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On fuzzy T-Neighborhood Groups

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Abstract: We shall generalize the concept of fuzzy neighborhood group, which is introduced by Ahsanullah in [1], using the concept of t-norm. We establish some basic results and prove some characterization theorem of fuzzy T-neighborhood groups.

1- Preliminaries

For the definition of fuzzy topology we refer to [10]. Our work mainly will deal with fuzzy T-neighborhood spaces introduced by Hashem and Morsi [5]. The notion of filterbasis which plays an essential role for the development of fuzzy T-neighborhood spaces can be found in [4]. The saturation operation for filterbasis β is define as

$$\beta^{\sim} = \{\mu \in I^X / \forall \varepsilon \in I_0 \exists \nu_\varepsilon \in \beta s.t. \mu_\varepsilon - \varepsilon \le \mu\}$$

For our convenience, however, we shall recall here the definition of fuzzy T- neighborhood spaces [5].

A family $\beta = (\beta(x))_{x \in X}$, of filterbases in I^X, will be a T-neighborhood base in X if and only if it satisfies the following two conditions for all $x \in X$:

(TNB1) $\upsilon(x) = 1$ for all $\upsilon \in \beta(x)$;

(TNB2) Every $\upsilon \in \beta(x)$ has a T-kernel in β. This is a family $(\upsilon_{y\varepsilon} \in \beta(y)),\ (y,\varepsilon) \in X \times Io$

which satisfies for all $(y, z, \varepsilon) \in X \times X \times I_o$,

$$[\upsilon_{x,\varepsilon}(z)\ T\ \upsilon_{z,\varepsilon}(y)] \le \upsilon(y) + \varepsilon.$$

If β is a (fuzzy) T-neighborhood system on X, then it provides a unique fuzzy closure operator given by

$$\mu^{-}(x) = \inf_{\upsilon \in \beta(x)} hgt(\mu\ T\ \upsilon) = \inf_{\upsilon \in \beta(x)} \sup_{y \in X} (\mu(y)\ T\ \upsilon(y)).$$

The fuzzy topology generated by this fuzzy closure is denoted by $t(\beta)$. Then the fuzzy topological space $(X, t(\beta))$ is called a fuzzy T-neighborhood space.

2-Fuzzy T-Neighborhood Groups

In the sequel we shall write a multiplicative group as $(G, .)$ or often as G, and e shall denote its identity element. Throughout, T is a continuous triangular norm.

Definition 2.1 Let $(G, .)$ be a group and β a fuzzy T-neighborhood system on G . Then the triple $(G, ., t(\beta))$ is called a fuzzy T-neighborhood group if and only if the following are fulfilled:

(FG1) The mapping $m : (G \times G,\ t(\beta) \times t(\beta)) \rightarrow (G, t(\beta))$:

$(x, y) \rightarrow xy$ is continuous.

(FG2) The mapping $r : (G, t(\beta)) \rightarrow (G, t(\beta))$:

$x \rightarrow x^{-1}$ is continuous.

A group structure and a fuzzy T-neighborhood system are said to be compatible if and only if (FG1) and (FG2) are satisfied.

Definition 2.2 Let $(G, .)$ be a group, then for all $\mu, v \in I^G$ and $x \in G$ we define

$$\mu . v\, (x) = \sup_{st=x} \mu\, (s) \wedge v\ \ (t)$$

we can also write

$$\mu . v\, (x) = m\, (\mu \times v\,)$$

where $\mu \times v \in I^{G \times G}$ is given by

$\mu \times v : G \times G \rightarrow I : (x, y) \rightarrow \mu(x) \wedge v(y).$

Remark that the above definition is obtained from the extension principle of Zadeh.

Definition 2.3 Let $(G, .)$ be a group, X a set, and $f \in X^G$, then we define

$$f^{-1} = f \circ r$$

and we call f symmetric if and only if $f = f^{-1}$.

Proposition 2.1 Let $(G, .)$ be a group and β a fuzzy T-neighborhood system on G. Then $(G, ., t(\beta))$ is fuzzy T-neighborhood group if and only if the mapping

$$h : (G \times G, t\, (\beta) \times t(\beta)) \rightarrow (G, t(\beta)) : (x, y) \mapsto x\, y^{-1} \quad \text{is continuous.}$$

<u>Proof</u>.

