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Parallel Initial-Value Algorithm for Quasilinear Shock Problems with Turning Points

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Abstract

In this paper, a parallel initial value algorithm is presented for quasilinear stationary shock problems with turning points exhibiting two-boundary layers or an internal layer. The method is based on obtaining two independent asymptotically equivalent first-order singularly-perturbed initial-value problems (SPIVPs) of the original problem. The error is estimated to be of order ε. The two SPIVPs are modified to obtain two boundary-layer correction problems. These non-stiff initial-value problems are solved simultaneously in parallel process using (RKV45) non-stiff code integrator. The obtained solutions are combined to approximate the solution of the original problem. Numerical experiments indicate the high accuracy and the efficiency of the method. Furthermore, the accuracy of numerical results improves as the small parameter ε tends to zero

Keywords: quasilinear singular perturbation problems; turning points problems; two boundary layer; internal layer; Initial value methods, parallel algorithms.

1. Introduction

Consider the quasilinear singular perturbation problem of the general form:

$$\varepsilon y'' + p(x,y)y' - q(x,y) = 0, \qquad x \in I = [-1,1], \tag{1.1.a}$$

$$y(-1) = A_-, \quad \text{and} \quad y(1) = A_+. \tag{1.1.b}$$

where ε is a small positive parameter $(0 < \varepsilon \ll 1)$, A_- and A_+ are given constants, $p(x,y)$, $q(x,y)$ are assumed to be sufficiently continuously differentiable functions for $x \in I$ and $y \in \mathbb{R}$.
Equation (1.1.a) can be rewritten in the conservative form

$$\varepsilon y'' + f(x,y)' - g(x,y) = 0, \qquad x \in [-1,1], \tag{1.2}$$

where

$$f_y(x,y) = p(x,y), \quad g(x,y) = q(x,y) + f_x(x,y).$$

Assume that

$$g_y(x,y) = p_x(x,y) + q_y(x,y) \ge 0, \quad x \in I, \quad y \in \mathbb{R}. \tag{1.3}$$

Condition (1.3) grantees that problem (1.1) has a unique solution y_ε [1]. As $\varepsilon \to 0$, y_ε tends to a bounded variation function, u, which may be discontinuous at some point $x^* \in (-1,1)$, corresponding to the location of a stationary shock [2]. For instance; u, defined in this way, is considered a solution of the reduced problem which results from (1.1) when ε is formally set to zero. Recently, Osher [3], Abrahamsson and Osher [4], Lorenz [5], [6] and Niijima [7] treated problem (1.1) for the case of two boundary layers where

$p(x,y)\equiv p(x)$. Vulanovic [8] discussed the case in which the coefficient of y' is written as $x\;p(x,y)$ with $p(x,y)\geq\beta>0$, and $q(x,y)\equiv 0$. Lin [9] treated problem (1.1) at a turning point $x=0$. Lorenz [10] treated problem (1.1) with an internal layer for the case of $p(x,y)\equiv p(y)$. Now, we want to achieve the following tasks. Firstly, deal with problem (1.1) in its general form with two-boundary layer or an internal layer and unknown turning point position. Secondly, reduce the order of the original problem to be solved as non-stiff first order initial value problems. Thirdly, extend parallel techniques to be applied on the general form of quasilinear problem (1.1) with turning points.

To reduce the order of problem (1.1) in its general form, we should take into account that, there are many techniques reduced the order of problem (1.1) without turning point. For example, Gasparo and Macconi [11] and Y.N. Reddy, P. P. Chakravarthy [12] replaced (1.1) by two equivalent IVP in two cases $p(x,y)\equiv p(x)$ or $p(x,y)\equiv p(y)$ .M.K.Kadabajoo, Y.N.Reddy [13] replaced (1.1) by an equivalent IVP in the case $p(x,y)\equiv p(y)$. Y.N. Reddy, P. Pramod Chakravarthy [14] treated the linear case by three equivalents IVP. Habib and El-Zahar [15] considered a semi-linear SPP which was integrated to obtain a scalar first-order initial-value problem. A variable step size technique using locally exact integration was implemented to solve the obtained first order (IVP). They later [16] treated the general case $p(x,y)\equiv p(x,y)$ and showed that the obtained formula is general to those in [11, 12, 13, 15]. Recently [17] they obtained an improved initial value technique to that in [16] to obtain the exact equivalent (IVP), under certain conditions. Therefore, we will reduce the order of problem (1.1) using a special case of that technique presented in [16] for problem (1.1) without turning point. The method is based on obtaining two independent asymptotically equivalent first-order SPIVPs of the original problem with bounded error of order ε .These SPIVPs are modified to obtain two independent boundary-layer correction problems. These non-stiff problems are solved simultaneously in parallel process using a non-stiff solver and the obtained solutions are combined to approximate the solution of the original problem. Numerical experiments indicate the high accuracy and the efficiency of the method. Furthermore, the accuracy of numerical results improves as the small parameter ε tends to zero

