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Optimization Model By Fuzzy Environment

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Abstract

A well known linear programming model and its equivalent model are introduced as a handling type of Fuzzy optimal problem. Linear programming (LP) with Crisp objective function and crisp constraints is modified and presented to be Fuzzy linear programming (FLP) with Fuzzy objective function and Fuzzy constraints (Zimmermann 1983, Wemers 1984).

For each value ri of the objective function r = cTx, ro < ri < r1, the membership $\mu_{\tilde{C}_r}$ of the constraints obtained by the min-max method and the membership $\mu_{\tilde{G}_r}$ of the objective function are to be found. The Fuzzy set { ri, $\mu_{\tilde{G}_r} . \mu_{\tilde{C}_r}$ } implies the optimal value ri which satisfies the equality approach $\mu_{\tilde{G}_r} = \mu_{\tilde{C}_r}$, or the product approach : $\max_i \{ \mu_{\tilde{G}_r} . \mu_{\tilde{C}_r} \}$.

Key words:

Fuzzy optimization, Min-max, membership function, fuzzy decision making, fuzzy linear programming.

{1} – Introduction:

The classical model of linear programming can be stated as :

$$\left.\begin{aligned} &\text{Maximize} \quad r(x) = c^T x \\ &\text{Such that} \quad A\,x \le b \\ &\qquad\qquad\quad x \ge 0 \end{aligned}\right\} \ldots\ldots\ldots \qquad \ldots\ldots (1)$$

With C, $x \in \Re_n$, $b \in \Re^m$, $A \in \Re^{m\times n}$

Before developing a specific model of linear programming in a fuzzy environment, we must take into consideration that fuzzy linear programming is not a uniquely defined type of model, many variations are possible, depending on the assumptions or features of the real situation to be modeled.

A first basic model for fuzzy linear programming (FLP) can be obtained from (1) by establishing an aspiration level Z for the value of the objective function to be achieved and by molding each constraint as a fuzzy set. We get a fuzzy LP model as follows:

For a fuzzy set to represent a crisp constraint $A_i x \leq b_i$, an interval P_i is to be introduced in such a manner that :

$$A_i x \leq b_i + P_i - \alpha_i P_i \ , \ o \leq \alpha_i \leq 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where P_i are chosen constants of admissible violations of the constraints and the objective function (Zimmermann, 1976)

Then :

$$\alpha_i = (b_i + P_i - A_i x) / P_i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ldots. (3)$$

Where α_i is interpretted as the degree to which x fulfills the Fuzzy inequality $A_i x < b_i$ (Bellman – Zadeh, Zimermann)

The membership function α_i of the fuzzy constraint $A_i x \leq b_i$ is:

$$\alpha_i = 1 - \frac{A_i x - b_i}{p_i} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (4)$$

If $\alpha_i = 1$ we get the fuzzy region $\tilde{R_o}$ and

If $\alpha_i = o$ we get the fuzzy region $\tilde{R_1}$

For the objective function $r = c^T x$, let $r = r_o$ be the maximum value of the objective function on the fuzzy region $\tilde{R}_o$ and $r = r_1$ be the max. value of the objective function on $\tilde{R}_1$. Then the optimal. value of the objective function $r = c^T x$ satisfies $r_o < r < r_1$, the membership function $\mu_{\tilde{G}}$ of the objective function (Wemers 1984, Zimmermann 1987) is defined as :

$$\mu_{\tilde{G}} = \frac{r - r_o}{r_1 - r_o} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \ldots\ldots (5)$$

To find the optimum of the objective function $r = c^T x$, we put:

$$\mu_{\tilde{c}_r} = \min_T \{ \mu_{\tilde{c}_r} \; ; \; i = 1, \dots, m\} \quad \dots (6)$$

And we find the point of intersection of (5) and (6) , i.e,.

$$\mu_{\tilde{G}_r} = \mu_{\tilde{c}_r} \quad \dots (7)$$

{ 2. A well known LP model Under equality approach:

Let us consider the LP model:

$$\left.\begin{array}{ll} \text{Maximize} & r = 2x_1 + x_2 \\ \text{Such that} & x_1 \leq 3 \\ & x_1 + x_2 \leq 4 \\ & 5x_1 + x_2 \leq 3 \\ & x_1 , x_2 > 0 \end{array}\right\} \quad \dots (8)$$

Taking the P intervals to be $P_1 = 6$, $P_2 = 4$, $P_3 = 2$

We get the parametric linear program:

$$\left.\begin{array}{ll} \text{Maximize} & r = 2x_1 + x_2 \\ \text{Such that} & x_1 \leq 9 - 6\alpha \\ & x_1 + x_2 \leq 8 - 4\alpha \\ & 5x_1 + x_2 \leq 5 - 2\alpha \\ & x_1, x_2 \geq o \end{array}\right\} \quad \dots (9)$$

for α = o we get the fuzzy region $\tilde{R}_1$

for α = 1 we get the fuzzy region $\tilde{R}_o$, (Fig. 1).

For the fuzzy sets representing the fuzzy constraints $\mu_{\tilde{c}_i}(x)$, the membership function is specified as:

$$\mu_{\tilde{c}_i}(x) = 1 - \frac{A_i x - b_i}{P_i} \; , (i = 1 , 2, 3) \quad \dots (10)$$

For the constraint $x_1 \leq 3$; $\mu_{\tilde{c}_1} = \dfrac{9 - x_1}{6}$

For the constraint $x_1 + x_2 \le 4$, $\mu_{\tilde{c}_2} = \dfrac{8 - x_1 - x_2}{4}$

For the constraint $5x_1 + x_2 \le 3$ $\mu_{\tilde{c}_3} = \dfrac{5 - 5x_1 - x_2}{2}$

Let $2x_1 + x_2 = r$ then,

$$\mu_{\tilde{C}_r} = \min_r \{ \max \{ \frac{9 - x_1}{6}, \frac{8 - x_1 - x_2}{4}, \frac{5 - 5x_1 - x_2}{2} \} \}$$

<u>For $r = 16$</u>

$x_1 = 8$, $x_2 = 0$ → $\mu_t = \max \{ \frac{1}{6}, 0 , 0 \} = \frac{1}{6}$

$x_1 = 7$, $x_2 = 2$ → $\mu_t = \max \{ \frac{1}{3} , 0 , 0 \} = 1/3$

$x_1 = 6$, $x_2 = 4$ → $\mu_t = \max \{ \frac{1}{2} , 0 , 0\} = 1/2$

$x_1 = 5$, $x_2 = 6$ → $\mu_t = \max \{ \frac{2}{3} , 0 , 0\} = 2/3$

$x_1 = 4$, $x_2 = 8$ → $\mu_t = \max \{ \frac{5}{6} , 0 , 0\} = 5/6$

then,

<u>for $r = 16$</u>, $\mu_{\tilde{C}_r} = \min \{ \frac{1}{6} , \frac{1}{3}, \frac{1}{2}, \frac{2}{3} , \frac{5}{6} \} = \frac{1}{6}$

<u>for $r = 15$</u>

$x_1 = 7.5$, $x_2 = 0$ → $\mu_t = \max \{ \frac{1}{4} , 0 , 0\} = 1/4$

$x_1 = 7$, $x_2 = 1$ → $\mu_t = \max \{ \frac{1}{3} , 0 , 0 \} = 1/3$

$x_1 = 6$, $x_2 = 3$ → $\mu_t = \max \{ \frac{1}{2} , 0 , 0\} = 1/2$

$x_1 = 5$, $x_2 = 5$ → $\mu_t = \max \{ \frac{2}{3} , 0 , 0 \} = 2/3$

then,

for r = 15, $\mu_{\tilde{C}_r} = \min \{ \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3} \} = \frac{1}{4}$

