objective functions, the algorithm solve minimum operrator model (8) again, and obtain the optimal solution

$x^o_{new}=[2.772,0,0,0,6.596]^T$, $\alpha^*_{new}=0.0873$

$F_1(x^o_{new})=$ -62, $\mu_1(x^o_{new})=$ 0.775

$F_2(x^o_{new})=$ 74.2, $\mu_2(x^o_{new})=$ 0.490

$\alpha'_{new}\in[0,0.0873]$ may be considered as a compromise index for all membership functions, with α^* is the optimization result of the min operator method**,** solve the compromise model again (9).

Table 4. The improved results of the compromise solution of iteration 2,for example 1.

α'_{new}	$F_1(x'_{new})$	$\mu_1(x'_{new})$	$F_2(x'_{new})$	$\mu_2(x'_{new})$
0.00	-67.00	0.840	80.50	0.440
0.02	-65.56	0.820	78.70	0.454
0.04	-64.40	0.805	77.24	0.470
0.06	-63.22	0.790	75.75	0.477
0.087	-61.77	0.772	73.915	0.500

Assume that the DM is satisfied with these results.

Example.2

$$\begin{aligned} &Max \quad F_1(x)=10x_1+11x_2+15x_3\\ &Max \quad F_2(x)=4x_1+5x_2+9x_3\\ &s.t. \quad g_1(x)=x_1+x_2+x_3\le \tilde{15}\\ &\qquad g_2(x)=7x_1+5x_2+3x_3\le \tilde{80}\\ &\qquad g_3(x)=3x_1+4.4x_2+10x_3\le \tilde{100}\\ &\qquad\qquad x_1,x_2,x_3\ge 0\end{aligned} \tag{17}$$

where the fuzzy resources with the corresponding maximal tolerances are $p_1=5$, $p_2=40$, and $p_3=30$.

▪ *According to step 0,* the membership function of the two objective functions can be estimated by solving problems (10) and (11) ,obtaining the possible range F^o_i, F^1_i for each objective function:

F^o_1, F^1_1 = [189.2861, 250],

F_2^o, F_2^1 = [99.286, 130].
Then the membership function of the two objective functions can be defined as follows:

$$\mu_1(x)=\begin{cases}1 & \text{if} \quad F_1(x)>250\\ \dfrac{F_1(x)-189.286}{60.714} & \text{if} \quad 189.286\le F_1(x)\le 250\\ 0 & \text{if} \quad F_1(x)<189.286\end{cases}$$

$$\mu_2(x)=\begin{cases}1 & \text{if} \quad F_2(x)>130\\ \dfrac{F_2(x)-99.286}{30.714} & \text{if} \quad 99.286\le F_2(x)\le 130\\ 0 & \text{if} \quad F_2(x)<99.286\end{cases}$$

For each of fuzzy constraints, the non-increasing linear membership functions are designed to as follows.

$$\mu_{g_1}=\begin{cases}1 & \text{if} \quad g_1(x)<15\\ \dfrac{20-g_1(x)}{5} & \text{if} \quad 15\le g_1(x)\le 20\\ 0 & \text{if} \quad g_1(x)>20\end{cases}$$

$$\mu_{g_2}=\begin{cases}1 & \text{if} \quad g_2(x)<80\\ \dfrac{120-g_2(x)}{40} & \text{if} \quad 80\le g_2(x)\le 120\\ 0 & \text{if} \quad g_2(x)>120\end{cases}$$

$$\mu_{g_3}=\begin{cases}1 & \text{if} \quad g_3(x)<100\\ \dfrac{130-g_1(x)}{30} & \text{if} \quad 100\le g_3(x)\le 130\\ 0 & \text{if} \quad g_3(x)>130\end{cases}$$

▪ When the membership functions of each objective and fuzzy constraint are determined, the min operator model as described in problem (8) will be ready to solve with this optimal solution .

$x^o=[4.04762,5.65476,7.79762]$, $\alpha^*=0.5$

$F_1(x^o)=219.655$, $\mu_1(x^o)=0.5$

$F_2(x^o)=114.649$, $\mu_2(x^o)=0.5$

$\mu_{g_1}(x^o)=0.4998$, $\mu_{g_2}(x^o)=1$, $\mu_{g_3}(x^o)=0.4998$

▪ *According* to step 3. The DM is ask to determine the compromise degree α' with $\alpha'\in[0,0.5]$. The compromise model (9) will be solved, with optimal solution $\bar{x}$.
With this solution the algorithm calculates the obective values $F_i(\bar{x})$ and the corresponding membership degree $\mu_i(\bar{x})$, and the DM is shown these results.