2. Turning point problems with two-boundary layer.

Consider problem (1.1) with

$$p(x,y),\qquad q(x,y)\in C^2\big([-1,1]\times \Re\big), \tag{2.1.a}$$

$$p(x^*,y)=0,\qquad p_x(x^*,y)<0,\;\; x^*\in(-1,1)\;,\quad y\in\Re\;, \tag{2.1.b}$$

$$p(x,y)\neq 0,\qquad \text{for } x\neq x^*,\quad y\in\Re\;, \tag{2.1.c}$$

$$q_y(x,y)\geq q_0>0,\qquad \text{on }\big([-1,1]\times\Re\big). \tag{2.1.d}$$

Thus, there is a turning point at $x=x^*$. Furthermore; problem (1.1) has a unique solution y services exponential boundary layer at each end point. The problem can be considered as two SPPs, each with a single boundary layer. Therefore, the solution y consists of two smooth curves y_+ *and* y_- that satisfy

$$\varepsilon y''_\pm+p(x_\pm,y_\pm)\,y'_\pm-q(x_\pm,y_\pm)=0,\qquad x_\pm\in I_\pm, \tag{2.2.a}$$

$$y_-(-1)=A_-,\;\; y_+(1)=A_+. \tag{2.2.b}$$

where $x_-\in I_-=[-1,x^*],\;x_+\in I_+=[x^*,1],\;y_-=y(x_-),\;y_+=y(x_+)$.

Problem (2.2) has a limited solution $u_\pm$ satisfying the reduced problem

$$p(x_\pm,u_\pm)u'_\pm-q(x_\pm,u_\pm)=0,\quad q(x^*,\,u_\pm(x^*))=0\;, \tag{2.3}$$

under condition (2.1) $u_-=u_+$.

2.1. Derivation of the Approximate Equation

Rewriting Eq (2.2.a) in the conservative form

$$\varepsilon y''_{\pm}+\frac{d}{dx_{\pm}}f(x_{\pm},y_{\pm})-g(x_{\pm},y_{\pm})=0, \quad x_{\pm}\in I_{\pm}, \tag{2.4}$$

where

$$f_{y_{\pm}}(x_{\pm},y_{\pm})=p(x_{\pm},y_{\pm}), \quad g(x_{\pm},y_{\pm})=q(x_{\pm},y_{\pm})+f_{x_{\pm}}(x_{\pm},y_{\pm}). \tag{2.5}$$

We can simply replace $y_{\pm}$ by $u_{\pm}$ in the $g(x_{\pm},y_{\pm})$ term of Eq (2.4) to obtain the approximate equation

$$\varepsilon y''_{\pm}+\frac{d}{dx_{\pm}}f(x_{\pm},y_{\pm})-g(x_{\pm},u_{\pm})=0, \quad x_{\pm}\in I_{\pm}\ , \tag{2.6}$$

Integrating Eq. (2.6) w.r.t $x_{\pm}$ we get

$$\varepsilon y'_{\pm}+f(x_{\pm},y_{\pm})=\int^{x_{\pm}} g(s,u(s))ds+k\ , \tag{2.7}$$

where k is the constant of integration.

Substituting Eq. (2.3) and Eq. (2.5) into Eq. (2.7), we get

$$\begin{aligned}\varepsilon y'_{\pm}+f(x_{\pm},y_{\pm})&=\int^{x_{\pm}}\left[q(s,u(s))+f_s(s,u(s))\right]ds+k\\ &=\int^{x_{\pm}}\left[p(y_{\pm},u(s))\frac{du(s)}{ds}+f_s(s,u(s))\right]ds+k\\ &=\int^{x_{\pm}}\left[f_u(s,u(s))\frac{du(s)}{ds}+f_s(s,u(s))\right]ds+k\end{aligned}$$

or

$$\varepsilon y'_{\pm}+f(x_{\pm},y_{\pm})=f(x_{\pm},u_{\pm})+k. \tag{2.8}$$

In order to determine k, we impose the condition that the reduced equation of (2.8) should satisfy the boundary condition $y_{\pm}(x^*)=u_{\pm}(x^*)+O(\varepsilon)=\lambda.$ i.e.

$$\begin{aligned}&f(x^*,\lambda)=f(x^*,\lambda-O(\varepsilon))+k\\ &f_y(x^*,\xi)\big(O(\varepsilon)\big)=k, \quad \lambda\le\xi\le\lambda+O(\varepsilon)\\ &p(x^*,\xi)\big(O(\varepsilon)\big)=k.\end{aligned}$$

Thus $k=O(\varepsilon)$

Upon integrating the first part of (2.5) and inserting it into Eq. (2.8) we obtain two independent SPIVPs

$$\varepsilon y'_{-}=\int_{y_{-}}^{u_{-}} p(x_{-},s)\,ds, \quad y_{-}(-1)=A_{-} \tag{2.9.a}$$

$$\varepsilon y'_{+}=\int_{y_{+}}^{u_{+}} p(x_{+},s)\,ds, \quad y_{+}(1)=A_{+} \tag{2.9.b}$$

or simply

$$\varepsilon y'_{-}+f(x_{-},y_{-})=f(x_{-},u), \quad y_{-}(-1)=A_{-}\ ,\ x_{-}\in[-1,x^*]\ , \tag{2.10.a}$$

$$\varepsilon y'_{+}+f(x_{+},y_{+})=f(x_{+},u), \quad y_{+}(1)=A_{+}\ ,\ x_{+}\in[x^*,1]\ , \tag{2.10.b}$$

Equations (2.10) is first order equations, which are asymptotically equivalent to the second-order nonlinear singularly-perturbed two-point boundary-value problem (2.2).