Let $k(x, y) = (x, y^{-1})$. Then continuity of k follows from (FG2). This together with (FG1) shows that h is continuous. The converse follows from the facts that $x^{-1} = e.x^{-1} = h(e, x)$ and $x\, y = x(y^{-1})^{-1} = h(x, y^{-1})$

Proposition 2.2 Let $(G, .)$ be a group and β a fuzzy T-neighborhood system on G. Then

(a) The mapping r is continuous at $e \in G$ if and only if for all $\mu \in \beta\, (e)$ and for all $\varepsilon \in I_0$ there exists $v \in \beta\, (e)$ such that $v - \varepsilon \leq \mu^{-1}$

(b) The mapping m is continuous at $(e,e) \in G\times G$ if and only if for all $\mu \in \beta(e)$ and for all $\varepsilon \in I_0$ there exists $v \in \beta(e)$ such that $v.v - \varepsilon \le \mu$

Proof.

(a) Follow from the fact that $r^{--1}(\mu) = \mu(r(x)) = \mu \circ r(x) = \mu^{-1}$

(b) m is continuous at $(e.e) \in G\times G$ *if* $\forall\mu \in \beta(e)$, *and* $\forall\varepsilon \in I_0\ \exists\ v \times v \in \beta(e) \times \beta(e)$ such that $m(v \times v) - \varepsilon \le \mu$. Since $m(v \times v) = v.\,v$, then the assertion follows.

Proposition 2.3 Let $(G, ., t(\beta))$ be a fuzzy T-neighborhood group, $x \in G$ and $y \in G$.

Then

(a) $\zeta_x : G \to G : z \mapsto xz$ (*resp.* $\Re_x : G \to G : z \mapsto zx$) the left (resp. right) translation, $\psi_{(\chi,y)} : G \to G : z \mapsto xzy$ and r are homeomorphisms.

(b) The inner automorphism operator $Int_x : G \to G : z \mapsto xzx^{-1}$ is a homeomorphism.

(c) $v \in \beta(e)$ if and only if $\zeta_x(v) \in \beta(x)$ if and only if $\Re_x(v) \in \beta(x)$.

(d) $v \in \beta(x)$ if and only if $\zeta_x^{-1}(v) \in \beta(e)$ if and only if $\Re_x^{-1}(v) \in \beta(e)$.

(e) If $v \in \beta(e)$ then $v^{-1} \in \beta(e)$.

(f) $v \wedge v^{-1}$, $v \vee v^{-1}$, and $v\,.\,v^{-1}$ are symmetric.

Proof.

(a) It is clear that $\Re_x$ is a 1:1 and onto mapping. Then by Definition 2.1 $\Re_x$ is continuous. Moreover, it is easy to see that the inverse $\Re_x^{-1}$ of $\Re_x$ is the mapping : $g \mapsto g\,x^{-1}$, which is continuous by the same argument as above. Hence $\Re_x$ is a homeomorphism. The fact that ζ_x is a homeomorphism follows similarly.

For the inversion mapping, let $r(x) = x^{-1}$. Then clearly r is 1:1, continuous and onto.

Since $r^{-1}(x) = x^{-1} = r(x)$ is continuous, hence r is a homomorphism. $\psi_{(x,y)}$ as composition of two homeomorphisms is homeomorphism.

(b) Follows from the fact that $\psi_{(x,x^{-1})}(z) = xzx^{-1}$ is a homeomorphism.

(c)-(d) Follows from the fact that ζ_x is homeomorphism and

$$\zeta_x(v)(x) = \sup_{ex=x} v(e) \wedge 1_x = 1_x.$$

(e) From the continuity of r

for all $v \in \beta(e)$ and $\varepsilon \in I_0$ there exist $\mu \in \beta(e)$ such that $\mu - \varepsilon \le v^{-1}$ and

$$\sup_{\varepsilon \in I_0} (v_\varepsilon - \varepsilon) \in \beta(e).$$

Then $(\mu - \varepsilon) \in \beta(e)$.

Thus $v^{-1} \in \beta(e)$.

(f) $[v \vee v^{-1}]^{-1} = r^{-1}[v \vee v^{-1}]$

$= r^{-1}[v(x)] \vee r^{-1}[v(x^{-1})]$

$= v(x^{-1}) \vee v[r(x^{-1})]$

$= v^{-1}(x) \vee v(x)$.

Then $v \wedge v^{-1}$, $v \,.\, v^{-1}$ are symmetric similarly.

Definition 2.4 A fuzzy T-neighbourhood space $(G, t(\beta))$ is called homogeneous if and only if for all $x, y, \in G$ there exists a homeomorphism $f : G \to G$ such that $f(x) = y$.

