Military Technical College
Kobry Elkobbah,
Cairo, Egypt
May 16-18,2006

3nd International Conference on
Engineering Mathematics and
Physics (ICMEP-3)

QUANTUM THEORY BASED ON FUZZY SETS

CHOVANEC F.[1] and JUREČKOVÁ[2] M.

ABSTRACT

A new probability model based on the theory of fuzzy sets is presented. In this model, a difference of comparable fuzzy sets is the primary operation. The idea of a difference of fuzzy sets (fuzzy events) is simple: If we have two comparable events a and b ($a \leq b$), then our knowledge on a and b entails the complete knowledge of the complement of a in b, i. e., $b \ominus a$.
The new algebraic structure of fuzzy sets is called a difference poset (a D-poset) of fuzzy sets. Some properties of a lattice ordered D-poset of fuzzy sets (a D-lattice of fuzzy sets) are studied. An MV-algebra of fuzzy sets (a Bold algebra) is characterized in the D-poset of fuzzy sets set-up. The sufficient and necessary conditions for a D-lattice of fuzzy sets to be a Bold algebra are given. The basic notions of the quantum logic theory - a state and an observable are defined in D-posets of fuzzy sets.

KEY WORDS

A D-poset of fuzzy sets, a D-lattice of fuzzy sets, a Bold algebra, a state, an observable.

[1] Associate professor, Dpt. of Natur. Sciences, Academy of Armed Forces, Liptovský Mikuláš, Slovakia
[2] Associate professor, Dpt. of Natur. Sciences, Academy of Armed Forces, Liptovský Mikuláš, Slovakia

1 INTRODUCTION

The model of fuzzy sets was created by Zadeh [23] to describe the events that are not given exactly, that are commented vaguely, non-uniquely. If A is a subset of a non-empty set X, then from the mathematical point of view the set A can be positively identified by its characteristic function $\chi_A : X \to \{0,1\}$ such that $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin A$.
On the other hand, a *fuzzy set* A can be characterized by a function $\mu_A : X \to [0,1]$. The function μ_A is called a *membership function* of a fuzzy set A and the value $\mu_A(x)$ is called the *grade of membership* of x in A. In the fuzzy set theory a fuzzy set is completely identified with its membership function. Relations between fuzzy sets and operations of fuzzy sets are defined by means of their membership functions. However, these operations of fuzzy sets should be defined in an appropriate way, i.e., they should coincide with the "classical" set operations in the case of the "classical" (the crisp) sets. The elementary operations of fuzzy sets A and B (fuzzy union, fuzzy intersection and fuzzy complementation) were defined by Zadeh as follows

$$A \cup B = C \text{ iff } \mu_C = \max_{x \in X}\{\mu_A(x), \mu_B(x)\} \text{ (denoted by } \mu_C = \mu_A \cup \mu_B\text{)},$$
$$A \cap B = D \text{ iff } \mu_D = \min_{x \in X}\{\mu_A(x), \mu_B(x)\} \text{ (denoted by } \mu_D = \mu_A \cap \mu_B\text{)},$$
$$A^c = X - A \text{ iff } \mu_{A^c}(x) = 1 - \mu_A(x) \text{ for all } x \in X \text{ (denoted by } \mu_A'\text{)}.$$

These operations are connected with the ordering $\subseteq$ of fuzzy sets, that is made identical with the natural ordering of membership functions

$$A \subseteq B \text{ iff } \mu_A(x) \le \mu_B(x) \text{ for all } x \in X.$$

Recall that these Zadeh's connectives are not the only possible ones (see, for example, Klement and Mesiar [12]).
In the mid Eighties of the last century there appeared attempts to build the quantum theory using ideas of the fuzzy sets theory. In the von Neumann's quantum logic theory [22], an important example is the set $\mathcal{L}(\mathcal{H})$ of all closed subspaces of a (real or complex) Hilbert space $\mathcal{H}$. From the algebraic point of view, the system $\mathcal{L}(\mathcal{H})$ is a complete orthomodular lattice. Another important example is a q-σ-algebra, suggested by Suppes [21], that is a non-empty collection Q of subsets of a non-empty set X which is closed with respect to the complementation and the countable unions of disjoint subsets. On the other hand, for the fuzzy sets theory, Piasecki [15] suggested a model of a *soft fuzzy σ-algebra.*
A soft fuzzy σ-algebra is a system M of fuzzy sets of a non-empty set X (i.e. M is a system of functions defined on X with values into the interval $[0,1]$) such that
(2.1) $1_X \in M$.
(2.2) If $\frac{1}{2}_X(x) = \frac{1}{2}$ for every $x \in X$, then $\frac{1}{2}_X \notin M$.
(2.3) If $f \in M$, then $f' = (1 - f) \in M$.
(2.4) If, $f_n \in M, n \ge 1$, then $\bigcup_{n \in N} f_n = \sup f_n \in M$.