2.2. Error estimate

From (2.1.a, b, c) it is easily proved that there exists a small positive numbers δ, β_0 and β_1 independent of ε such that

$$p(x,y) \ge \beta_0 > 0, \qquad x \in [-1,-\delta] \tag{2.11.a}$$

$$p(x,y) \le -\beta_1 < 0, \qquad x \in [\delta, 1-\delta] \tag{2.11.b}$$

Then for the solution y of Eq (2.2) we have the following bound

LEMMA 1. *Assume that (2.1) holds true. Then* form [1]

$$|y(x) - u(x)| \le C\varepsilon, \qquad |x| \le \delta$$

$$|y(x) - u(x)| \le C\left[\varepsilon + e^{-m(1 \mp x)/\varepsilon}\right], \qquad x \in [\pm 1, \pm\delta].$$

where m is a positive constant independent of ε.

LEMMA 2. *Let condition (2.1) be satisfied. Then problem (2.2) can be reduced to the following asymptotic initial-value problem*

$$\varepsilon y'_{\pm} = \int_{y_{\pm}}^{u_{\pm}} p(x_{\pm}, s)\, ds + O(\varepsilon), \qquad y(\pm 1) = A_{\pm}$$

Proof. Integrating Eq. (2.4) w.r.t $x_{\pm}$ results in

$$\varepsilon y'_{\pm} + f(x_{\pm}, y_{\pm}) = \int^{x_{\pm}} g(s, y(s))ds + k = G(x_{\pm}) + k. \tag{2.12}$$

The solution of problem (2.12) is the same as that of (2.2) for $x_{\pm} \in I_{\pm}$. In what follows we construct an approximate solution of (2.12). Let

$$G(x_{\pm}) = \int^{x_{\pm}} g(s, y(s))ds, \qquad \overline{G}(x_{\pm}) = \int^{x_{\pm}} g(s, u(s))ds.$$

Replacing $G(x_{\pm})$ *by* $\overline{G}(x_{\pm})$ in the above expression and from LEMMA1 we get the following bounded error equation.

$$\left|G(x_{\pm}) - \overline{G}(x_{\pm})\right| = \left|\int^{x_{\pm}} \big(g(s, y(s)) - g(s, u(s))\big)ds\right|.$$

$$\le \int^{|x_{\pm}|} \left|g_y(s, \xi_{\pm})\right| \, |y(s) - u(s)| ds \;\le \begin{cases} C\,|x|\,\varepsilon & for \;\; |x| \le \delta \\ C\,\varepsilon & for \;\; \delta \le |x| \le 1 \end{cases}$$

Where $\xi_{\pm}$ lies between $u(x_{\pm})$ and $y(x_{\pm})$

Equation (2.12) can be approximated by Eq (2.7) with $O(\varepsilon)$, which results in asymptotic first-order initial-value problems (2.10).

3. Interior shock problems

The following summarizes some results from [2,18, 19].
Consider problem (1.1) with

$$p(x^*, y^*) = 0, \quad q_y(x,y) \ge q_* > 0, \qquad x \in I, \;\; y \in ‘ \,. \tag{3.1}$$

The solution has a shock layer at $x = x^*$ where $x^* \in (-1,1)$, $y^* = y(x^*)$. The problem has a solution consisting of two smooth curves y_+ *and* y_-, which satisfy

$$\varepsilon y''_{\pm} + p(x_{\pm},y_{\pm})\, y'_{\pm} - q(x_{\pm},y_{\pm}) = 0,\ y_-(x^*) = y_+(x^*) = y^*. \tag{3.2}$$

The corresponding reduced problem has a discontinuous solution consisting of two smooth curves, u_+ *and* u_-, which satisfy:

$$p(x_{\pm},u_{\pm})\, u'_{\pm} - q(x_{\pm},u_{\pm}) = 0,\quad u_-(-1) = A_-,\quad u_+(1) = A_+. \tag{3.3}$$

Applying the proposed method to Eq (2.9) through Eq (3.2), we obtain two independent SPIVPs, which can be solved separately as follows.

$$\varepsilon y'_- = \int_{y_-}^{u_-} p(x_-,s)\, ds, \qquad y_-(x^*) = y^*, \tag{3.4.a}$$

$$\varepsilon y'_+ = \int_{y_+}^{u_+} p(x_+,s)\, ds, \qquad y_+(x^*) = y^*, \tag{3.4.b}$$

or simply

$$\varepsilon y'_- + f(x_-,y_-) = f(x_-,u_-), \qquad y_-(x^*) = y^*,\ x_- \in [0,x^*], \tag{3.5.a}$$

$$\varepsilon y'_+ + f(x_+,y_+) = f(x_+,u_+), \qquad y_+(x^*) = y^*,\ x_+ \in [x^*,1], \tag{3.5.b}$$

To obtain the position of the turning point x^*, we proceed in the following way.

