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SOLVING INTEGRAL EQUATIONS ON GENERAL REGIONS

Marwa Hani Maneea *   Reda Abdou Abd El- Monem **   Gamal M. Attia ***

BSTRACT

There exist many numerical methods for solving the integral equations, but most of these methods deal with integration over rectangular or triangular regions. This technique is modified to approximate the integral equation over general regions either on two or three dimensions supported by some numerical examples.

KEY WORDS

Numerical integration, quadratic interpolation, integral equations.

1. INTRODUCTION

We consider the problem of approximating an integral equation of the form

$$\lambda\rho(P)-\int_S K(P,Q)\,\rho(Q)\,dS_Q=\psi(P),\quad P\in S \tag{1}$$

where S is a subset of m - dimensional Euclidean space R^m, and dS_Q is the element of volume, area, or length on S. The function $K(P,Q)$ is called the kernel of the integral equation and it is given, also $\psi(P)$ is given, and the unknown is $\rho(P)$ [1].

In this paper, equation (1) can be assigned in two and three dimensions, so it can be written in the form

$$\lambda\rho(s,t)-\int_a^b\int_{c(x)}^{d(x)} K(x,y,s,t)\,\rho(x,y)dy\,dx=\psi(s,t) \tag{2}$$

or

$$\lambda \rho(s,t) - \int_{c}^{d} \int_{a(y)}^{b(y)} K(x,y,s,t)\, \rho(x,y)\, dx\, dy = \psi(s,t) \tag{3}$$

if it is in two dimensions, and it takes the form

$$\lambda \rho(s,t,u) - \int_{a}^{b} \int_{c(x)}^{d(x)} \int_{e(x,y)}^{f(x,y)} K(x,y,z,s,t,u) \rho(x,y,z)\, dz dy dx = \psi(s,t,u) \tag{4}$$

if it is in space (general form).

In fact, integral equations on regions or in space of this type or others can be approximated by performing appropriate partitions of the given region or area [2].

First, let us give some discussion about solving integral equations on fixed regions using Nyström method.

For the equation (1), consider the case that $K(P,Q)$ is continuous over S . let a composite integration scheme be based on

$$\int_{S} g(Q)\, dS_Q \approx \sum_{j=1}^{q_n} w_{n,j}\, g(t_{n,j}), \quad g \in C(S) \tag{5}$$

with an increasing sequence of values of n. We assume that for every $g \in C(S)$, the numerical integrals converge to the true integral as $n \to \infty$.

Using the above quadrature scheme, approximate the integral in (1), obtaining a new equation:

$$\lambda \rho_n(P) - \sum_{j=1}^{q_n} w_{n,j}\, K(P,t_{n,j})\, \rho_n(t_{n,j}) = \psi(P), \quad P \in S \tag{6}$$

It is written as an exact equation with a new unknown function $\rho_n(t)$. To find the solution at the node points, this yields the linear system

$$\lambda \rho_n(t_{n,i}) - \sum_{j=1}^{q_n} w_{n,j}\, K(t_{n,i},t_{n,j})\, \rho_n(t_{n,j}) = \psi(t_{n,i}), \quad i = 1, \dots, q_n \tag{7}$$

Which is of order q_n. The unknown is a vector of $\rho_n(t_{n,j})$.

If one solves for $\rho_n(t)$ in (6), then $\rho_n(t)$ is determined by its values at the node points. Therefore, the solution ρ_n is obtained by Nyström interpolation

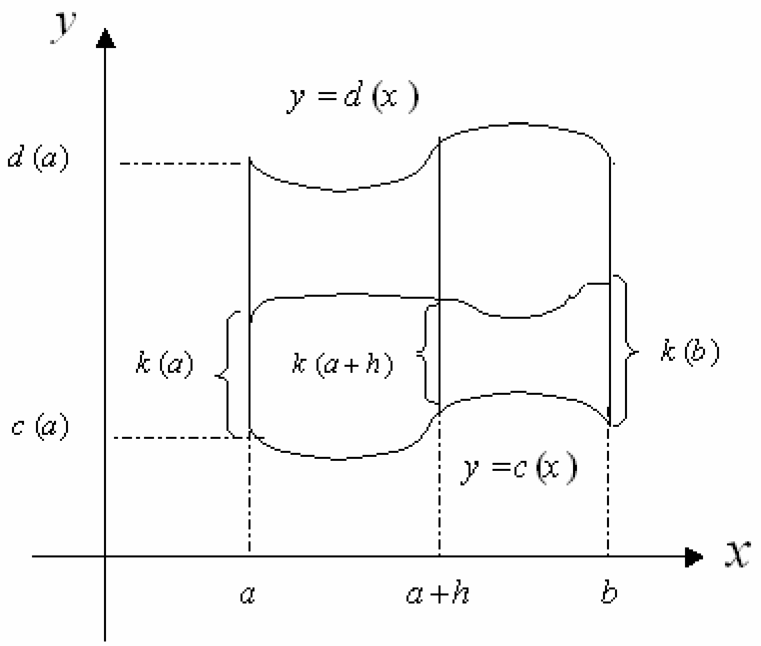
$$\rho_n(P) = \frac{1}{\lambda}\left[\psi(P) + \sum_{j=1}^{q_n} w_{n,j}\, K(P,t_{n,j})\, \rho_n(t_{n,j})\right], \quad i = 1, \dots, q_n \tag{8}$$