Similarly,

for $r = 14$, $\mu_{\tilde{C}_r} = \dfrac{1}{3}$

<u>for $r = 12$</u> , $\mu_{\tilde{C}_r} = \dfrac{1}{2}$

<u>for r = 10</u>

$x_1 = 5, x_2 = 0 \rightarrow \mu_t = \max\{\frac{2}{3}, \frac{3}{4}, 0\} = 3/4$

$x_1 = 4, x_2 = 2 \rightarrow \mu_t = \max\{\frac{5}{6}, \frac{1}{2}, 0\} = 5/6$

$x_1 = 3, x_2 = 4 \rightarrow \mu_t = \max\{1, \frac{1}{4}0, 0\} = 1$

for $r = 10$, $\mu_{\tilde{C}_r} = \min\{\frac{3}{4}, \frac{5}{6}, 1\} = 3/4$

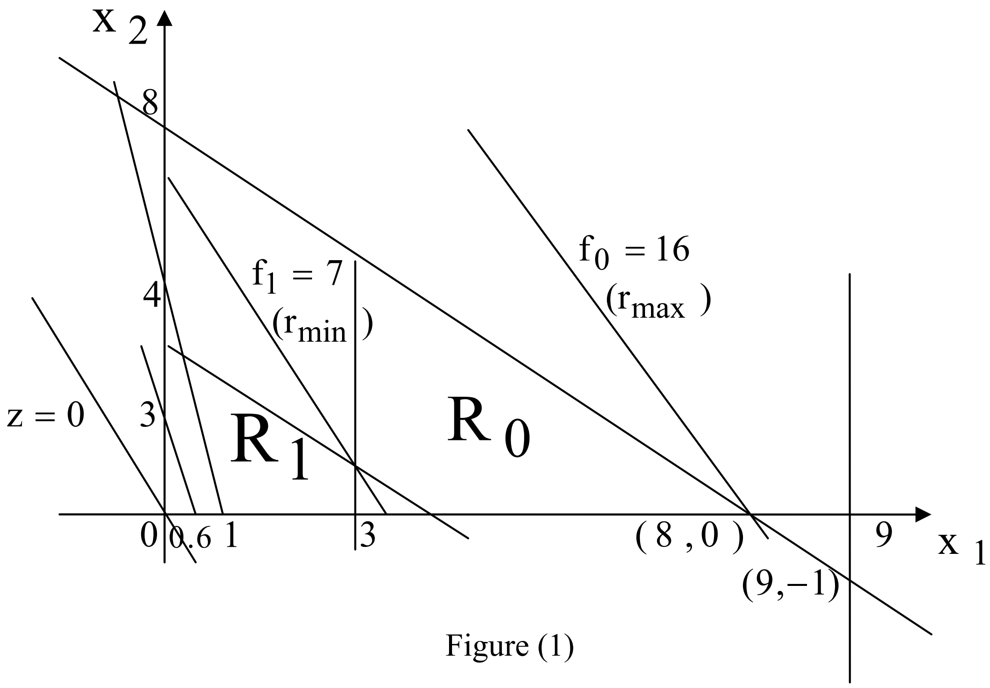

Figure (1)

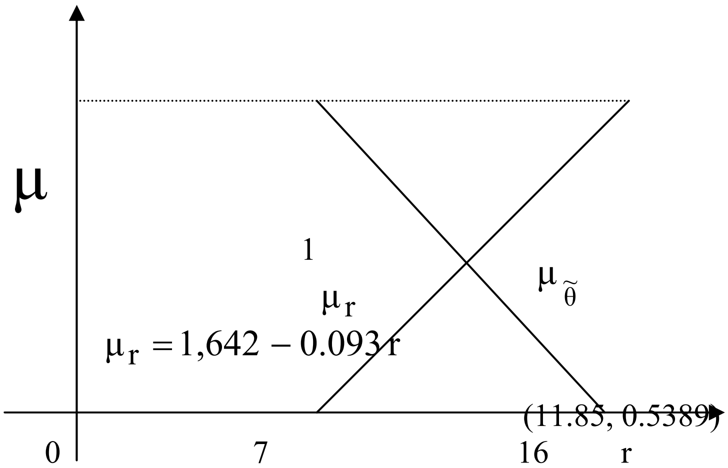

Figure (2)

<u>for $r = 7$</u> , $\mu_{\tilde{C}_r} = 1$

We get the tabulated data:

r	7	10	12	14	15	16
μ_r	1	0.75	0.5	0.333	0.25	0.167

By the least squares method we get a line:

$$\mu_{\tilde{C}_r} = 1.642 - 0.093\,r \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11),$$

fitting the given points, (Fig. 2)