Recalling the continuity conditions at the turning point x^*, $y_-(x^*) = y_+(x^*)$ *and* $y'_-(x^*) = y'_+(x^*)$ in Eq (3.4), to obtain

$$\int_{u_-(x^*)}^{u_+(x^*)} p(x^*,s)\, ds = 0\ , \tag{3.6}$$

4. Parallel initial value algorithm

The SPIVPs Eq.(2.10.a) and Eq.(2.10.b) or Eq. (3.5.a) and Eq.(3.5.b) could be rewritten as

$$\varepsilon Y' = \psi(X,Y),\ Y(X_0) = Y_0 \qquad Y_0 = \begin{cases} A_m & at\ X_0 = m1\ \ with\ Eq(2.10) \\ y^* & at\ X_0 = x^*\ \ with\ Eq(3.5) \end{cases} \tag{4.1}$$

where $X \equiv x_{\pm}$, $Y \equiv y_{\pm}$, $Y' \equiv y'_{\pm}$.

Setting $Y(X) = U(X) + V(\tau) + O(\varepsilon)$, where $U(X) = u_{\pm}$ and $\tau = (X - X_0)/\varepsilon$ in (4.1), we obtain the boundary layer corrected equation

$$\frac{dV}{d\tau} = \psi(\varepsilon\tau + X_0, U(\varepsilon\tau + X_0), V(\tau) + U(\varepsilon\tau + X_0)) \tag{4.2}$$

with the boundary condition $U(0) + V(0) = Y_0$

letting $\varepsilon = 0$ we have

$$\frac{dV}{d\tau} = \psi(X_0, U(X_0), V(\tau) + U(X_0))\ \ with\ \ V(X_0) = Y_0 - U(X_0) \tag{4.3}$$

Notice that since the perturbation parameter ε is not present in Eq. (4.3), any standard techniques for non-stiff problems could be used. We prefer (RKV56) code of Matlab to get the solution.
In the view of practical experiments, Eq. (4.3) should be solved only over a bounded region containing the transition layer .i.e. from $\tau = X_0$ to $\tau = \tau_{out}$, where $V(\tau_{out}) \approx 0$. In our algorithm, the integration of process is stopped when $V(\tau) \le \min(10^{-7}, \varepsilon)$.

Note that. The Initial-value problem (4.3) is simple formula, so their analytical solutions can be obtained easily. This enables us from obtaining an asymptotic analytical solution for the nonlinear original problem.

These details will be performed by the following algorithms

For problems with two-boundary layer

Algorithm1

Step1. Set, $X_0 = -1$, $Y_0 = A_-$, $U(X_0) = u(-1)$, and integrate Eq. (4.3) using (RKV56) code in the forward direction and stop when $V(\tau) \le \min(10^{-7}, \varepsilon)$
Step 2. set $y_- = U(X) + V(\tau)$,
Step 3. Set, $X_0 = 1$, $Y_0 = A_+$, $U(X_0) = u(1)$,and integrate Eq. (4.3) using (RKV56, Abtol=10^{-8}) code in the backward direction and stop when $V(\tau) \le \min(10^{-7}, \varepsilon)$
Step 4. set $y_+ = U(X) + V(\tau)$,
Step 5. Combine the two solution y_- and y_+ to obtain the solution overall the problem domain

For problems with internal layer

Algorithm2

Step1. Set, $X_0 = x^*$, $Y_0 = y^*$, $U(X_0) = u(x^*)$,and integrate Eq. (4.3) using (RKV56) code in the backward direction and stop when $V(\tau) \le \min(10^{-7}, \varepsilon)$
Step 2. set $y_- = U(X) + V(\tau)$,
Step 3. Set, $X_0 = x^*$, $Y_0 = y^*$, $U(X_0) = u(x^*)$,and integrate Eq. (4.3) using (RKV56) code in the forward direction and stop when $V(\tau) \le \min(10^{-7}, \varepsilon)$
Step 4. set $y_+ = U(X) + V(\tau)$,
Step 5. Combine the two solution y_- and y_+ to obtain the solution overall the problem domain

In fact, the integration of the initial-value problems (2.10.a) and (2.10.b) or (3.5.a) and (3.5.b) are completely independent and can be performed simultaneously.

Parallel algorithm. Two-Processor

Task1. Perform **Step 1** on processor **P1**
Task2. Perform **Step 3** on processor **P2**
Task3. Perform **Step 5** using the results of **Step 2** and **Step 4**.

Task 1 and Task 2 are simultaneous and require quite the same amount of work. The computational cost of Task 3 depends on the number of output points in Task 1 and Task 2.

5. Numerical examples

In order to assess both the applicability and the accuracy of the parallel initial-value algorithm presented in this paper for singular perturbation problems, we applied it to a variety of singularly perturbed problems with turning point, as indicated in the following examples.