Piasecki investigated the fuzzy probability measures from the Bayes principle point of view and he showed that the fuzzy probability measures fulfilling the Bayes principle are only so called *fuzzy P-measures*.

A mapping $p : M \to [0,1]$ is a fuzzy *P*-measure, if the following properties are satisfied.

(i) $p(f \cup f') = 1$ for all $f \in M$.

(ii) If $f_n \in M, n \geq 1, f_n \leq f_m', n \neq m$, then $p\left(\bigcup_{n\in N} f_n\right) = \sum_{n\in N} p(f_n)$.

The triplet X, M, p is called a *soft fuzzy probability space*.

Riečan $[18]$ combining the von Neumann approach with the Piasecki concept of the fuzzy soft-σ-algebra proposed to study a model of *F-quantum spaces* (*fuzzy quantum spaces*).

An *F*-quantum space is a couple (X, M), where X is a non-empty set (a *universum*) and $M \subset [0,1]^X$, which fulfils the conditions (2.1) – (2.4).

If $\mathcal{S}$ is a σ-algebra of subsets of a non-empty set X and $M = \{\chi_A : A \in \mathcal{S}\}$, then (X, M) is an *F*-quantum space.

Pykacz $[17]$ suggested to substitute the property (2.4) by a weaker one

(2.4)* If $f_n \in M, n \geq 1, f_n \leq f_m', \ n \neq m$, then $\bigcup_{n\in N} f_n \in M$.

The set M fulfilling the conditions (2.1) – (2.3) and (2.4)* is said to be a *fuzzy-q-σ-algebra* and the couple (X, M) an *F-quantum poset* (a *fuzzy quantum poset*).

Let Q be a q-σ-algebra of subsets of a non-empty set X and $\mathcal{F} = \{\chi_A : A \in Q$. Then $\mathcal{F}$ is a fuzzy-q-σ-algebra and (X , $\mathcal{F}$) is an *F*-quantum poset.

F-quantum spaces and *F*-quantum posets were studied by many authors. They mainly investigated problems which were important namely from quantum logic point of view. Therefore, there were introduced such notions as *F*-states, *F*-observables, compatibility and sum ability of *F*-observables, mean value, entropy, etc. (see, for example $[4],[6],[7],[8],[9],[10],[19]$).

In the early 1990s the attempts to create fuzzy sets probability models without the obligation of the domain of fuzzy sets to be a lattice occurred. Kôpka $[13]$ introduced a new probability model based on the theory of fuzzy sets, a difference poset of fuzzy sets (a D-poset of fuzzy sets), in which a difference of comparable fuzzy sets is the primary operation.

In this paper, we introduce some properties of a D-lattice of fuzzy sets (i.e. a D-poset of fuzzy sets, that is a lattice as well), we characterize MV- algebras of fuzzy sets (Bold algebras) in the D-poset of fuzzy sets set-up and we give the sufficient and necessary conditions for a D-lattice of fuzzy sets to be a Bold algebra.

2 DIFFERENCE POSETS OF FUZZY SETS

Let $\mathcal{F}$ be a partially ordered system of fuzzy sets. We say that $\mathcal{F}$ is lattice ordered (or $\mathcal{F}$ is a lattice), if for every $f, g \in \mathcal{F}$ the least upper bound $f \vee g$ and the greatest lower bound $f \wedge g$ exist in the system $\mathcal{F}$. Let us note that $f \vee g$ need not coincide with $f \cup g$ and dually $f \wedge g \neq f \cap g$, in general. However, if $f \cup g \in \mathcal{F}$ ($f \cap g \in \mathcal{F}$) then $f \cup g = f \vee g$ ($f \cap g = f \wedge g$).

Example 1 *Let* $\mathcal{F} \subseteq [0,1]^{[0,1]}$, $\mathcal{F} = \{0, 1, f, g\}$, *where* $0(x) = 0$, $1(x) = 1$, $f(x) = x$, $g(x) = 1 - x$ *for all* $x \in [0,1]$. *Then* $(f \cup g)(x) = \max_{x \in [0,1]}(x, 1-x) = \frac{1}{2} + \left| x - \frac{1}{2} \right|$ *for every* $x \in [0,1]$ *and so* $f \cup g \notin \mathcal{F}$, *but* $f \vee g = 1$ *and* $1 \in \mathcal{F}$.

If a system of fuzzy sets is lattice ordered, we can define the difference of fuzzy sets equivalently to the difference of crisp sets by the formula

$$f - g = f \wedge g' = \min\{f, g'\}.$$

In this case,

$$f - (f - g) = g \tag{1}$$

is not true for comparable fuzzy sets $g \leq f$, in general.

Kôpka $[13]$ defined the difference of comparable fuzzy sets such that the property (1) is fulfilled.

Definition 2 *Let* $\mathcal{F} \subseteq [0,1]^X$ *be a system of fuzzy subsets of a non-empty set* X. *A partial binary operation* ⊖ *is said to be a difference on* $\mathcal{F}$, *if the element* $f \ \theta \ g$ *is defined in* $\mathcal{F}$ *for* $g \leq f$, *and the following conditions are satisfied.*

(D1) $f \ \theta \ g \leq f$.

