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TWO-GRID ITERATION METHOD FOR SOLVING MULTIVARIABLE INTEGRAL EQUATIONS ON POLYGONAL REGIONS

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ABSTRACT

The Nyström method for solving integral equations on polygonal regions reduces to the solution of a finite system of linear equations. The size of this system can easily very large, prohibitively so for solution by direct methods, and this necessitates the use of iterative methods. In this paper two-grid iterative method is applied for fast solution of these large systems.

KEY WORDS

Integral equations, Nyström methods, Iterative techniques.

1. INTRODUCTION

Consider the integral equation

$$\lambda\,\rho(x,y) - \int_R K(x,y,\xi,\eta)\,\rho(\xi,\eta)\,d\xi\,d\eta = \psi(x,y), \quad (x,y) \in R \tag{1}$$

We assume R is a polygonal region in R^2, and assume the integral operator is compact on $C(R)$ into $C(R)$, and the integral equation is assumed to be uniquely solvable.

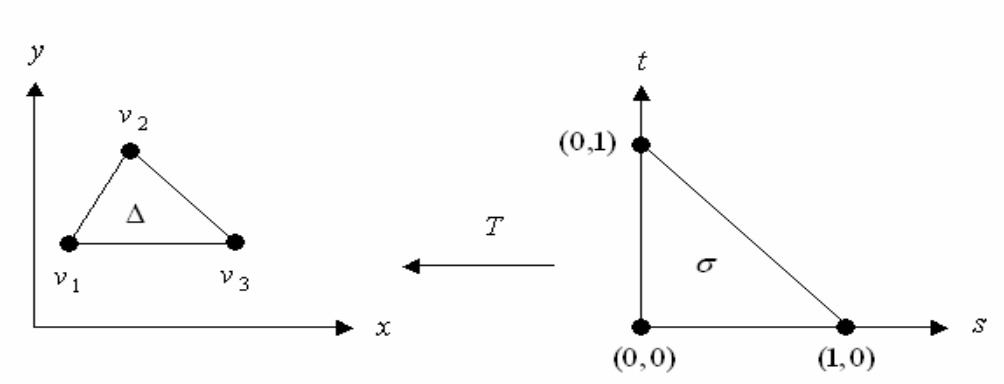
Equation (1) is written symbolically as $(\lambda - \mathbb{k})\rho = \psi$.
We will use the framework for Nyström method of solving the above equation, with the discretizations based on numerical integration over triangulations $\mathcal{T}_n = \{\Delta_{n,1}, \cdots, \Delta_{n,n}\}$ of R, so we begin by reviewing some of that framework.
To define a Nyström method for (1), we will first define a numerical integration operator $\mathbb{k}$ of (1).
We need to define numerical integration on Δ, we first consider Δ as the image of the standard unit simplex $\sigma = \{(s,t) : 0 \le s,t,s+t \le 1\}$.

Let the vertices of Δ be denoted by $\{v_1, v_2, v_3\}, v_j = (x_j, y_j)$.

Define $T : \sigma \xrightarrow[onto]{1-1} \Delta$ by $(x, y) = T(s,t) = u\,v_1 + t\,v_2 + s\,v_3, u = 1 - s - t.$

The vertices of σ namely $\{(0,0),(1,0),(0,1)\}$, map to the vertices of Δ [2].
Let a composite integration scheme be based on

$$\int_{\sigma} G(s,t)\, d\sigma \approx \sum_{i=1}^{\ell} \omega_i \; G(\mu_i) \tag{2}$$



with $\{\mu_1, \ldots, \mu_\ell\}$ the quadrature nodes within σ and $\{\omega_1, \ldots, \omega_\ell\}$ the weights. For specific formula see [1], [3].

In fact,

$$\int_{\Delta} g(x,y)\, dA = 2Area(\Delta) \int_{\sigma} g(T(s,t))\, d\sigma \approx \sum_{i=1}^{\ell} \omega_i \; g(T(\mu_i)) \tag{3}$$

in which the area of Δ is computed by

$$Area(\Delta) = \frac{1}{2}\left|det \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}\right|$$

Now we can approximate $\mathbb{k}\rho(x,y)$ using (3) by

$$\mathbb{k}\rho(x,y) = \int_R K(x,y,\xi,\eta)\, \rho(\xi,\eta) d\xi\, d\eta = \sum_{p=1}^{n} \int_{\Delta_{n,p}} K(x,y,\xi,\eta)\, \rho(\xi,\eta) d\xi\, d\eta$$

$$= 2\sum_{p=1}^{n} Area(\Delta_{n,p}) \int_{\sigma} K\left(x,y,T_{n,p}(s,t)\right) \rho\left(T_{n,p}(s,t)\right) d\sigma$$

$$\approx 2\sum_{p=1}^{n} Area(\Delta_{n,p}) \sum_{i=1}^{\ell} \omega_i\, K\left(x,y,T_{n,p}(\mu_i)\right) \rho\left(T_{n,p}(\mu_i)\right) \equiv \mathbb{k}_n\rho(x,y), (x,y) \in R$$