Theorem 2.1 If $(G, ., t(\beta))$ is a fuzzy T-neighbourhood group, then the fuzzy T-neighbourhood space $(G, t(\beta))$ is homogeneous.

Proof.

Follows from Proposition 2.3 (a), since $\Re_{x^{-1}y}$ is a homeomorphism for all $x, y \in G$

Theorem 2.2 $(G, ., \mathrm{T})$ is a topological group if and only if $(G, ., w(\mathrm{T}))$ is a fuzzy T-neighborhood group, where $w(\mathrm{T})$ is the topologically generated fuzzy topology.

Proof.

This follows from the facts that $(G, w(\mathrm{T}))$ is T-neighborhood space (Theorem 4.1 in [5]) and that the map

$$h : (G \times G, \mathrm{T} \times \mathrm{T}) \to (G, \mathrm{T}) : (x, y) \mapsto xy^{-1}$$

is continuous if and only if

$$h : (G \times G, w(\mathrm{T}) \times w(\mathrm{T})) \to (G, w(\mathrm{T})) : (x, y) \mapsto xy^{-1}$$

is continuous[5].

Proposition 2.4 If $(G, ., t(\beta))$ is a fuzzy T-neighbourhood group, μ is an open fuzzy set and $v \in I^G$, then $\mu.\, v$ (*resp.* $v.\mu$) is an open fuzzy set.

Proof. Let $x \in G$ then $\mu.\, v = \sup_{st=x} \mu(s) \wedge v(t)$

$= \sup_{t \in G} \mu(xt^{-1}) \wedge v(t)$

$= \sup_{t \in G} \mu(\Re_t^{-1}(x)) \wedge v(t)$

$= \sup_{t \in G} \Re_t(\mu)(x) \wedge v(t)$

$\therefore \mu.\, v = \bigvee_{t \in G} [\Re(\mu) \wedge v(t)] = (\bigvee_{t \in G} \Re_t(\mu) \wedge v(t))(x)$

where each $v(t)$ is constant. Since μ is open and every $\Re_t$ is homeomorphism then $\mu.\,v$ is an open fuzzy set.

The proof of the next two lemmas is verbally the same as in the classical case, so we omit the proofs.

Lemma 2.1 Let (G, Δ) and (G', Δ') be fuzzy topological spaces and $f: G \to G'$.

Then f is continuous and closed if and only if $\overline{f(\overline{\mu})} = \overline{\overline{f(\mu)}}$ for all $\mu \in I^G$.

Lemma 2.2 If (G, Δ) and (G', Δ') are fuzzy topological space and $f: G \to G'$ is a homeomorphism then

$$\overline{f^{-1}(v)} = f^{-1}(\overline{v}) \quad \forall v \in I^{G'}$$

Proposition 2.5 In fuzzy T-neighbourhood group $(G, ., t(\beta))$ the following properties hold:

(a) $\overline{\overline{\mu^{-1}}} = \overline{\mu}^{-1} \quad \forall \mu \in I^G$

(b) $\overline{1_x \mu 1_y} = 1_x \overline{\mu} 1_y \qquad \forall \mu \in I^G$ *and* $\forall x,.y \in G.$

Proof.

(a) $(\overline{\mu})^{-1} = r^{-1}(\overline{\mu}) = \overline{\left[r^{-1}(\mu)\right]} = \overline{\mu^{-1}}$ using Lemma 2.2

(b) We have $\psi_{(x,y)}(\mu) = 1_x.\mu.1_y$ is homomorphism and using Lemma 2.2 the assertion follows.

Lemma 2.3 [1] If $(G, .)$ and $(G', .)$ are groups and $f: G \to G'$ is a group-homomorphism, then

$$f(\mu . \zeta) = f(\mu) . f(\zeta) \text{ for all } \mu, \zeta \in I^G.$$

Corollary 2.1 [1] If $(G, .)$ and $(G', .)$ are groups and $f: G \to G'$ a group-homomorphism, then

$$f(1_x . 1_a^{-1} . \mu) = 1_{f(x)} . 1_{f(a^{-1})} . f(\mu) \qquad \text{for all } \mu \in I^G \text{ and } x, a \in G.$$

Proposition 2.6 Let $(G, ., t(\beta))$ and $(G', ., t(\beta))$ be fuzzy T-neighborhood groups, and $f: G \to G'$ a continuous group-homomorphism, then

(a) If $\mu \in I^G$ is symmetric then $f(\mu)$ and $\overline{f(\mu)}$ are symmetric.