Example 5.1 Consider the nonlinear singular-perturbation problem [9] given by

$$\varepsilon y'' - x(y^2+1)y' - y(y^2+1) = 0 \quad x \in [-1,1]. \tag{5.1}$$

with boundary conditions $y(-1) = 1$ *and* $y(1) = 1$.
The asymptotic expansion solution is given by

$$y(x) = \frac{\sqrt{3}\exp((x^2-1)/(2\varepsilon))}{\sqrt{4-\exp((x^2-1)/\varepsilon)}}.$$

The problem has a turning point at $x^* = 0$ and $y(x^*) = 0$.The reduced problem solution is $u(x) = 0$
The corresponding initial value problems are given by

$$\varepsilon y_-' - x_-(y_-^3/3 + y_-) = 0 \quad , \quad y_-(-1) = 1, \quad x_- \in [-1,0]$$
$$\varepsilon y_+' - x_+(y_+^3/3 + y_+) = 0 \quad , \quad y_+(1) = 1, \quad x_+ \in [0,1]$$

and the corresponding corrected boundary layer problems are given by

$$V_-' + (V_-^3/3 + V_-) = 0 \quad , \quad V_-(0) = 1, \quad x_- \in [-1,0]$$
$$V_+' - (V_+^3/3 + V_+) = 0 \quad , \quad V_+(0) = 1, \quad x_- \in [0,1]$$

The parallel algorithm was applied to obtain the numerical results shown in Fig.1, Fig.2, Fig.3 and Table 1

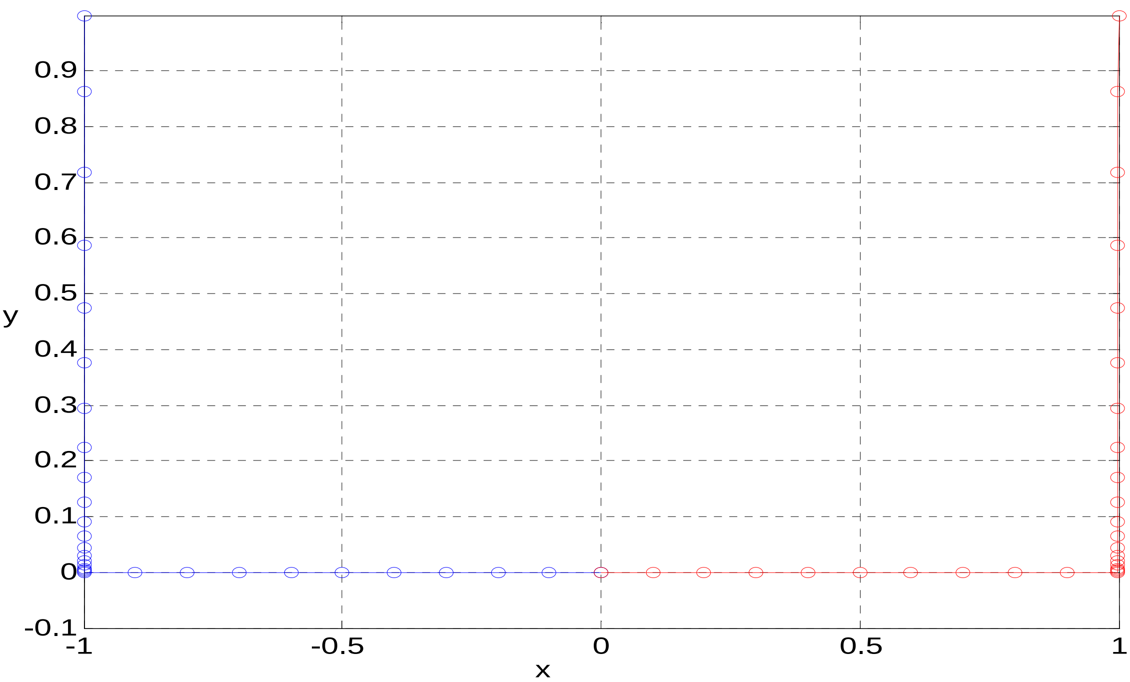

Fig.1.Shows the numerical solution of example 1, at $\varepsilon = 10^{-5}$ over the region $x \in [-1,1]$

Fig 1. Shows that the solution of the problem 1 is symmetric about the origin, and that is clear in Fig.2 and Fig3. The given exact solution and the boundary layer corrected problems (Recati Equations) confirm that. This indicates why the numerical error over the left region $x \in [-1,0]$ Fig 1 is identical to that obtained over the right region $x \in [0,1]$ Fig.2.