The integral equation (1) is approximated by $(\lambda - \mathbb{k}_n)\rho_n = \psi$, or equivalently,

$$\lambda\rho_n(x,y) - 2\sum_{p=1}^{n} \text{Area}(\Delta_{n,p}) \sum_{i=1}^{\ell} \omega_i K\left(x,y,T_{n,p}(\mu_i)\right) \rho_n\left(T_{n,p}(\mu_i)\right) = \psi(x,y),$$

$$(x,y) \in R \qquad (4)$$

Solve for ρ_n by first solving the linear system

$$\lambda\rho_n\left(T_{n,q}(\mu_j)\right) - 2\sum_{p=1}^{n} \text{Area}(\Delta_{n,p}) \sum_{i=1}^{\ell} \omega_i K\left(T_{n,q}(\mu_j),T_{n,p}(\mu_i)\right) \rho_n\left(T_{n,p}(\mu_i)\right) = \psi\left(T_{n,q}(\mu_j)\right),$$

$$q = 1,\dots,n, j = 1,\dots,\ell \dots$$

(5)

The solution ρ_n is obtained at the remaining points $(x,y) \in R$ by using the Nyström interpolation formula

$$\rho_n(x,y) = \frac{1}{\lambda}\left[\psi(x,y) + 2\sum_{p=1}^{n} \text{Area}(\Delta_{n,p}) \sum_{i=1}^{\ell} \omega_i K\left(x,y,T_{n,p}(\mu_i)\right) \rho_n\left(T_{n,p}(\mu_i)\right)\right] \dots \qquad (6)$$

The analysis of this method is given in the following theorem.

Theorem: [1, p.206] Let R be a polygonal region in R^2, and let $\{T_n\}$ be a sequence of triangulations of R. Assume

$$\delta_n = \max_{k=1,\dots,n} diameter(\Delta_{n,k}) \to 0 \text{ as } n \to \infty.$$

Assume the integral equation $(\lambda - \mathbb{k})\rho = \psi$ of (1) is uniquely solvable for $\psi \in C(R)$. Assume the integration formula (2) has degree of precession $d \geq 0$. Then

a) For all sufficiently large n, say $n \geq n_0$, the approximating equation (4) is uniquely solvable, and the inverses $(\lambda - \mathbb{k}_n)^{-1}$ are uniformly bounded on $C(R)$. For the error in ρ_n,

$$\rho - \rho_n = (\lambda - \mathbb{k}_n)^{-1}(\mathbb{k}\rho - \mathbb{k}_n\rho)$$

and $\rho_n \to \rho$ as $n \to \infty$.

b) Assume $K(x,y,\dots) \in C^{d+1}(R), for\ all\ (x,y) \in R, and\ \rho \in C^{d+1}(R).$

Then

$$\|\rho - \rho_n\| \in c\,\delta_n^{d+1}, n \geq n_0.$$

From this discussion we conclude that the order of the linear system to be solved can become very large with only a few refinements of the triangulation, and this necessitates the use of iterative methods for solving the system. The purpose of this paper is to consider iterative variant of (4)-(6). It will be assumed that (5) can be solved directly for some small number $n\ell$ of nodes; this will be used in defining iterative method for solving (5) with $N > n$.

2. GENERAL FORM

Let us assume that we want to solve $(\lambda - \mathbb{k}_N)\rho_N = \psi$ for some integers N. Rewrite this equation as

$$0 = \psi - (\lambda - \mathbb{k}_N)\rho_n$$

Assume $\rho_N^{(0)}$ is an initial estimate of the solution ρ_n for (5). Define the residual

$$r_N^{(0)} = \psi - (\lambda - \mathbb{k}_N)\rho_N^{(0)}$$
$$= (\lambda - \mathbb{k}_N)\left[(\lambda - \mathbb{k}_N)^{-1}\psi - \rho_N^{(0)}\right]$$
$$= (\lambda - \mathbb{k}_N)\left[\rho_n - \rho_N^{(0)}\right]$$
$$\rho_n = \rho_N^{(0)} + (\lambda - \mathbb{k}_N)^{-1} r_N^{(0)}$$