For the membership function of the objective function; recall (5) then:

$$\mu_{\tilde{G}} = \frac{r - r_o}{r_1 - r_o} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12)$$

From (Fig. 1) : on $\tilde{R}_1$ the max. objective function $r_o = 2x_1 + x_2$ passing thr, the point (3,1)

gives $r_o = 7$ and on $\tilde{R}_o$ the max obj. Function

$r_1 = 2x_1 + x_2$ passing thr, (8,0) gives $r_1 = 16$.

Then from (12):

$$\mu_{\tilde{G}} = \frac{r - 7}{9} = 0.1111\,r - 0.7777 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

The solution satisfies $\mu_{\tilde{G}} = \mu_{\tilde{C}_r}$ then from (11) and (13) we get:

$r_{\tilde{D}} = 11.85$

Then $\mu_{\tilde{G}} = \mu_{\tilde{C}_r} = \mu_{\tilde{D}} = \dfrac{11.85-7}{9} = 0.5389$

But $\mu_{\tilde{C}_r} = \dfrac{9-x_1}{6} = 0.5389$

Then $x_1 = 5.7666$

But $2x_1 + x_2 = 11.85$

Then $x_2 = 0.317$

The resulting solution:

$x_1 = 5.7666$, x2 = 0.317, $r = 11.85$,

$\mu = 0.5389$

{ 3} . An equivalent LP Model:

The classical model (1) can be transformed into an equivalent model as follows:

Maximize $r(x) = c^T x$.

Such that $Ax \le b$,

$x \ge o$

with intervals P_i , i = 1, ……….. , m.

The membership functions of the fuzzy constraints are:

$\mu_{\tilde{C}_i} = (b_i - P_i - A_i x) / p_i$ …………………………………………… (14)

$\mu_{\tilde{C}_r} = \min \{ \mu_{\tilde{C}_i}$, i = 1, ………, m}

And the membership of the objective function is:

$\mu_{\tilde{G}(x)} = \dfrac{C^T X - r_o}{r_1 - r_o} = \dfrac{r - r_o}{r_1 - r_o}$ ……………………………………..……. (15)

Since the optimum value of $r = c^T x$ is attained under the condition:

$\mu_{\tilde{C}_r}(x) = \mu_{\tilde{G}(x)} \ge \lambda$

Then by (14)

$\lambda_\iota \le (b_i + p_i - A_i (x) / p\,\iota$, …………………………...………..…… (16)

(ι = 1, …… , m)

And $\lambda = \min\limits_i \{ \lambda_\iota$. i = 1, …. , m) ……………………….……… … (17)

And by (15)

$\lambda \leq \dfrac{C^T X - r_o}{r_1 - r_o}$ ……………………………………….……………… (18)

From (16) and (18) we get:

$\lambda_i \, p_i + A_i \, x \leq b_i + P_i$., (i= 1…., m)……………………………….……… . (19)

$\lambda \; (r_1 - r_o) - c^T x \leq - r_o$ …………………………...…………… . (20)

The equivalent model is introduced as :

maximize λ such that

$$\left.\begin{aligned} &\lambda \; (r_1 - r_o) - c^T x \leq - r_o \\ &\lambda_i \, p_i + A_i \, x \leq b_i + P_i., \; (i = 1……., m) \\ &\lambda \leq 1 \\ &\lambda \; , x_1 \; , x_2 \geq 0 \end{aligned}\right\} \quad ………… (21)$$

{ 4} – A well known equivalent LP model under equality approach

Let us consider as an example the same model in { 2} :

maximize

$r = 2 \, x_1 + x_2$ such that

$$\begin{aligned} &x_1 \leq 3 \\ &x_1 + x_2 \leq 4 \\ &5 \, x_1 + x_2 \leq 3 \\ &x_1, x_2 \geq 0 \end{aligned}$$

with the intervals of constraints being :

$P_1 = 6$, $P_2 = 4$, $P_3 = 2$ and with $r_o = 7$ and $r_1 = 16$ as before.

