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RESTRICTIVE APPROXIMATIONS FOR INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract:

Various previous works for Restrictive Approximations are applied for Integrations and Initial Boundary Value Problems (IBVP) for Partial Differential Equations (PDE). We try here for Initial Value Problem (IVP) for Ordinary Differential Equations (ODE). We show that its restrictive approach is of zero truncation error for problems of solutions belongs to semi-group of linear operator. Enclosed trials for solving some IVP for ODE by single and multistep methods with different types of stability conditions. Acceptable numerical results are given.

Keywords: Restrictive approximations, Numerical methods for O.D.E., A–Stability, Semi – group of linear operator.

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1. Introduction

The restrictive Padé Approximations for Initial Boundary Value Problems for parabolic and hyperbolic Partial Differential Equations derived in many papers [4-13]. In our work we use the Restrictive Euler method for initial value problem of Ordinary Differential Equations with numerical examples in different step size. The stability condition of The Euler method proved in [2], and we prove the stability condition of the Restrictive Euler method. The A–Stability Analysis is very important for the treatment of the stiff system, Dahlquist (1963) [3] defined the A–Stability of a given numerical method for the solution of initial value problem of ordinary differential equations as follows: "If α is a complex constant with negative real part, any solution of the difference equations which arises by applying the given method to the differential equation y' = α y converges to zero as n → ∞".

The explicit Euler method is not A–Stable while the implicit Euler method is A–Stable, we try in our work to know if the restrictive explicit Euler is A–Stable but we show that it is not.

If the solution of the I.V.P. belongs to a semi – group of linear operator , then the restrictive Euler method produces zero local truncation error i.e. exact solution.

2. Restrictive Euler method and Reduction to an Almost Exact Solution of I.V.P. for O.D.E.

The idea of restrictive approximations succeeded when the solution belongs to a semi – group of linear operator y(x) of the following properties [1]:

i) $y(\alpha + \beta) = y(\alpha)\, y(\beta)$

ii) $y(0) = 1$

iii) $\lim_{\Delta t \to 0} \left\| y(\Delta t)X - X \right\| = 0$

Solving the initial value problem:

$$y'' = f(x, y) \qquad , \qquad y(x_0) = y_0$$

Firstly : By The Euler method

$y_{n+1} = y_n + h f_n$

We estimate its local truncation error L.T.E. to the form :

$$T_{nE} = y_{n+1} - y_n - h y'_n \tag{2.1}$$

but by Taylor expansion :

$$y_{n+1} = y_n + h y'_n + \frac{h^2}{2} y''_n + \frac{h^3}{6} y'''_n + \cdots$$

Then the series form of the local truncation error L.T.E. is

$$T_{nE} = \frac{h^2}{2} y''_n + \frac{h^3}{6} y'''_n + \cdots \tag{2.2}$$

Secondly : By The Restrictive Euler method

$y_{n+1} = y_n + h \varepsilon f_n$

We estimate its local truncation error L.T.E to the form:

$$T_{nRE} = y_{n+1} - y_n - h \varepsilon y'_n \tag{2.3}$$

putting $\sigma_n = y_{n+1} - y_n$

$$\therefore T_{nRE} = \sigma_n - h \varepsilon y'_n \tag{2.4}$$

Where

$T_{0RE} = y_1 - y_0 - h \varepsilon y'_0$

putting $\sigma_0 = y_1 - y_0$

$$\therefore T_{0RE} = \sigma_0 - h \varepsilon y'_0 \tag{2.5}$$

$\because$ The main idea of restrictive Euler method is to put $T_{0RE} = 0$ to get ε, then

$$\Rightarrow \quad \sigma_0 = h \varepsilon y'_0 \tag{2.6}$$

by equation (2.4) $\Rightarrow$ $y'_0 \; T_{nRE} = y'_0 \sigma_n - h \varepsilon y'_0 y'_n$ (2.7)

And using (2.6) into (2.7) we get

$y'_0 \; T_{nRE} = y'_0 \sigma_n - \sigma_0 y'_n$

as the L.T.E for the restrictive Euler method

$$\Rightarrow y'_0 \; T_{nRE} = y'_0 [y'_n h + \frac{1}{2} y''_n h^2 + \frac{1}{6} y'''_n h^3 + \cdots] - y'_n [y'_0 h + \frac{1}{2} y''_0 h^2 + \frac{1}{6} y'''_0 h^3 + \cdots]$$

$$= \frac{1}{2}[y'_0 y''_n - y'_n y''_0] h^2 + \frac{1}{6}[y'_0 y'''_n - y'_n y'''_0] h^3 + \frac{1}{24}[y'_0 y^{(4)}_n - y'_n y^{(4)}_0] h^4 + \cdots \tag{2.8}$$

the case of proving that the coefficient of various powers of h tends to zero, it prove the zero local truncation error of restrictive Euler method. As mentioned before it can be proved that the solution is belongs to a semi – group of linear operator.i.e. all coefficients of several powers of h is zero.