(b) If $\mu' \in I^G$ is symmetric then $f^{-1}(\mu')$ and $\overline{f^{-1}(\mu')}$ are symmetric.

Proof. Straightforward.

We shall now give some characterization theorems of fuzzy T-neighborhood groups. The first gives necessary and sufficient conditions for a group structure and fuzzy T-neighborhood system to be compatible, and the second gives necessary and sufficient conditions for a filter to be the neighborhood filter of e in a fuzzy T-neighbourhood group.

Theorem 2.3 Let $(G, .)$ be a group and β a fuzzy T-neighborhood base on G. then $(G, ., t(\beta))$ is a fuzzy T-neighborhood group if and only if the following are fulfilled:

(a) for every $a \in G$ we have

$$\beta^{-1}(a) = \{\zeta_a(\mu) \,/\, \mu \in \beta(e)\}^{-1}$$

(res . $\beta(a) = \{\Re_a(\mu) \,/\, \mu \in \beta(e)\}$) and $\beta(a) = \{\zeta_a(\mu) \,/\, \mu \in \beta(e)\}$ is a T-neighborhood base at a.

(b) For all $\mu \in \beta(e)$ and for all $\varepsilon \in I_0$ there exists $\nu \in \beta(e)$ such that $\nu - \varepsilon \le \mu^{-1}$, i.e., r is continuous at e.

(c) For all $\mu \in \beta(e)$ and for all $\varepsilon \in I_0$ there exists $\nu \in \beta(e)$ such that $\nu.\ \nu - \varepsilon \le \mu$, i.e., m is continuous at (e, e).

(d) For all $\mu \in \beta(e)$, for all $\varepsilon \in I_0$ and for all $x \in G$ there exist $\nu \in \beta(e)$ such that $1_x.\ \nu.1_{x^{-1}} - \varepsilon \le \mu$, i.e., int_x is continuous at e.

Proof. If $(G, ., t(\beta))$ is fuzzy T-neighbourhood group, then (a) follows from Proposition 2.3; (a); (b) and (c) are immediate from Proposition 2.2; and (d) is just the fact that
$Int_x = \zeta_x^{-1} .\ \Re_x^{-1}$ is continuous at e.

To prove the converse, we first remark that from (b) it follows that $\nu \in \beta^{-1}(e) \Rightarrow \nu^{-1} \in \beta^{-1}(e)$, and therefore, replacing ν by ν^{-1} in (c), that for all $\mu \in \beta(e)$, and for all $\varepsilon \in I_0$ there exists a $\nu \in \beta(e)$ such that
$\nu.\nu^{-1} - \varepsilon \le \mu$. This proves that

$$h : G \times G \to G : (x, y) \mapsto xy^{-1} \qquad \text{is continuous in } (e,e).$$

As it follows from (a) that ζ_a $(res. \Re_a)$ is continuous at a and e, we immediately obtain the continuity of m at (a,b) from scheme:

$$G \times G \xrightarrow{\zeta_{a^{-1}} \times \zeta_{b^{-1}}} G \times G \xrightarrow{m} G \xrightarrow{Int_b} G \xrightarrow{\zeta_{ab}^{-1}} G$$

(as successively $(a, b) \mapsto (e,e) \mapsto e \mapsto e \mapsto ab^{-1}$). Then apply Proposition 2.1.

Theorem 2.4 Let $(G, .)$ be a group and $\Im$ a family of fuzzy subset of G such that the following hold:

(a) $\Im$ is a filterbasis, such that $\mu(e) = 1$ for all $\mu \in \Im$.

(b) For all $\mu \in \Im$ and for all $\varepsilon \in I_0$ there exists $\nu \in \Im$ such that $\nu - \varepsilon \le \mu^{-1}$.

(c) For all $\mu \in \Im$ and for all $\varepsilon \in I_0$ there exists $\nu \in \Im$ such that $\nu.\ \nu - \varepsilon \le \mu$.

(d) For all $\mu \in \Im$, for all $\varepsilon \in I_0$ and for all $x \in G$ there exists $\nu \in \Im$ such that
$1_x.\ \nu.1_{x^{-1}} - \varepsilon \le \mu$.

Then there exists a unique fuzzy T-neighborhood system β such that $\Im$ is a basis for the fuzzy T-neighbourhood system at e, $\beta(e)$ and β is compatible with the group structure.