(D2) $f \ \theta \ (f \ \theta \ g) = g$.

(D3) *If* $f, g, h \in \mathcal{F}$, $h \leq g \leq f$, *then* $f \ \theta \ g \leq f \ \theta \ h$ *and* $(f \ \theta \ h) \theta (f \ \theta \ g) = g \ \theta \ h . \theta$

With respect to the probability theory, we need consider such system $\mathcal{F}$ of fuzzy sets for which the next conditions hold.

(D4) *If* $1_{\mathcal{F}}(x) = 1$ *for any* $x \in X$, *then* $1_{\mathcal{F}} \in \mathcal{F}$.

(D5) *If* $(f_n)_{n \in N} \subseteq \mathcal{F}$, $f_n \leq f_{n+1}, n \geq 1$, *then* $\bigvee_{n \in N} f_n \in \mathcal{F}$.

Definition 3 *A system* $\mathcal{F}$ *of fuzzy sets fulfilling the conditions* (D1) – (D4) *is called a difference poset of fuzzy sets (shortly a D-poset of fuzzy sets). Moreover, if a D-poset of fuzzy sets* $\mathcal{F}$ *is lattice ordered, then* $\mathcal{F}$ *is called a D-lattice of fuzzy sets. The system* $\mathcal{F}$ *fulfilling the conditions* (D1) – (D5) *is called a D-σ-poset of fuzzy sets.*

It is evident, that the element $1_{\mathcal{F}}$ is the greatest element of $\mathcal{F}$ and the element $1_{\mathcal{F}} \ \theta \ 1_{\mathcal{F}}$ is the least element of $\mathcal{F}$, denoted by $0_{\mathcal{F}}$ and, moreover, $0_{\mathcal{F}}(x) = 0$ for any $x \in X$.

Example 4 *Let $\mathcal{F}$ be a system of fuzzy subsets of a non-empty set X. Let $\Phi:[0,1]\to[0,\infty)$ be an injective increasing continuous function such that $\Phi(0)=0$. A partial binary operation $\ominus$ defined by the formula*

$$(f \,\theta\, g)(t)=\Phi^{-1}(\Phi(f(t))-\Phi(g(t)))$$

for every $f,g\in\mathcal{F}$, $g\le f, t\in X$, is a difference on $\mathcal{F}$.

Specifically, if $\Phi(x)=kx, k>0$, then $\ominus$ is the usual difference of real functions

$$(f \,\theta\, g)(t)=f(t)-g(t),$$

and if $\Phi(x)=x^2$ then

$$(f \,\theta\, g)(t)=\sqrt{f^2(t)-g^2(t)}.$$

A function Φ is called a generator of a difference. Moreover, if $\Phi(1)=1$, then Φ is called a normed generator.

Proposition 5 *Let $\mathcal{F}$ be a D-poset of fuzzy sets and let $f,g,h,k\in\mathcal{F}$. Then the following assertions are true*.

(i) *If $g\le f\le h$, then $f\,\theta\, g\le h\,\theta\, g$, and $(h\,\theta\, g)\,\theta\,(f\,\theta\, g)=h\,\theta\, f$.*
(ii) *If $g\le h$ and $f\le h\,\theta\, g$, then $g\le h\,\theta\, f$, and $(h\,\theta\, g)\,\theta\, f=(h\,\theta\, f)\,\theta\, g$.*
(iii) *If $g\le f\le h$, then $g\le h\,\theta\,(f\,\theta\, g)$, and $(h\,\theta\,(f\,\theta\, g))\,\theta\, g=h\,\theta\, f$.*
(iv) *If $g\le h$ and $f\le h$, then $h\,\theta\, f=h\,\theta\, g$ if and only if $f=g$.*
(v) *If $k\le f\le h$, $k\le g\le h$, then $h\,\theta\, f=g\ominus k$ if and only* if $h\,\theta\, g=f\,\theta\, k$.
(vi) *If $g\le f$, then $f\,\theta\, g=0_{\mathcal{F}}$ if and only if $f=g$.*
(vii) *If $g\le f$, then $f\,\theta\, g=f$ if and only if $f=0_{\mathcal{F}}$.*

For any $f\in\mathcal{F}$ we put

$$f^{\perp}=1_{\mathcal{F}}\,\theta\, f.$$

The unary operation $\perp$ is an involution (i.e. $(f^{\perp})^{\perp}=f$) and an order-reversing operation (i.e., if $g\le f$ then $f^{\perp}\le g^{\perp}$.

If $\mathcal{F}$ is a D-lattice of fuzzy sets, then we can define a (total) binary operation – on $\mathcal{F}$ by the formula

$$f-g=f\,\theta\,(f\wedge g). \tag{2}$$

It is easy to prove that the binary operation – has the following properties.