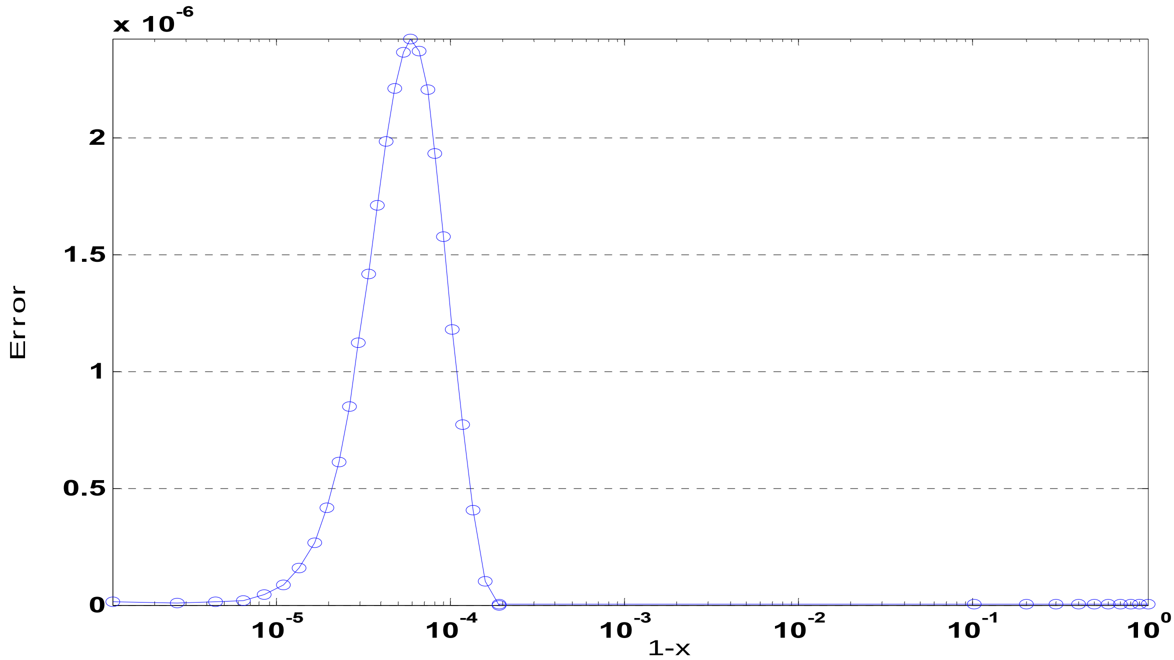

Fig.1. Error distribution of the obtained numerical solution of example 1, at $\varepsilon = 10^{-5}$ over the region $x \in [-1, 0]$

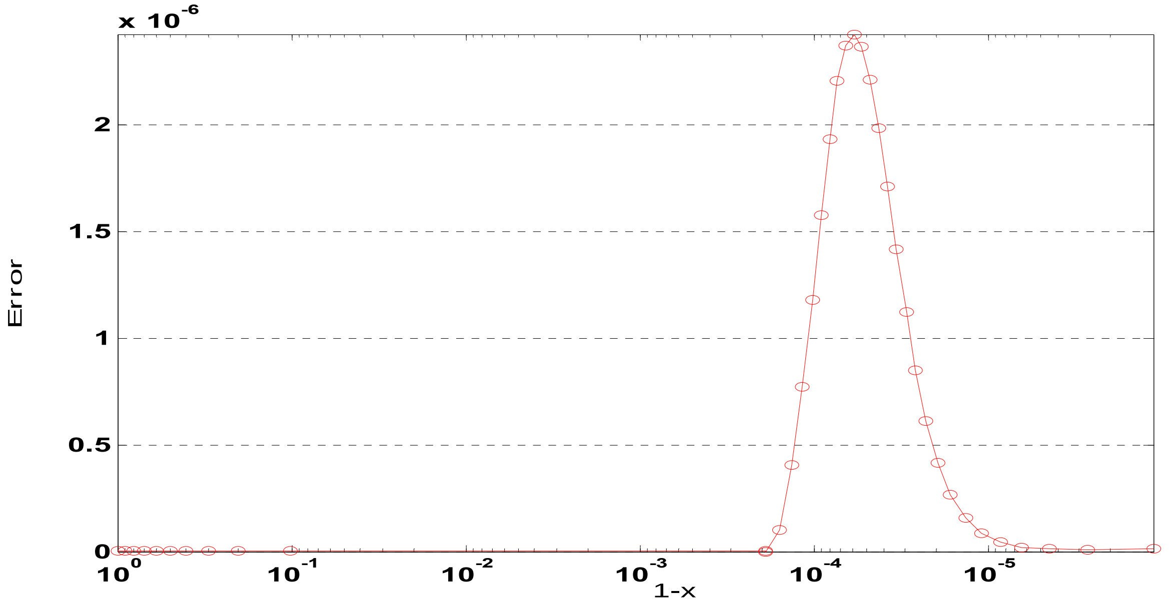

Fig.2. Error distribution of the obtained numerical solution of example 1, at $\varepsilon = 10^{-5}$ over the region $x \in [0,1]$

Example 5.2 Consider the nonlinear singular perturbation problem [18] given by

$$\varepsilon y'' + yy' - y = 0 \quad x \in [0,1], \qquad (5.2)$$
$$y(0) = -1/2 \;\; and \;\; y(1) = 1.$$

The problem has an approximate solution [18] for comparison

$$y_+ = u_+ - \frac{1}{4}\left[1 + \tanh(-(x_+ - 1/4)/8\varepsilon)\right], \qquad y_- = u_- + \frac{1}{4}\left[1 + \tanh((x_- - 1/4)/8\varepsilon)\right].$$