This neighbourhood system is given by

$$\beta(x) = \{1_x.\mu \,/\, \mu \in \Im\}^{-1} = \{\mu\,.1_x \,/\, \mu \in \Im\}^{-1},\ x \in G. \qquad (1)$$

Proof. It follows already from the preceding theorem that if fuzzy T-neighborhood system exists, compatible with the group structure of G, it must be given by (1), and so it is unique. It follows also from the preceding theorem that if $(\beta(x))_{x\in G}$**,** defined by

(1) Is a fuzzy T-neighborhood system, it is compatible with the group structure. So we have only to check the condition (FNS3).

Given $\beta = 1_x.\ \mu \in \beta(x)$ with $\mu \in \Im$, and given $\varepsilon \in I_0$, we take $v \in \Im$ such that $v.\ v - \varepsilon \le \mu$, and for all $z \in G$ we take $\beta^{\varepsilon}_z = 1_z\,.\ v$. We have for all $y \in G$ that

$$\begin{aligned}\sup_{z\in G} \beta^{\varepsilon}_x(z) \wedge \beta^{\varepsilon}_x(y) - \varepsilon &\le \sup_{z\in G} \beta^{\varepsilon}_z(z) \wedge \beta^{\varepsilon}_z(y) - \varepsilon\\ &= \sup_{z\in G} v(x^{-1}z) \wedge v(y^{-1}z) - \varepsilon\\ &= \sup_{t\in G} v(x^{-1}yt) \wedge v(t^{-1}) - \varepsilon\\ &= v.\ v(x^{-1}y) - \varepsilon\\ &\le \mu(x^{-1}y)\\ &= 1_x.\mu(y)\\ &= \beta(y)\end{aligned}$$

and this ends the proof

Remark that if G is a commutative group then $1_x.v.1x^{-1} = v$ for all $v \in I^G$ and $x \in G$. Remark also that if G is a group and β a fuzzy T-neighborhood system on G, then it is immediate that the continuity of h at (e,e) is equivalent to the continuity of m at (e,e) togther with that of r at e. there for the conditions (b) and

(c) in Theorem 2.3 can be replaced by the unique condition:

(c') For all $\mu \in \beta(e)$ and for all $\varepsilon \in I_0$ there exist $v \in \beta(e)$ such that $v.v^{-1} - \varepsilon \le \mu$,
i.e., h is continuous at (e,e).

Theorem 2.5 If $(G, ., t(\beta))$ is a fuzzy T-neighborhood group then the closure operator of $(G, t(\beta))$ is given for all $\mu \in I^G$ and $x \in G$ by

$$\bar{\mu}(x) = \inf_{v\in\beta(x)} \sup_{s\in G} \mu\, T\, v(s) = \inf_{v\in\beta(e)} \sup_{st^{-1}=x} \mu(s)\, T\, v(t) = \inf_{v\in\beta(e)} \sup_{t^{-1}s=x} \mu(s)\, T\, v(t)$$

Proof. Let $\mu \in I^G$ and $x \in G$. then from proposition 2.3 [13] we have

$$\bar{\mu}(x) = \inf \quad \sup \mu T v(s)$$

$$\begin{aligned} &\quad v \in \beta(x) \;\; s \in G \\ &= \inf_{v \in \beta(e)} \sup_{s \in G} \mu(s)\, T\, \zeta_x(v)(s) \\ &= \inf_{v \in \beta(e)} \sup_{s \in G} \mu(s)\, T\, v(x^{-1} s) \\ &= \inf_{v \in \beta(e)} \sup_{st^{-1} = x} \mu(s)\, T\, v(t) \end{aligned}$$

the third formula is proved analogously.

Theorem 2.6 Let $(G, ., t(\beta))$ and $(G', ., t(\beta))$ be fuzzy T-neighbourhood groups, and $f : G \to G'$ a group-homomorphism. Then f is continuous if and only if f is continuous at one point.

Proof. We have only to show that if f is continuous at a, it is continuous at each $x \in G$. If $x' = f(x)$,let $\mu' \in \beta'(x')$ then $v' = 1_{x'a'}{}^{-1}.\mu' \in \beta'(a')$ (with $a' = f(a)$), so we can find $v \in \beta(a)$ such that $f(v) \le v'$.

Now $\mu = 1_{xa}{}^{-1}\, v \in \beta(x)$ while from Corollary 2.1, it follows that

$$\begin{aligned} f(\mu) &= f(1_x.\, 1_a\, .\, v) \\ &= 1_{f(x)}\, .\, 1_{f(a)}{}^{-1}\, .f(v) \\ &\le 1_{x'}\, 1_{a'}{}^{-1}\, .\, v' \\ &= \mu', \end{aligned}$$

hence the result follows.

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