By (21) , the equivalent model is defined as

Maximize λ

Such that

$$\lambda \sim_G = \frac{2x_1 + x_2 - 7}{9} = \frac{r - 7}{9}$$

$$\lambda \sim_{C_1} = \frac{9 - x_1}{6}$$

$$\lambda \sim_{C_3} = \frac{8 - x_1 - x_2}{4}$$

$$\lambda \, \tilde{c}_3 = \frac{5 - 5x_1 - x_2}{2}$$

$\lambda \leq 1$

$\lambda \ , x_1 \ , x_2 \ \geq 0$

For r = $2x_1 + x_2 = 16$

x_1	x_2	$\lambda_{\tilde{c}_1}$	$\lambda_{\tilde{c}_2}$	$\lambda_{\tilde{c}_3}$	MAX $\lambda_{\tilde{c}_i}$	$\lambda_{\tilde{G}}$
8	0	1/6	0	0	1/6	1
7	2	1/3	0	0	1/3	1
6	4	1/2	0	0	1/2	1
5	6	2/3	0	0	2/3	1
4	8	5/6	0	0	5/6	1

$\lambda_{\tilde{c}_r} = \frac{1}{6} \ , \ \lambda_{\tilde{G}_r} = 1$ for $r = 16$

Similarly for r = 15 :

$\lambda_{\tilde{c}_r} = \frac{1}{4} \ , \ \lambda_{\tilde{G}_r} = \frac{8}{9}$ for $r = 15$

Similarly for r = 14:

$\lambda_{\tilde{c}_r} = \frac{1}{3} \ , \ \lambda_{\tilde{G}_r} = \frac{7}{9}$ for $r = 14$ …… etc.

We can construct the following tabulated data:

r	16	15	14	13	12	11
$\lambda_{\tilde{c}_r}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$	$\frac{1}{2}$	$\frac{5}{8}$
$\lambda_{\tilde{G}_r}$	1	$\frac{8}{9}$	$\frac{7}{9}$	$\frac{6}{9}$	$\frac{5}{9}$	$\frac{4}{9}$

We notice that $\lambda_{\tilde{c}_r} = \lambda_{\tilde{G}_r}$ in the interval 11 < r < 12.

The first line $\lambda_{\tilde{c}_r}$ = f (r) in this interval is determined by the two points (11, $\frac{5}{8}$), (12 , $\frac{1}{2}$),

$$\lambda_{\tilde{c}_r} = \frac{16 - r}{8}$$

The second line $\lambda_{\tilde{G}_r}$ = g (r) in this interval is determined by the two points (11, $\frac{4}{9}$) , (12 , $\frac{5}{9}$)

,

$$\lambda_{\tilde{G}_r} = \frac{r-7}{9} \quad \text{………………………………………………………….. (29)}$$

Putting $\lambda_{\tilde{c}_r} = \lambda_{\tilde{G}_r}$ in (23) and (24) then,

r = 11.764

and from (24) we get : λ = 0.5298, (Fig. 3).

From $\lambda = \frac{9-x_1}{6} = \frac{8-x_1-x_2}{4}$ we get :

$x_1 = 5.824$

$x_2 = 0.116$

comparing the first model with the equivalent one under the equality approach we get:

	First Model	Equivalent model
x_1	5.7666	5.824
x_2	0.317	0.116
r	11.85	11.764
λ	0.5389	0.5293

5 – Product Approach

For each value r = c^{T}x we find the membership

$\mu_{\tilde{C}_r}$ = min {max $\mu_{\tilde{C}_r}$ i = 1,…..,m}

and the membership of the objective function

$$\mu_{\tilde{G}_r} = \frac{r-r_o}{r_1-r_o}$$

We calculate $\mu_{\tilde{C}_r} \, . \, \mu_{\tilde{G}_r}$ for each r = c^{T}x and find $\max_r . \{ \mu_{\tilde{C}_r} \, . \, \mu_{\tilde{G}_r} \}$, the corresponding

value r is the optimum value.