(1) If $g\le f$ then $f-g=f\,\theta\, g$.
(2) $f-g\le f$ for any $f,g\in\mathcal{F}$.
(3) $f-(f-g)=f\wedge g$.
(4) If $g\le f$, then $g-f=0_{\mathcal{F}}$.
(5) $f\wedge g=0_{\mathcal{F}}$ if and only if $f-g=f$.

A dual binary operation $+$ to the operation – on a D-lattice of fuzzy sets is defined by

$$f+g=(f^{\perp}-g^{\perp}) \text{ for any } f,g\in\mathcal{F}. \tag{3}$$

Evidently $f+0_{\mathcal{F}}=f$, $f+1_{\mathcal{F}}=1_{\mathcal{F}}$ and $f+f^{\perp}=1_{\mathcal{F}}$ for every $f\in\mathcal{F}$.

Proposition 6 *Let $\mathcal{F}$ be a D-lattice of fuzzy sets such that $f \cap g \in \mathcal{F}$ for every $f, g \in \mathcal{F}$. Then the operation $+$ is commutative and associative*.

Proof. This result follows from the fact that the operation $\cap$ is defined point wisely. Indeed,

$$(f+g)(x) = f(x) + g(x) = \left(f^{\perp}(x) - g(x)\right)^{\perp} = \left(f^{\perp}(x)\,\theta\,\left(f^{\perp}(x) \wedge g(x)\right)\right)^{\perp}$$
$$= \left(f^{\perp}(x)\,\theta\,\left(f^{\perp}(x) \cap g(x)\right)\right)^{\perp} = \left(f^{\perp}(x)\,\theta\,\min\{f^{\perp}(x), g(x)\}\right)^{\perp}.$$

There are always just two possibilities for real numbers $f^{\perp}(x)$ and $g(x)$: either $f^{\perp}(x) \le g(x)$ or $g(x) \le f^{\perp}(x)$. The first inequality implies $(f+g)(x) = 1$ and the same result we obtain for

$$(g+f)(x) = \left(g^{\perp}(x) - f(x)\right)^{\perp} = \left(g^{\perp}(x)\,\theta\,\min\{g^{\perp}(x), f(x)\}\right)^{\perp} = \left(g^{\perp}(x)\,\theta\,g^{\perp}(x)\right)^{\perp} = 1.$$

On the other hand, from $g(x) \le f^{\perp}(x)$ and (ii) of Proposition 5 we have

$$(f+g)(x) = \left(f^{\perp}(x) - g(x)\right)^{\perp} = \left(g^{\perp}(x) - f(x)\right)^{\perp} = (g+f)(x).$$

Similarly $(f+g)+h = f+(g+h)$.

An MV-algebra (introduced by C. Chang [2]) is a very important algebraic model of many-valued logics.

Definition 7 *An MV-algebra $\mathcal{A}$ is an algebra ($\mathcal{A}$, $+$, $*$, $0, 1$), where $\mathcal{A}$ is a non-empty set, 0 and 1 are constant elements of $\mathcal{A}$, $+$ is a binary operation, and $*$ is a unary operation, satisfying the following axioms*:

(MVA1) $a+b=b+a$.
(MVA2) $(a+b)+c=a+(b+c)$.
(MVA3) $a+0=a$.
(MVA4) $a+1=1$.
(MVA5) $(a^*)^*=a$.
(MVA6) $0^*=1$.
(MVA7) $a+a^*=1$.
(MVA8) $(a^*+b)^*+b=(a+b^*)^*+a$.

The lattice operations $\vee$ and $\wedge$ are defined in an MV-algebra $\mathcal{A}$ by

$$a \vee b = (a^*+b)^*+b \quad \text{and} \quad a \wedge b = \left((a+b^*)^*+b^*\right)^*. \tag{4}$$

We write $a \le b$ if $a \vee b = b$. The relation $\le$ is a partial ordering over $\mathcal{A}$ and $0 \le a \le 1$, for every $a \in \mathcal{A}$. An MV-algebra is a distributive lattice with respect to the operations $\vee, \wedge$.

Let a, b be any two elements of an MV-algebra $\mathcal{A}$. If we put

$$b\;\theta\;a = (a+b^*)^* \quad \text{for} \quad a \le b, \tag{5}$$

then θ is a difference on $\mathcal{A}$ with properties (D1) – (D3).

Example 8 *Let $\mathcal{G} = [0,1]^X$. We put $0(x)=0$, $1(x)=1$,*

$$(f+g)(x) = \min\{f(x)+g(x), 1\}, \tag{6}$$

$$f^*(x) = 1 - f(x), \tag{7}$$

for any $x \in X$ *. Then* ($\mathcal{G}$, $+$, $*$, 0, 1) *becomes an MV-algebra*.

Every sub algebra of $\mathcal{G}$ is according to $[1]$ called a *Bold algebra* (of fuzzy sets).

Proposition 9 *Every Bold algebra is a D-lattice of fuzzy sets*.