Formula (8) is called the *Nyström interpolation formula*.

Now, let us give some discussion about multiple integrals:

1.1. Multiple Numerical Integration [3]

Consider the double integral



We use composite Simpson's rule to integrate with respect to two variables. The step size for the variable x is $h = \dfrac{b-a}{n}$, but the step size for y varies with x is

$$k(x) = \frac{d(x) - c(x)}{m}$$

where n and m are the number of partitions on x and y directions respectively.

Consequently, Simpson's rule states

$$\int_a^b \int_{c(x)}^{d(x)} f(x, y)\, dy\, dx$$

$$\approx \int_a^b \frac{k(x)}{3} \; [f(x, c(x)) + 4f(x, c(x) + k(x)) + f(x, d(x))]\, dx$$

$$\approx \frac{h}{3} \{ \frac{k(a)}{3} [f(a, c(a)) + 4f(a, c(a) + k(a)) + f(a, d(a))]$$

$$+\frac{4k(a+h)}{3}[f\ (a+h, c(a+h)) + 4f\ (a+h, c(a+h)$$

$$+ k(a+h)) + f\ (a+h, d(a+h))]$$

$$+\frac{k(b)}{3}[f\ (b, c(b)) + 4f\ (b, c(b) + k(b)) + f\ (b, d(b))]\}$$

2. SOLVING INTEGRAL EQUATIONS IN TWO-DIMENSIONS

To describe technique involved with solving the integral equation in form (2)
The integration part of equation (2) is approximated by numerical integration formula [4-5].

Define $x_i = a + i\,h,\ i = 0, 1, ..., n.,\ h = \frac{b-a}{n}$

$$\lambda\,\rho(s,t) - \sum_{i=0}^{n} w_i \int_{c(x_i)}^{d(x_i)} K(x_i, y, s, t)\ \rho(x_i, y)\,dy \ \approx \psi(s,t)$$

Define $k_i = \dfrac{d(x_i) - c(x_i)}{m},\ y_{ij} = c(x_i) + j\,k_i,\ j = 0,1, ...,m.$

$$\lambda\,\tilde{\rho}(s,t) - \sum_{i=0}^{n}\sum_{j=0}^{m} w_i\, w_{ij}\ K(x_i, y_{ij}, s, t)\ \tilde{\rho}(x_i, y_{ij}) = \psi(s,t)$$

where

$$w_i = \begin{cases} \frac{h}{3}, & i = 0,\ i = n \\ \frac{4h}{3}, & i = 1,3,...,n-2, \\ \frac{2h}{3}, & i = 2,4,...,n-1 \end{cases} \qquad w_{ij} = \begin{cases} \frac{k_i}{3}, & j = 0,\ j = m \\ \frac{4k_i}{3}, & j = 1, 3, ..., m-2 \\ \frac{2k_i}{3}, & j = 2,4,...,m-1 \end{cases}$$

where n, m are the number of partitions on x and y directions recpectivily .

Solve the system

$$\lambda\,\tilde{\rho}(x_u, y_{uv}) - \sum_{i=0}^{n}\sum_{j=0}^{m} w_i\, w_{ij}\ K(x_i, y_{ij}, x_u, y_{uv})\ \tilde{\rho}(x_i, y_{ij}) = \psi(x_u, y_{uv}),$$

for $u = 0,1,...,n,\ v = 0,1,...,m.$
The solution is obtained by using the Nyström interpolation formula

$$\tilde{\rho}(s,t) = \frac{1}{\lambda}[\ \psi(s,t) + \sum_{i=0}^{n}\sum_{j=0}^{m} w_i\, w_{ij}\ K(x_i, y_{ij}, s, t)\ \tilde{\rho}(x_i, y_{ij})\ ] \tag{9}$$

Formula (9) is the approximation of Nyström method for solving the integral equation over general region of form (2).
In the same manner, the integral equations in form (3) can also be approximated using Nyström interpolation formula. Hence