Consider the approximation
$(\lambda - \mathbb{k}_N)^{-1} \approx (\lambda - \mathbb{k}_n)^{-1}$
where $n \ll N$,
we define
$\rho_N^{(1)} = \rho_N^{(0)} + (\lambda - \mathbb{k}_n)^{-1} r_N^{(0)}$

Define $(\lambda - \mathbb{k}_n)^{-1} r_N^{(0)} = \varphi_N^{(1)}$,
$\rho_N^{(1)} = \rho_N^{(0)} + \varphi_N^{(1)}$

Finally, the general iteration is defined by

$$\rho_N^{(k+1)} = \rho_N^{(k)} + \varphi_N^{(k+1)} \tag{7}$$

where

$$\varphi_N^{(k+1)} = (\lambda - \mathbb{k}_n)^{-1} r_N^{(k)} \tag{8}$$

$$r_N^{(k)} = \psi - (\lambda - \mathbb{k}_N)\rho_N^{(k)} \tag{9}$$

where $k \geq 0$, and $\rho_N^{(0)}$ is properly selected [4-6].

Theorem:
Assume the integral equation $(\lambda - \mathbb{k})\rho = \psi$ of (1) is uniquely solvable for all $\psi \in C(R)$, and let $K(x, y, \xi, \eta)$ *be continuous for* $(x, y), (\xi, \eta) \in R$. Assume the integration formula (2) is convergent for all $g(x, y) \in C(R)$. Then if n is chosen sufficiently large, the iteration method (7) is convergent, that is,
$\rho_N^{(k)} \to \rho_N$ *as* $k \to \infty$. *for all* $N \gg n$.
Proof. Similar to [1, p. 250] .

3. IMPLEMENTATION

In this subsection we indicate how the iteration scheme given by (7) can be implemented.
The linear system to be solved is (5), with unknown
$\rho_N\left(T_{N,q}(\mu_j)\right), q = 1, \ldots, N, j = 1, \ldots, \ell$

Turning to the iteration formula (7), assume $\left\{\rho_N^{(k)}\left(T_{N,p}(\mu_i)\right)\right\}$ is known. First, calculate the residual $r_N^{(k)}$ at $\{T_{N,q}(\mu_j) \cup T_{n,q}(\mu_j)\}$:

$$r_N^{(k)}(x,y) = \psi(x,y) + 2\sum_{p=1}^{N} \text{Area}(\Delta_{N,p}) \sum_{i=1}^{\ell} \omega_i K\left(x, y, T_{N,p}(\mu_i)\right) \rho_N^{(k)}\left(T_{N,p}(\mu_i)\right) - \lambda \rho_N^{(k)}(x,y),$$

Second, calculate the correction $v_N^{(k+1)}$ on the coarse mesh by solving the system

$$\lambda\, \varphi_N^{(k+1)}\left(T_{n,q}(\mu_j)\right) - 2\sum_{p=1}^{n} \text{Area}(\Delta_{n,p}) \sum_{i=1}^{\ell} \omega_i K\left(T_{n,q}(\mu_j), T_{n,p}(\mu_i)\right) \varphi_N^{(k+1)}\left(T_{n,p}(\mu_i)\right) = r_N^{(k)}\left(T_{n,q}(\mu_j)\right), \qquad q = 1, \dots, n, j = 1, \dots, \ell.$$

Third, extended this correction to the fine mesh

$$\varphi_N^{(k+1)}\left(T_{N,q}(\mu_j)\right) = \frac{1}{\lambda}\left[2\sum_{p=1}^{n} \text{Area}(\Delta_{n,p}) \sum_{i=1}^{\ell} \omega_i K\left(T_{N,q}(\mu_j), T_{n,p}(\mu_i)\right) \varphi_N^{(k+1)}\left(T_{n,p}(\mu_i)\right) + r_N^{(k)}\left(T_{N,q}(\mu_j)\right)\right], \qquad q = 1, \dots, N, j = 1, \dots, \ell.$$

Finally, define the new iteration $\rho_N^{(k+1)}$ on the fine mesh by

$$\rho_N^{(k+1)}\left(T_{N,q}(\mu_j)\right) = \rho_N^{(k)}\left(T_{N,q}(\mu_j)\right) + \varphi_N^{(k+1)}\left(T_{N,q}(\mu_j)\right), \qquad q = 1, \dots, N, j = 1, \dots, \ell.$$

We stop the iteration when

$$\left\|\rho_N^{(k+1)}\left(T_{N,q}(\mu_j)\right) - \rho_N^{(k)}\left(T_{N,q}(\mu_j)\right)\right\|_\infty = \left\|\varphi_N^{(k+1)}\left(T_{N,q}(\mu_j)\right)\right\|_\infty \le \varepsilon,$$

where ε is a given tolerance.