Proof. Let $\mathcal{B} \subseteq [0,1]^X$ be a Bold algebra. By the above, $\mathcal{B}$ is lattice ordered. First we prove that $f \le g$ (in $\mathcal{B}$) if and only if $f(x) \le g(x)$ for any $x \in X$.
Due to (4), (6) and (7) we have for any $x \in X$.

$$(f \vee g)(x) = \left((f^* + g)^* + g\right)(x) = \min\left\{(f^* + g)^*(x) + g(x), 1\right\}$$
$$= \min\{1 - \min\{1 - f(x) + g(x), 1\} + g(x), 1\}.$$

If $f(x) \le g(x)$ then $1 \le 1 - f(x) + g(x)$ and therefore,

$$(f \vee g)(x) = \min\{g(x), 1\} = g(x).$$

Conversely, if $f \vee g = g$ then

$$1 - \min\{1 - f(x) + g(x), 1\} + g(x) = g(x),$$
$$\min\{1 - f(x) + g(x), 1\} = 1,$$
$$1 - f(x) + g(x) \ge 1,$$
$$g(x) \ge f(x).$$

Suppose that $f \le g$. In view of (5) we get

$$(g - f)(x) = (f + g^*)^*(x) = 1 - \min\{1 + f(x) - g(x), 1\} = 1 - (1 + f(x) - g(x)) = g(x) - f(x)$$

for any $x \in X$. Now the validity of axioms (D1) – (D3) is evident.

The converse assertion is not true, in general.

Example 10 *Let* $\mathcal{H} \subseteq [0,1]^{[0,1]}$, $\mathcal{H} = \{0, 1, a, b, c\}$, *where* $0(x) = 0$ *,* $1(x) = 1$*,* $a(x) = x$*,* $b(x) = 1 - x$*,* $c(x) = \frac{1}{2}$ *for all* $x \in [0,1]$*. Evidently* $b = 1 - a$ *and* $c = 1 - c$ *.* $\mathcal{H}$ *is a D-lattice of fuzzy sets, but it is not a Bold algebra. Indeed,*

$$(a \vee (1 - a) \wedge c) = 1 \wedge c = c \text{ and } (a \wedge c) \vee ((1 - a) \wedge c) = 0 \vee 0 = 0,$$

so $\mathcal{H}$ *is not a distributive lattice.*

If $\mathcal{F}$ be a D-lattice of fuzzy sets such that $f \cup g \in \mathcal{F}$ for every $f, g \in \mathcal{F}$, then $\mathcal{F}$ is a Bold algebra. Indeed, it suffices to put $f + g = (f^* - g)^*$ for any $f, g \in \mathcal{F}$ and $f^* = 1_{\mathcal{F}} - f$. The converse assertion is not true, in general.

Example 11 *Let* $\mathcal{H} \subseteq [0,1]^{[0,1]}$, $\mathcal{H} = \{0, 1, a, b, c, d\}$, *where* $0(x) = 0$ *,* $1(x) = 1$*,* $a(x) = x$*,* $b(x) = 1 - x$*,* $c(x) = \frac{1}{2}x$*,* $d(x) = 1 - \frac{1}{2}x$ *and operations* $+$ *and* $*$ *are as above. Then* ($\mathcal{H}$, $+$, $*$, 0, 1) *is a Bold algebra, in which* $a \cup b$ *does not belong to* $\mathcal{H}$.

There is a natural question: When will a D-lattice of fuzzy sets be a Bold algebra? The answer relates to the notion of the compatibility of fuzzy sets.

3 COMPATIBILITY IN D-POSETS OF FUZZY SETS

Very important relation from the physical applications point of view is the compatibility relation. It is well known that an orthomodular lattice of pair wise compatible elements creates a Boolean algebra. A similar result was obtained in orthomodular posets, where a stronger relation of so called *f*-compatibility has to be used instead of the pairwise compatibility (see [16]). Very interesting results were attained during the research of the compatibility relation in D-posets of fuzzy sets introduced by Kôpka [14].

Definition 12 *Let $\mathcal{F}$ be a D-poset of fuzzy sets. We say that fuzzy sets $f, g \in \mathcal{F}$ are compatible, and write $f \leftrightarrow g$, if there exist fuzzy sets $k, h \in \mathcal{F}$ such that $k \le f \le h$, $k \le g \le h$ and $h \ \theta \ f = g \ \theta \ k$ (equivalently $h \ \theta \ g = f \ \theta \ k$).*

Theorem 13 *Let $\mathcal{F}$ be a D-lattice of fuzzy sets. Then fuzzy sets $f, g \in \mathcal{F}$ are compatible if and only if $(f \vee g)\theta \ g = f \ \theta \ (f \wedge g)$.*