The reduced problem solution is $u_-(x_-) = x_- - 1/2, \quad u_+(x_+) = x_+$. Equation (3.6) is turned into a quadratic equation. Thus $x^* = 1/4$ and $y^* = [u_-(1/4) + u_+(1/4)]/2 = 0$
The corresponding initial-value problem is given by

$$\varepsilon y'_- + 0.5y_-^2 = 0.5(x_- - 1/2)^2 , \qquad y_-(1/4) = 0, \quad x_- \in [0,1/4]$$
$$\varepsilon y'_+ + 0.5y_+^2 = 0.5x_+^2 , \qquad y_+(1/4) = 0, \quad x_- \in [1/4,1]$$

and the corresponding corrected boundary layer problems are given by

$$V_-' + 0.5V_-^2 - 0.25V_- = 0 \quad , \quad V_-(0) = 0.25 ,$$
$$V_+' + 0.5V_+^2 + 0.25V_+ = 0 \quad , \quad V_+(0) = -0.25 .$$

The numerical results are presented in the following figures and table1

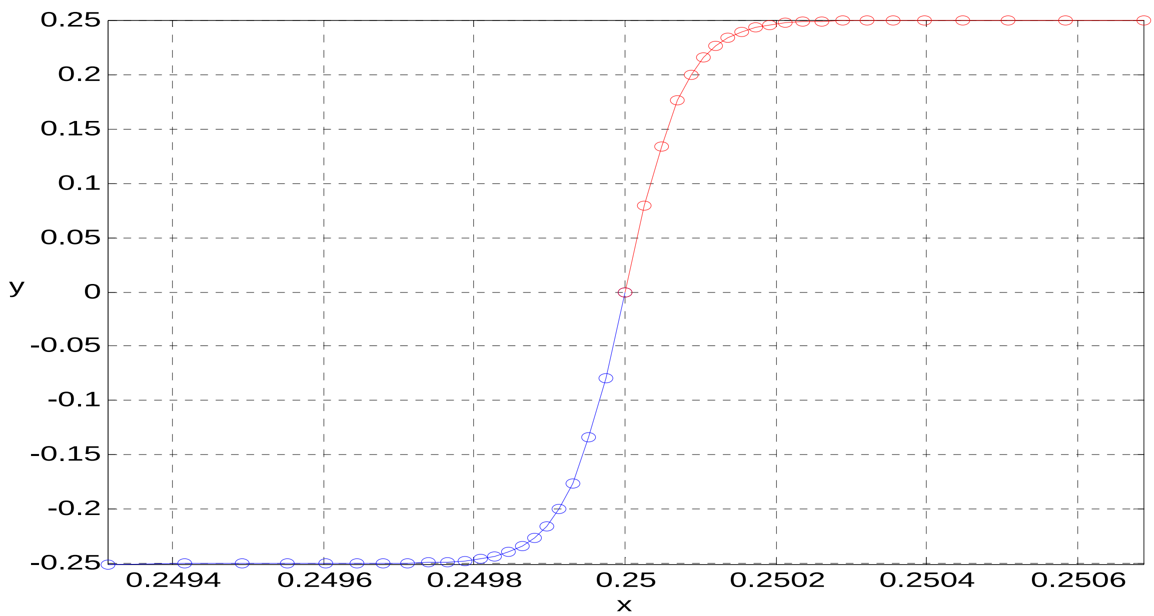

Fig.4.Shows the numerical solution of example 2, at $\varepsilon = 10^{-5}$ over the layer region