Define $y_j = c + j\,k,\ j = 0, 1, ..., m.,\ k = \frac{d-c}{m}$

$$\lambda\,\rho(s,t) - \sum_{j=0}^{m} w_j \int_{a(y_j)}^{b(y_j)} K(x, y_j, s, t)\,\rho(x, y_j)\,dx \approx \psi(s,t)$$

Define $h_j = \dfrac{b(y_j) - a(y_j)}{n},\ x_{ji} = a(y_j) + i\,h_j,\ i = 0,1,...,n.$

$$\lambda\,\tilde{\rho}(s,t) - \sum_{j=0}^{m}\sum_{i=0}^{n} w_j\, w_{ji}\, K(x_{ji}, y_j, s, t)\,\tilde{\rho}(x_{ji}, y_j) = \psi(s,t)$$

where

$$w_j = \begin{cases} \dfrac{k}{3}, & j = 0,\ j = m \\ \dfrac{4k}{3}, & j = 1,3,...,m-2 \\ \dfrac{2k}{3}, & j = 2,4,...,m-1 \end{cases}, \qquad w_{ji} = \begin{cases} \dfrac{h_j}{3}, & i = 0, i = n \\ \dfrac{4h_j}{3}, & i = 1,3,...,n-2 \\ \dfrac{2h_j}{3}, & i = 2,4,...,n-1 \end{cases}$$

Solve the system

$$\lambda\,\tilde{\rho}(x_{vu}, y_v) - \sum_{j=0}^{m}\sum_{i=0}^{n} w_j\, w_{ji}\, K(x_{ji}, y_j, x_{vu}, y_v)\,\tilde{\rho}(x_{ji}, y_j) = \psi(x_{vu}, y_v),$$

for $u = 0,1,...,n,\ v = 0,1,...,m.$

The solution is obtained by using the Nyström interpolation formula

$$\tilde{\rho}(s,t) = \frac{1}{\lambda}\Big[\,\psi(s,t) + \sum_{j=0}^{m}\sum_{i=0}^{n} w_j\, w_{ji}\, K(x_{ji}, y_j, s, t)\,\tilde{\rho}(x_{ji}, y_j)\Big] \tag{10}$$

Formula (10) is the approximation of Nyström method for solving the integral equation over general region of form (3).

3. SOLVING INTEGRAL EQUATIONS IN THREE-DIMENSIONS

To describe technique involved with approximating the integral equation of form (4).
To solve this integral equation, as two dimensions transform the integration part by numerical integration formula. [6-7]

Define $x_i = a + i\,h,\ i = 0,1,\ldots,n.,\ h = \dfrac{b-a}{n}$

$$\lambda\,\rho(s,t,g) - \sum_{i=0}^{n} w_i \int_{c(x_i)}^{d(x_i)} \int_{e(x_i,y)}^{f(x_i,y)} K(x_i,y,z,s,t,g)\,\rho(x_i,y,z)\,dz\;dy \approx \psi(s,t,g)$$

Define $k_i = \dfrac{d(x_i) - c(x_i)}{m},\ y_{ij} = c(x_i) + j\,k_i,\ j = 0,1,\ldots,m.$

$$\lambda\,\rho(s,t,g) - \sum_{i=0}^{n} \sum_{j=0}^{m} w_i\,w_{ij} \int_{e(x_i,y_{ij})}^{f(x_i,y_{ij})} K(x_i,y_{ij},z,s,t,g)\,\rho(x_i,y_{ij},z) \approx \psi(s,t,g)$$

Define $S_{ij} = \dfrac{f(x_i,y_{ij}) - e(x_i,y_{ij})}{1},\ z_{ijr} = e(x_i,y_{ij}) + r\,S_{ij},\ r = 0,1,\ldots,1.$

$$\lambda\tilde{\rho}(s,t,g) - \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{r=0}^{1} w_i\,w_{ij}\,w_{ijr}\,K(x_i,y_{ij},z_{ijr},s,t,g)\,\tilde{\rho}(x_i,y_{ij},z_{ijr}) = \psi(s,t,g)$$

where

$$w_i = \begin{cases} \dfrac{h}{3}, & i = 0,\ i = n \\ \dfrac{4h}{3}, & i = 1,3,\ldots,n-2, \\ \dfrac{2h}{3}, & i = 2,4,\ldots,n-1 \end{cases} \qquad w_{ij} = \begin{cases} \dfrac{k_i}{3}, & j = 0,\ j = m \\ \dfrac{4k_i}{3}, & j = 1,3,\ldots,m-2 \\ \dfrac{2k_i}{3}, & j = 2,4,\ldots,m-1 \end{cases}$$