Proof. First we prove that $((f \vee g)\theta \ f) \wedge ((f \vee g)\theta \ g) = 0_{\mathcal{F}}$ for arbitrary f, g from a D-lattice of fuzzy sets.
From the inequalities $f \le f \vee g, g \le f \vee g$ and (D3) we have

$$0_{\mathcal{F}} = (f \vee g)\theta \ (f \vee g) \le (f \vee g)\theta \ f$$

and $0_{\mathcal{F}} \le (f \vee g)\theta \ g$. If there exists $w \in \mathcal{F}$, $w \le (f \vee g)\theta \ f$, $w \le (f \vee g)\theta \ g$, then $f = (f \vee g)\theta \ ((f \vee g)\theta \ f) \le (f \vee g)\theta \ w$, $g \le (f \vee g)\theta \ w$, therefore,

$$f \vee g \le (f \vee g)\theta \ w \le f \vee g$$

and so $(f \vee g)\theta \ w = f \vee g$, which implies that $w = 0_{\mathcal{F}}$. We proved that $0_{\mathcal{F}}$ is the greatest lower bound of the set $\{(f \vee g)\theta \ f, (f \vee g)\theta \ g\}$.
Let $f \leftrightarrow g$. Then there exist fuzzy sets $k, h \in \mathcal{F}$ such that $k \le f \le h$, $k \le g \le h$ and $h \ \theta \ f = g \ \theta \ k$. From the inequalities $f \le f \vee g \le h, g \le f \vee g \le h$ it follows that $(f \vee g)\theta \ f \le h \ \theta \ f = g \ \theta \ k \le g$, and similarly $(f \vee g)\theta \ g \le f$. Then

$$f \ \theta \ ((f \vee g)\theta \ g) = ((f \vee g)\theta \ ((f \vee g)\theta \ f))\theta \ ((f \vee g)\theta \ g) = g \ \theta \ ((f \vee g)\theta \ f).$$

Denote $u = f \ \theta \ ((f \vee g)\theta \ g)$. It is clear that $u \le f \vee g$ and $f \ \theta \ u = (f \vee g)\theta \ g$, $g \ \theta \ u = (f \vee g)\theta \ f$.
Calculate

$$(f \vee g)\theta \ u = (f \ \theta \ u) \wedge (g \ \theta \ u) = ((f \vee g)\theta \ g) \wedge ((f \vee g)\theta \ f) = 0_{\mathcal{F}},$$

hence $u = f \wedge g$, which gives $f \ \theta \ (f \wedge g) = (u \le f \vee g)\theta \ g$.
The sufficient condition is evident.

Theorem 14 *In a Bold algebra, any two fuzzy sets are mutually compatible. Conversely, a D-lattice of mutually compatible fuzzy sets is a Bold algebra.*

Proof. Let $\mathcal{B} \subset [0,1]^X$ be a Bold algebra and $f, g \in \mathcal{B}$. There are always just two possibilities for real numbers $f(x), g(x)$: either $f(x) \le g(x)$ or $g(x) \le f(x)$. The first inequality implies

$$((f \vee g)\theta \ g)(x) = (f \vee g)(x)\theta \ g(x) = g(x)\theta \ g(x) = 0$$

and $f \ \theta \ (f \wedge g)(x) = f(x)\theta \ f(x) = 0$, for any $x \in X$.

If $g(x) \le f(x)$, then $((f \vee g)\theta\, g)(x) = f(x)\theta\, g(x) = f\ \theta\ (f \wedge g)(x)$ for any $x \in X$, so $f \leftrightarrow g$.

On the contrary, suppose that $\mathcal{B} \subset [0,1]^X$ is a D-lattice of mutually compatible fuzzy sets. Let us put $f^* = 1 - f$ and $f + g = (f^* - g)^*$ for every $f, g \in \mathcal{B}$, where – is a total binary operation on $\mathcal{B}$ defined by the formula (2). The completion of the proof requires routine verifications of the axioms (MVA1) – (MVA8). N

Let us assume fuzzy sets a, b, c from the Example 10. Then $a \leftrightarrow b$, but $a\ \theta\ c$. Indeed, $(a \vee c)\theta\ c = 1\ \theta\ c = c$ and $a\ \theta\ (a \wedge c) = a\ \theta\ 0 = a$.

Definition 15 *A maximal subset M of mutually compatible fuzzy sets of a D-lattice of fuzzy sets $\mathcal{F}$ is called a block of $\mathcal{F}$.*

Theorem 16
(i) *Every subset $\mathcal{A}$ of mutually compatible fuzzy sets of a D-lattice of fuzzy sets $\mathcal{F}$ is contained in a block.*
(ii) *Every D-lattice of fuzzy sets $\mathcal{F}$ is a set-theoretical union of its blocks.*

Proof. (i) Let $\emptyset \neq \mathcal{A} \subseteq \mathcal{F}$ be a set of mutually compatible fuzzy sets of $\mathcal{F}$ and $\mathcal{A} = \{\mathcal{B} \subseteq \mathcal{F} : \mathcal{A} \subseteq \mathcal{B}$, $\mathcal{B}$ is a set of mutually compatible fuzzy sets$\}$. Then for every chain $\mathcal{B} \subseteq \mathcal{A}$ (i.e., for $X, Y \in \mathcal{B}$ we have $X \subseteq Y$ or $Y \subseteq X$), the set $\bigcup \mathcal{B}$ belongs to $\mathcal{A}$. By maximal principle there exists a maximal element $M \in \mathcal{A}$.
(ii) Let $f \in \mathcal{F}$. Denote by M_f the block containing the set $A = \{0_{\mathcal{F}}, f, f^{\perp}, 1_{\mathcal{F}}\}$.
Then $\bigcup_{f \in F} M_f = \mathcal{F}$.
The exact proof of this theorem can be found in [20] for a general case of a D-lattice.