Table 5. The results of the compromise solution of iteration 1,for example 2.

α'	$F_1(\bar{x})$	$\mu_1(\bar{x})$	$F_2(\bar{x})$	$\mu_2(\bar{x})$	$\mu_{g1}(\bar{x})$	$\mu_{g2}(\bar{x})$	$\mu_{g3}(\bar{x})$
0.0	231.973	0.703	125.741	0.861	0.459	1	0.00
0.1	228.289	0.642	123.236	0.780	0.498	1	0.10
0.3	220.905	0.521	118.215	0.616	0.577	1	0.30
0.5	219.655	0.500	114.649	0.500	0.500	1	0.50

The algorithm will try to improve the above results using the attainable reference point method as follow:

- **At** $\alpha' = 0$

The result of $F_1(x)$ is not satisficind to DM. Thus, the algorithm improve it by relaxing $F_2(x)$.According to the original tolerance of $F_2(x)$ and the optimization result, the new tolerant limit $\hat{F}_2^o$ is determined, as follow: $99.286 < \hat{F}_2^o \le 125.741$

Assume that the new tolerant limit $\hat{F}_2^o = 125$

The following auxiliary problem will be solved:

$$\begin{aligned} Max \quad & F_1(x) = 10x_1 + 11x_2 + 15x_3 \\ s.t \quad & F_2(x) \le 125 \\ & (Ax)_j \le b_j + p_j \ , \quad j = 1,\ldots,m \ , \\ & x \ge 0 \end{aligned} \tag{18}$$

The result of $F_1(\hat{x}) = 245$, is satisficing such that its new tolerant limit is taken as $231.973 \le \hat{F}_1^o \le 245$, let $\hat{F}_1^o = 235$.Then the corresponding membership function of the two objective functions are rewritten as:

$$\mu_1(x) = \begin{cases} 1 & if \quad F_1(x) > 250 \\ \dfrac{F_1(x) - 235}{15} & if \quad 235 \le F_1(x) \le 250 \\ 0 & if \quad F_1(x) < 235 \end{cases}$$

$$\mu_2(x) = \begin{cases} 1 & if \quad F_2(x) > 130 \\ \dfrac{F_2(x) - 125}{5} & if \quad 125 \le F_2(x) \le 130 \\ 0 & if \quad F_2(x) < 125 \end{cases}$$

with the above new membership function of the two objective functions, the algorithm solve minimum operrator model (8) again, and obtain the optimal solution

$\alpha^* = 0.14$, $\alpha' \in [0, 0.14]$ may be considered as a compromise index for all membership functions, with α^* is the optimization result of the min operator method**,** solve the following compromise model again.

$$
\begin{aligned}
Max \quad & \tilde{\alpha} = \tfrac{1}{5}\left(0.991667x_1 + 1.261667x_2 + 2.191667x_3 - 29.333\right) \\
s.t. \quad & 10x_1 + 11x_2 + 15x_3 \le 250 \\
& 10x_1 + 11x_2 + 15x_3 - 15\alpha' \ge 235 \\
& 4x_1 + 5x_2 + 9x_3 \le 130 \\
& 4x_1 + 5x_2 + 9x_3 - 5\alpha' \ge 125 \\
& x_1 + x_2 + x_3 \ge 15 \\
& x_1 + x_2 + x_3 + 5\alpha' \le 20 \\
& 7x_1 + 5x_2 + 3x_3 \ge 80 \\
& 7x_1 + 5x_2 + 3x_3 + 40\alpha' \le 120 \\
& 3x_1 + 4.4x_2 + 10x_3 \ge 100 \\
& 3x_1 + 4.4x_2 + 10x_3 + 30\alpha' \le 130 \\
& x_1, x_2, x_3 \ge 0
\end{aligned} \qquad (19)
$$

Table 6. The improved results of the compromise solution of iteration 2, for example 2.

α'	$F_1(x'_{new})$	$\mu_1(x'_{new})$	$F_2(\hat{x}_{new})$	$\mu_2(x'_{new})$	$\mu_{g_1}(x'_{new})$	$\mu_{g_2}(x'_{new})$	$\mu_{g_3}(x'_{new})$
0.00	250.00	1.00	130.00	1.000	0.00	0.880	0.00
0.05	246.99	0.95	128.48	0.951	0.05	0.903	0.05
0.10	243.97	0.90	126.96	0.901	0.10	0.930	0.10
0.14	241.50	0.86	125.70	0.860	0.14	0.953	0.14