and $$w_{ijr} = \begin{cases} \dfrac{S_{ij}}{3}, & r = 0, r = 1 \\ \dfrac{4S_{ij}}{3}, & r = 1,3,\ldots,1-2 \\ \dfrac{2S_{ij}}{3}, & r = 2,4,\ldots,1-1 \end{cases}$$

where $n, m, 1$ are the number of partitions on x, y, and z directions respectivily.
Solve the system

$$\lambda \tilde{\rho}(x_u, y_{uv}, z_{uvp}) - \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{r=0}^{1} w_i w_{ij} w_{ijr} K(x_i, y_{ij}, z_{ijr}, x_u, y_{uv}, z_{uvp}) \tilde{\rho}(x_i, y_{ij}, z_{ijr}) = \psi(x_u, y_{uv}, z_{uvp})$$

for $u = 0,1,\ldots,n$, $v = 0,1,\ldots,m$, $p = 0,1,\ldots,1$.

The solution is obtained by using the Nyström interpolation formula

$$\tilde{\rho}(s,t,g) = \frac{1}{\lambda}[\psi(s,t,g) + \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{r=0}^{1} w_i w_{ij} w_{ijr} K(x_i, y_{ij}, z_{ijr}, s, t, g) \tilde{\rho}(x_i, y_{ij}, z_{ijr})] \tag{11}$$

Formula (11) is the approximation of Nystrom method for solving the integral equation over general region of the form (4).

4. NUMERICAL EXAMPELES

We give numerical examples using the methods analyzed in (2), (3). The method was implemented with a package of programs using MATLAB.

Example (1). Consider the integral equation

$$\lambda \rho(s,t) - \int_0^1 \int_0^{x^2+1} 8s^2 \sin(st)\, e^{s(y-x^2)} \rho(x,y)\, dy\, dx = \psi(s,t)$$

For illustrative purposes, we choose $\rho(x,y) = y x^3 - x^5$ and define $\psi(x,y)$ accordingly.
Apply formula (9) to the example, the results are in tables (1), (2)

Table 1. at $\lambda = 0.001$

n	m	N	RMS error
10	10	121	6.0221e-003
20	20	441	4.8301e-004

30	30	961	9.6571e-005

where N is the number of basic nodes of the system.

Table 2. at $n = m = 20$ and so $N = 441$

λ	RMS error
0.001	4.83017e-004
1.0	1.79391e-005
10	6.81206e-006
100	4.93417e-007
1000	4.80529e-008

From table (1), it is clear that when λ is fixed and as the number of intervals n, m increases hence the number of basic nodes of the system N increases as the error decreases and so the accuracy increases.
From table (2), when the partitions n, m and hence N are fixed, as λ increases as the error decreases.

Example (2): Consider the integral equation

$$\lambda \rho(s,t) - \int_0^1 \int_0^{y^2+1} 8 s^2 \sin(st) e^{s(x-y^2)} \rho(x,y)\, dx\, dy = \psi(s,t)$$

For illustrative purposes, we choose $\rho(x, y) = x y^3 - y^5$ and define $\psi(x, y)$ accordingly.
Apply formula (10) to the example, the results are in tables (3), (4)

Table 3. at $\lambda = 0.001$

n	m	N	RMS error
10	10	121	4.3691e-002
16	16	289	8.7378e-003
20	20	441	5.3005e-003
30	30	961	4.5878e-003

Table 4. at $n = m = 20$ and so $N = 441$

λ	RMS error
0.001	5.30051e-003
1.0	3.55062e-003
10	2.60412e-003
100	2.20032e-004
1000	1.97978e-005

Tables (3), (4) give the same conclusions of tables (1), (2).

Example(3): Consider the integral equation

$$\lambda \rho(s,t,u) - \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \sin(s+t+u)\rho(x,y,z)\, dz\, dy\, dx = \psi(s,t,u)$$

For illustrative purposes, we choose $\rho(x,y,z) = x\,y\,z$ and define $\psi(x,y,z)$ accordingly.

Apply formula (11) to the example, the results are in table (5)

Table . 5

n	m	l	λ	RMS error
6	6	6	10	7.4290e-005
8	8	8	10	7.2855e-005
6	6	6	100	7.4290e-006
8	8	8	100	7.1855e-006

From table (5), note that as the number of partitions increases as the error decreases, also as λ increases as the system converges.

5. CONCLUSIONS

This paper represents the solution of multivariable integral equations on general and different regions. We solve the integral equations in R^2 and R^3. From the results on tables, it is obvious that when λ is constant and as the number of intervals increases as the error decreases. Also when the number of intervals is fixed and λ increases as the error decreases.

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