4 STATES AND OBSERVABLES ON D-POSETS OF FUZZY SETS

States and observables are the fundamental notions of the quantum logics probability theory. D-posets of fuzzy sets have been studied as carriers of states or probability measures in the fuzzy probability theory.

Definition 17 *Let $\mathcal{F}$ be a D-σ-poset of fuzzy sets. A probability measure (a state) on $\mathcal{F}$ is a mapping $m : \mathcal{F} \to [0,1]$ such that*
(P1) $m(1_{\mathcal{F}}) = 1$.
(P2) *If $(f_n)_{n \in N} \subseteq \mathcal{F}$, $f_n \le f_{n+1}$ and $\bigvee_{n \in N} f_n = f$, then $m(f) = m(f_1) + \sum_{n=2}^{\infty} m(f_n \ominus f_{n-1})$.*

Example 18 *Let $\mathcal{F} \subseteq [0,1]^X$ be a D-poset of fuzzy sets. Let $t_0 \in X$ such that $f(t_0)$ there exists for every $f \in \mathcal{F}$. Then the mapping $s : \mathcal{F} \to [0,1]$ defined by*
$$s(f) = f(t_0) \text{ for any } f \in \mathcal{F},$$
is a state on $\mathcal{F}$.

An *observable* is a quantum paraphrase of a random variable.

Definition 19 *Let $\mathcal{F}$ be a D-σ-poset of fuzzy sets and $\mathcal{B}(R)$ be the Borel σ-algebra of the real line R. The mapping $x:\mathcal{B}(R)\to\mathcal{F}$ is said to be an observable on $\mathcal{F}$, if the following conditions are fulfilled.*

(O1) $x(R)=1_X$

(O2) *If $(A_n)_{n\in N}$ is a sequence of Borel sets such that $A_n\subseteq A_{n+1}$ for every $n\in N$, then*

$$x(A_n)\le x(A_{n+1}),\ \textit{and}\ x\left(\bigcup_{n\in N}A_n\right)=\bigvee_{n\in N}x(A_n).$$

(O3) If A,B are Borel sets, $A\subseteq B$, then $x(B\setminus A)=x(B)\ \theta\ x(A)$.

Example 20 *Let $\mathcal{F}$ be a D-poset of fuzzy sets and $f\in\mathcal{F}$. A mapping $x_f:\mathcal{B}(R)\to\mathcal{F}$ defined by*

$$x_f(E)=\begin{cases}1, & if\quad \{0,1\}\cap E=\{0,1\},\\ f, & if\quad \{0,1\}\cap E=\{1\},\\ 1-f, & if\quad \{0,1\}\cap E=\{0\},\\ 0, & if\quad \{0,1\}\cap E=\varnothing\end{cases}$$

is an observable on $\mathcal{F}$ called an indicator of f.

The set $\mathcal{R}(x)=\{x(E):E\in\mathcal{B}(R)\}$ is said to be the range of an observable x. We note that if x is an observable in a σ-orthomodular poset $\mathcal{L}$, then the range $\mathcal{R}(x)$ is always a Boolean sub-σ-algebra of $\mathcal{L}$. But the range of an observable on a D-poset of fuzzy sets is not a sub-D-poset, in general.

Example 21 *Let $\mathcal{F}$ be a D-poset of fuzzy sets (see Example 4), where $\Phi(t)=t$ for every $t\in[0,1]$. Let x be the observable on $\mathcal{F}$ defined as in Example 20, where $f\in\mathcal{F}$ is a constant function, for example, f = 0.8. Then $\mathcal{R}(x)=\{0,0.2,0.8,1\}$, but $0.8\ \theta\ 0.2=0.6$ is not contained in $\mathcal{R}(x)$.*

Proposition 22 *Let x be an observable on a D-poset of fuzzy sets $\mathcal{F}$. Then the range $\mathcal{R}(x)$ is contained in the set of pair wise compatible elements.*

Proof. Let $f,g\in\mathcal{R}(x)$, $f=x(A),g=x(B)$, where $A,B\in\mathcal{B}(R)$. We put $h=x(A\cup B)$ and $k=x(A\cap B)$. Evidently $k\le f\le h, k\le g\le h$ and

$$h\ \theta\ f=x(A\cup B)\ \theta\ x(A)=x(A\cup B\setminus A)=x(B\setminus A\cap B)=x(B)\ \theta\ x(A\cap B)=g\ \theta\ k,$$

therefore, $f\leftrightarrow g$. ∎

It is easy to prove that if $x:\mathcal{B}(R)\to\mathcal{F}$ is an observable on $\mathcal{F}$, then the mapping $m_x:\mathcal{B}(R)\to[0,1]$, $m_x(E)=m(x(E))$, is a probability measure on $\mathcal{B}(R)$. The mapping m_x is said to be a *probability distribution* of the observable x in the state m.