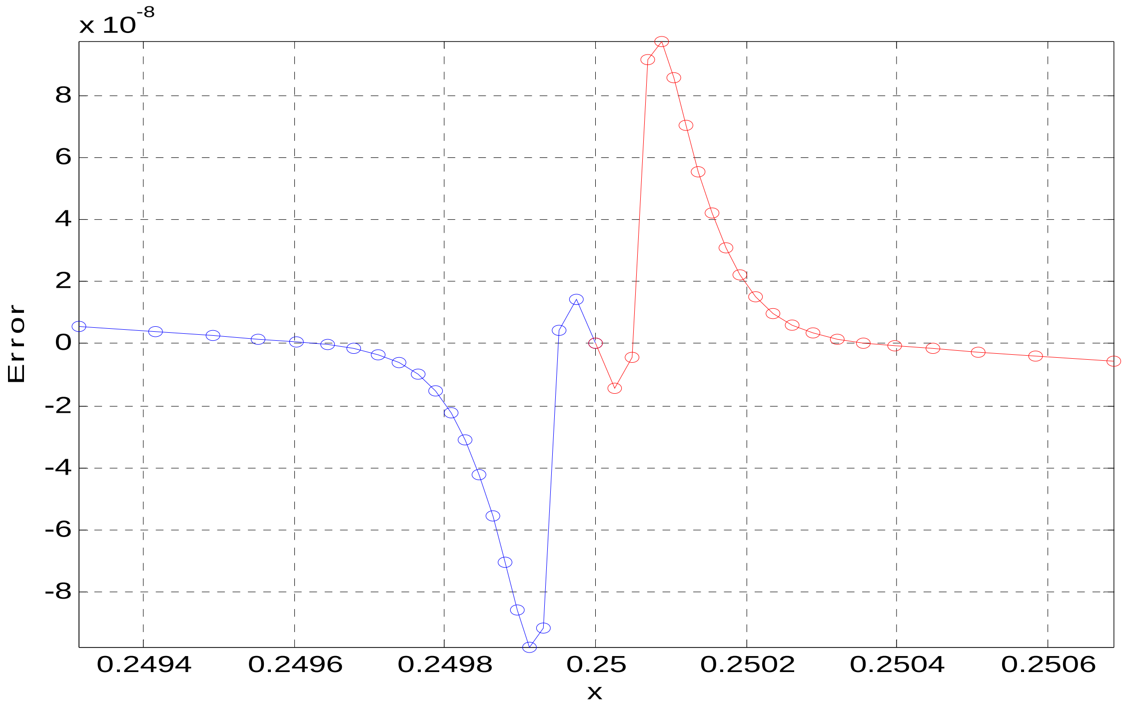

Fig.5. Error distribution of the obtained numerical solution of example 2, at $\varepsilon = 10^{-5}$ over the layer region

Example 5.3 Consider the nonlinear singular perturbation problem given by

$$\varepsilon y'' + yy' - y^3 = 0 \quad x \in [0,1], \qquad (5.3)$$
$$y(0) = 2/3 \;\; and \;\; y(1) = -1/2.$$

From Smith[20] and Whittman[21] . It is known that a shock, which occurs as the singular perturbation parameter ε decreases, causes the interior boundary layer

The problem has an approximate solution

$$y(x) \cong \begin{cases} \dfrac{2}{3-2x} - \dfrac{8e^{4(x-x^*)/5\varepsilon}}{5\left[1+e^{4(x-x^*)/5\varepsilon}\right]}, & x < x^* \\[2ex] \dfrac{-1}{1+x} - \dfrac{8e^{-4(x-x^*)/5\varepsilon}}{5\left[1+e^{-4(x-x^*)/5\varepsilon}\right]}, & x > x^* \end{cases}$$

The reduced problem solution is $u_-(x_-) = \dfrac{2}{3-2x_-}$, $\quad u_+(x_+) = \dfrac{-1}{1+x_+}$ .Equation (3.6) is turned into a quadratic equation. Thus $x^* = 1/4$ and $y^* = \left[u_-(1/4) + u_+(1/4)\right]/2 = 0$

The corresponding initial-value problem is given by

$$\varepsilon y_-' + 0.5 y_-^2 = 0.5\left(\frac{2}{3-2x_-}\right)^2, \qquad y_-(1/4) = 0, \;\; x_- \in [0, 1/4]$$

$$\varepsilon y_+' + 0.5 y_+^2 = 0.5\left(\frac{-1}{1+x_+}\right)^2, \qquad y_+(1/4) = 0, \;\; x_- \in [1/4, 1]$$

and the corresponding corrected boundary layer problems are given by

$$V_-' + 0.5V_-^2 - 0.8V_- = 0 \quad , \quad V_-(0) = 0.8 \,,$$
$$V_+' + 0.5V_+^2 + 0.8V_+ = 0 \quad , \quad V_+(0) = -0.8 \,.$$

The numerical results are presented in table1

Test Problem	$\varepsilon = 10^{-2}$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-4}$	$\varepsilon = 10^{-5}$	$\varepsilon = 10^{-6}$	$\varepsilon = 10^{-7}$
5.1	2.3793e-003	2.3582e-004	2.3568e-005	2.3617e-006	2.4128e-007	3.0000e-008
5.2	9.7828e-008	9.7828e-008	9.7828e-008	9.7828e-008	9.7828e-008	9.7828e-008
5.3	8.4296e-008	8.4296e-008	8.4296e-008	8.4296e-008	8.4296e-008	8.4296e-008

Table1. Maximum numerical solution error, at different values of ε for each test problem over the entire domain $x \in [0,1]$

The numerical results in table1 indicate how efficient is the algorithm in approximating the exact solution. Moreover, the numerical results confirm that as the small parameter ε tends to zero the error involved in the equivalent (IVPs) decreases and the approximate solution obtained improves in accuracy. In fact the analytical solution of the obtained equivalent (IVPs) of example 2 and example 3 are exactly the given problem solutions. Therefore, the numerical solver (RKV56) reaches its maximum accuracy regardless the value of the perturbation parameter ε .

6. Conclusions

A parallel initial value algorithm is presented for quasilinear stationary shock problems with turning points exhibiting two boundary layers or an internal layer and unknown turning point position. The original SPP is reduced to two equivalent SPIVPs and the error is estimated to be of order ε .Then, the two SPIVPs are modified to obtain two boundary-layer correction problems which are solved simultaneously in parallel algorithm using the non-stiff integrator (RKV56). The obtained solutions are combined to approximate the solution of the original problem overall the entire domain. The numerical results compare very well with the available exact solutions. Furthermore, the accuracy of numerical results improves as the small parameter ε tends to zero.

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