From the first F.L.P model we have :

$\mu_{\tilde{C}_r}$ = 1.642 – 0.093r

$\mu_{\tilde{G}_r} = 0.111r - 0.7777$

Then $\delta = \mu_{\tilde{C}_r} \cdot \mu_{\tilde{G}_r}$;

for max. δ , $\frac{d\delta}{dr} = 0 \Rightarrow r = 12.32$

At r = 12.32, $\mu_{\tilde{C}} = 0.49624$, $\mu_{\tilde{G}} = 0.59105$ then. $\mu_{\tilde{C}_r} \cdot \mu_{\tilde{G}_r} = 0.2933$

and we get for the first F.L.P model:

r	$\mu_{\tilde{C}}$	$\mu_{\tilde{G}}$	$\mu_{\tilde{C}} \cdot \mu_{\tilde{G}}$	Remarks
7	7	0	0	
10	0.75	0.33	0.25	
11	0.583	0.444	0.2589	
11.85	0.5389	0.5389	0.2904	$\mu_{\tilde{C}} = \mu_{\tilde{G}}$
12	0.5	0.555	0.278	
12.32	0.4962	0.5911	0.2933	Max $\mu_{\tilde{C}} \cdot \mu_{\tilde{G}}$
15	0.25	0.888	0.222	
16	0.167	1	0.167	

But in the case of the equivalent F.L.P model

since $\mu_{\tilde{C}_r} = 2 - 0.125\, r$, $\mu_{\tilde{G}_r} = 0.111\, r - 0.7777$

then,

$\phi = \mu_{\tilde{C}_r} \cdot \mu_{\tilde{G}_r}$, for max. ϕ , $\frac{d\phi}{dr} = o \Rightarrow r = 11.5$

At r =11.5 , $\mu_{\tilde{C}} = 0.563$, $\mu_{\tilde{G}} = 0.4989$

Then : $\mu_{\tilde{C}} \cdot \mu_{\tilde{G}} = 0.2809$ at r = 11.5

And we get for the equivalent F.LP model:

r	$\mu_{\tilde{C}}$	$\mu_{\tilde{C}}$	$\mu_{\tilde{C}} \cdot \mu_{\tilde{C}}$	Remarks
11	0.625	0.4444	0.277	
11.5	0.5631	0.4989	0.2809	Max $\mu_{\tilde{C}} \cdot \mu_{\tilde{G}}$
11.764	0.5293	0.5293	0.2809	$\mu_{\tilde{C}} = \mu_{\tilde{G}}$
12	0.5000	0.5555	0.277	

13	0.4167	0.6667	0.277	
14	0.3333	0.7778	0.259	
15	0.25	0.8889	0.2222	
10	0.167	1.0000	0.167	

Comparing the first model with the equivalent one under the product approache we get:

	First model	Equivalent model
x_1	6.0225	5. 622
x_2	0.275	0.256
r	12.32	11.5
λ	0.2933	0.2809

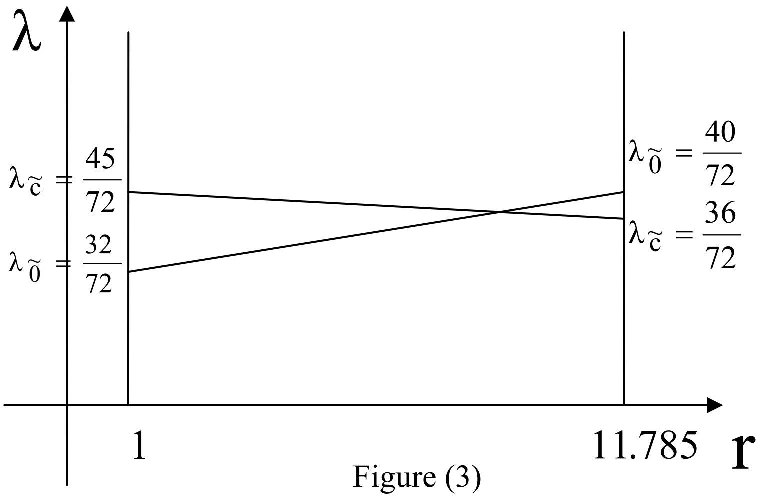

Figure (3)

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