Now a mean value of the observable x in the state m can be defined by the integral

$$E(x)=\int_R t\,dm_x(dt),$$

if it exists and is finite.

The dispersion (variance) can be defined in a D-poset of fuzzy sets in a similar manner. So, a D-poset of fuzzy sets is a suitable model for the probality theory on non-Boolean structures.

THE OPEN PPROBLEM

It is known (see Theorem 16) that every D-lattice of fuzzy sets is a set-theoretical union of its blocks (Bold algebras). Now, a natural dual question arises: How can we construct a D-poset of fuzzy sets from a given collection of Bold algebras?
We note that a method of quantum logics construction based on the "pasting" of Boolean algebras was originally suggested by R. Greechie [11]. A generalization of this method for pasting of MV-algebras has been performed in [3].

ACKNOWLEGEMENT

This work has been partially supported by the Slovak Academy of Sciences via the project Center of Excellence – Physics of Information, by the grant 1/2/2005 and by the grants VEGA 2/6088/6, 1/2002/05 SAV, Bratislava, Slovakia.

REFERENCES

[1] Belluce, L. P., Semisimple algebras of infinite valued logic and bold fuzzy set theory, Can. J. Math. **38** (1986), No. 6, 1356-1379.
[2] C. C. Chang, *Algebraic analysis of many-valued logics*, Trans. Amer. Math. Soc. **88** (1958), 467-490.
[3] F. Chovanec and M. Jurečková, MV-algebra pasting. International Journal of Theoretical Physics, **42**(2003), No. 9, 1913-1926.
[4] F. Chovanec and F. Kôpka, *Fuzzy equality and convergences for F-observables in F-quantum spaces*, Applications of mathematics **36**(1991), 32-45.
[5] F. Chovanec and F. Kôpka, *Boolean D-posets*, Tatra Mountains Mathematical Publications **10**(1997), 183-197.
[6] A. Dvurečenskij and F. Chovanec, *Fuzzy quantum spaces and compatibility*, International Journal of Theoretical Physics **27** (1988), 1069-1082.
[7] A. Dvurečenskij and F. Chovanec, *Compatibility and summability of observables in fuzzy quantum spaces*, BUSEFAL **56** (1993), 4-14.
[8] A. Dvurečenski, F. Chovanec and F. Kôpka, *On mean value additivity on fuzzy quantum spaces*, Acta Math. Univ. Comen. **58-59** (1990), 107-117.
[9] A. Dvurečenskij and B. Riečan, *On joint observables for F-quantum spaces*, BUSEFAL **35** (1988), 10-14.
[10] A. Dvurečenskij and B. Riečan, *On joint distribution of observables for F-quantum spaces*, Fuzzy Sets and Systems **20** (1991), 65-73.
[11] Greechie, R.: *Orthomodular lattices admitting no states*. J. Comb. Theor., **10**(1971), 119-132.
[12] E. P. Klement, R. Mesiar and E. Pap, *Triangular Norms*, Springer-Verlag, 2000.
[13] F. Kôpka, *D-posets of fuzzy sets*, Tatra Mountains Mathematical Publications **1**(1992), 83-89.

[14] F. Kôpka, *Compatibility in D-posets of fuzzy sets*, Tatra Mountains Mathematical Publications **6**(1995), 95-102.
[15] K. Piasecki, *Probability of fuzzy events defined as denumerable additivity measure*, Fuzzy Sets and Systems **17** (1985), 271-284.
[16] P. Pták and S. Pulmannová, *Orthomodular Structures as Quantum Logics*, Kluwer Academic Publ., Dordrecht, Boston, London, 1991.
[17] J. Pykacz, *Quantum logics and soft fuzzy probability spaces*, BUSEFAL **32** (1987), 150-157.
[18] B. Riečan, *A new approach to some notions of statistical quantum mechanics*, BUSEFAL **35** (1988), 4-6.
[19] B. Riečan, *On mean value in F-quantum spaces*, Apl. Mat. **35** (1990), 209-214.
[20] Z. Riečanová, *Generalization of Blocks for D-Lattices and Lattice - Ordered Effect Algebras*, International Journal of Theoretical Physics **39**(2000), Iss. 2, pp. 231—237
[21] Suppes, P.: *The probability argument for a non-classical logic of quantum mechanics*, Phil. Science **33** (1966), 14-21.
[22] J. von Neumann, *Mathematische Grundlagen der Quantummechanik*, Springer-Verlag, Berlin, 1932.
[23] L. A.Zadeh, *Fuzzy sets*, Inform. Control **8** (1965), 338-353.