4. NUMERICAL EXAMPELES

In this section, we give a numerical example and solve it using Nystrom and iterative technique to show the difference between them. This example implemented with a package of programs using MATLAB.

Consider the integral equation

$$\lambda\, \rho(x,y) - \int_0^1 \int_0^1 \sin\left(x + y + e^{\xi} + e^{\eta}\right) \rho(\xi,\eta)\, d\xi\, d\eta = \psi(x,y), \quad (x,y) \in R$$

For illustrative purposes we choose $\rho(x, y) = e^{x+y}$ and define $\psi(x,y)$ accordingly.
In defining the numerical integration operator $\mathbb{k}_n$ or $\mathbb{k}_N$, the basic integration rule being used over σ is given by [7-8]

$$\int_\sigma G(s,t)\, d\sigma \approx \sum_{i=1}^{7} \omega_i\, G(\mu_i)$$

where

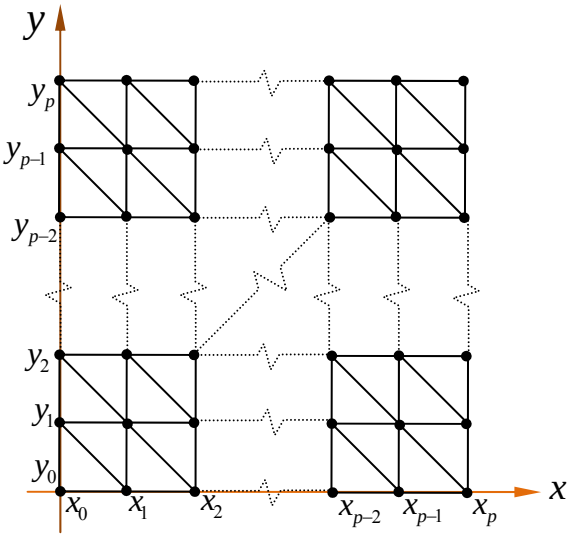
$$\omega_1 = \frac{9}{80}, \omega_2 = \omega_3 = \omega_4 = \frac{155-\sqrt{15}}{2400}, \omega_5 = \omega_6 = \omega_7 = \frac{155+\sqrt{15}}{2400}$$

$$\mu_1 = \left(\frac{1}{3}, \frac{1}{3}\right), \mu_2 = (\alpha, \alpha), \mu_3 = (\alpha, \beta), \mu_4 = (\beta, \alpha), \mu_5 = (\gamma, \gamma), \mu_6 = (\gamma, \delta), \mu_7 = (\delta, \gamma)$$

$$\alpha = \frac{6-\sqrt{15}}{21}, \beta = \frac{9+2\sqrt{15}}{21}, \gamma = \frac{6-\sqrt{15}}{21}, \delta = \frac{9+2\sqrt{15}}{21}$$

This quadrature formula has degree of precision $d = 5$.

The triangulation of the region of integration is illustrated in the following figure.



Where $x_i = y_i = i\,h, i = 0,1,\ldots,p;\ h = 1/p$. The number of triangles $= 2p^2$ and the number of nodes $= 14p^2$. The vertices of the triangles are given in the following algorithm

u=0;
for i=0:p-1
for j=0:p-1
u=u+1
the vertices of $\Delta_{n,u}$: $v_{u,1}^{(n)} = (x_i, y_j)$,
$v_{u,2}^{(n)} = (x_{i+1}, y_j), v_{u,3}^{(n)} = (x_{i+1}, y_{j+1})$

u=u+1
the vertices of $\Delta_{n,u}$:
$v_{u,1}^{(n)} = (x_{i+1}, y_{j+1})$,
$v_{u,2}^{(n)} = (x_{i+1}, y_j), v_{u,3}^{(n)} = (x_{i+1}, y_{j+1})$
End of i and j

Numerical results are given in Table 1.

Table . 1

λ	p	RMS error	Time of Nyström method (sec)	Time of iterative method (sec)
10	350	5.948e-009	0.8290	0.3240
10	2016	3.197e-011	146.64	7.6410
10	3150	8.399e-012	585.93	18.172
100	350	6.304e-010	0.8290	0.3240
100	2016	3.391e-012	146.65	7.6420
100	3150	8.908e-013	585.96	18.176

5. CONCLUSIONS

Nyström and Iterative technique can be applied on any integration formula. From the results, it is clear that as the number of basic points of the system increases as the error decreases. but in Nyström method, it consumed very large time to be executed. On the contrary, iterative method consumes small time to be executed compared with Nyström method. In addition, as λ increases as the system converge.

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