5. Conclusions

In this paper, we make a study of the multiobjective linear programming problems with Fuzzy goals in the objectives and constraints and study compromise model introduced by Yan Wu et.al,[8].
This compromise model to improve the solution yielded by min operator. Moreover, to generate fuzzy-efficient solutions between non-compensatory and fully compensatory, and make an improvement of the generated fuzzy-efficient solutions by compromise model. The improvemrnt is based on reconstructing the membership functions by changing tolerances of the objectives using the principle of the interactive Attainable Reference Point Method introduced by Xiaomin M. Wang et.al,[16], to guarantee the optimization problem feasibility
The optimization results of the numerical examples show that this method can get improved results to the compromise model.
In addition to the simple example in this paper, *The proposed method can be applied to an engineering applications.*

REFERENCES

[1] Y. J., Lai, and C. L. Hwang, Fuzzy Multiple Objective Decision Making, Springer, New York (1994).

[2] Y. J., Lai, and C. L. Hwang, Fuzzy Mathematical Programming, Springer, New York (1992).

[3] H. Rommelfanger. Fuzzy linear programming and applications, European Journal of Operational Research, vol. 92, pp. 512-528, (1996).

[4] H. J., Zimmermann, “Fuzzy mathematical Computer and Operations Research, 10(4), 45-55 (1983).

[5] H. J., Zimmermann, “Fuzzy programming and linear programming with several objective functions,” Fuzzy Sets and Systems, 1, 45-55 (1978).
[6] S. M. Guu, and Y. K. Wu, “Weighted coefficients in two-phase approach for solving the multiple objective programming problems,” *Fuzzy Sets and Systems*, 85, 45-48 (1997).
[7] E. S. Lee, and R. J. Li, “Fuzzy multiple objective programming and compromise programming with Pareto optimum,” *Fuzzy Sets and Systems*, **53**, 275-288 (1993).
[8] Yan-Kuen Wu. and Sy-Ming Guu " A Compromise Model for Solving Fuzzy Multiple Objective Linear Programming Problems ", Journal of the Chinese Institute of Industrial Engineers, Vol. 18, No. 5, pp. 87-93 (2001)
[9] M. Sakawa. Fuzzy Sets and Interactive Multiobjective Optimization, Plenum Press, NewYork, (1993).
[10] M. Sakawa, K. Kato, and T. Suzuki. "An interactvie fuzzy satisficing method for multiobjective non-convex programming problems through genetic algorithms", Proceedings of 8th Japan Society for Fuzzy Theory and Systems Chugoku / Shikoku Branch Office Meeting, pp. 33-36, (2002).
[11] T. Matsui, M. Sakawa, K. Kato, T. Uno, K. Tamada.. "An interactive fuzzy satisficing method through particle swarm optimization for multiobjective nonlinear programming problems", Proceedings of the 2007 IEEE Symposium Series on Computational Intelligence, pp. 71-76, (2007)
[12] B., Werners, " An interactive fuzzy programming ", Fuzzy Sets and Systems, 23, 131-147 (1987).
[13] S.Y. Li, Y.P. Yang, and C.J. Teng, Fuzzy goal programming with multiple priorities via generalized varying-domain optimization method, IEEE Trans. Fuzzy Syst., pp. 596-605, (2004).
[14] S.Y.,Chaofang, and S. Li, "An interactive satisficing method based on alternative tolerance for fuzzy multiple objective optimization", Applied Mathematical Modelling 33, pp. 1886–1893, (2009).
[15] E.S. Lee, R.J. Li, Fuzzy multiple objective programming and compromise programming with Pareto optimum, Fuzzy Sets Syst. 53 (1993) 275–288.
[16] M.X.,Wang, Z L. Qin, and Y.D. Hu, "An Interactive Algorithm for Multicriteria Decision Making: The Attainable Reference Point Method". IEEE TRANSACTIONS ON SYSTEMS, MAN, AND CYBERNETICS—PART A: SYSTEMS AND HUMANS, VOL. 31, NO. 3, MAY (2001).
[17] K., Miettinen, "Nonlinear Multiobjective Optimization". Kluwer Academic Publishers, Massachusetts, (1999).
[18] P. Vincke. Multicriteria Decision-Aid. John Wiley & Sons, Inc., Chichester,(1992).
[19] V,Chankong and Haims YY, Multiobjective Decision Making: Theory and Methodology, North-Holland, Amsterdam, (1983).
[20] J.M. Cadenas, J.L. Verdegay," Using ranking functions in multiobjective fuzzy linear programming ", Fuzzy Sets and Systems 111, pp. 47-53, (2000).
[21] B., Werners, “Mathematical models for decision in Aggregation Models in Mathematical Programming, G. Mitra (ed.), Spring, Berlin (1988).