Example 1:

Considering the following I.V.P for first order O.D.E. with a given exact solution.

$y' = 3y \qquad ; \qquad y(0) = 1 \qquad (2.9)$

$y_{Ex}(x) = e^{3x} \qquad (2.10)$

We try to use the restrictive Euler method for h = 0.1, 0.01 and 0.001

i) Firstly for h = 0.1:

$y_{Ex}(0.1) = 1.349858807576$

$y_E(x+h) = y_E(x) + h\, y'_E(x) \qquad (2.11)$

$y_E(0.1) = 1.3$

$y_{R.E}(x+h) = y_{R.E}(x) + \varepsilon\, h\, y'_{R.E}(x) \qquad (2.12)$

Where ε can be determined by using the exact solution at x = 0.1, i.e.:

$y_{R.E}(0.1) = y_{Ex}(0.1) \qquad (2.13)$

$\therefore \varepsilon = 1.16619602525333$

After which the solution and the absolute errors for Euler and restrictive Euler methods for x=0.1, 0.2 and 0.3 can be tabulated in table 2.1

$y(x)$	x = 0.1	x = 0.2	x = 0.3
y_{Ex}	1.349858807576	1.822118800390510	0.21892675367434
y_E	1.3	1.69	0.242269594720319
$y_{R.E}$	ε=1.16619602525333	1.8221188003905	0.221215908247985
$\lvert E_{(y_E)}\rvert$	—	1.321E-01	2.626E-01
$\lvert E_{(y_{R.E})}\rvert$	—	9.770E-15	2.043E-14

Table 2.1 (Absolute errors for Euler and restrictive Euler methods for h=0.1)

ii) Secondly for h = 0.01:

$y_{Ex}(0.01) = 1.03045453395353$

$y_E(0.01) = 1.03$

Using equations (2.12) and (2.13)

$\therefore \varepsilon = 1.015151131784$

In the same way we can find the results in table 2.2 as:

$y(x)$	x = 0.01	x = 0.02	x = 0.03
y_{Ex}	1.03045453395352	1.06183654654536	1.09417428370521
y_E	1.03	1.0609	1.092727
$y_{R.E}$	ε=1.015151131784	1.06183654654537	1.09417428370522
$\lvert E_{(y_E)}\rvert$	—	9.365E-04	1.447E-03
$\lvert E_{(y_{R.E})}\rvert$	—	6.439E-15	9.992E-15

Table 2.2 (Absolute errors for Euler and restrictive Euler methods for h=0.01)

iii) Thirdly for h = 0.001

$y_{Ex}(0.001) = 1.00300450\ 450338$

$y_E(0.001) = 1.003$

Using equations (2.12) and (2.13)

$$\therefore \varepsilon = 1.00151501\ 12666$$

In the same way we can find the results in table 2.3 as:

$y(x)$	x = 0.001	x = 0.002	x = 0.003
y_{Ex}	1.00300450450338	1.00601803605406	1.00904062177387
y_E	1.003	1.006009	1.009027027
$y_{R.E}$	ε=1.00150150112666	1.00601803605407	1.00904062177388
$\lvert E_{(y_E)} \rvert$	—	9.036E-06	1.359E-05
$\lvert E_{(y_{R.E})} \rvert$	—	5.773E-15	8.660E-15

Table 2.3 (Absolute errors for Euler and restrictive Euler methods for h=0.001)

Example 2:

Considering the following I.V.P for second order O.D.E. with a given exact solution.

$$y'' - 5y' + 6y = 0 \quad , \quad y(0) = 1 \ , \ y'(0) = 2 \tag{2.14}$$

Let $y' = z$

$z' = 5z - 6y$

* Its exact solution will take the form:

$$y_{Ex}(x) = e^{2x}$$
$$z_{Ex}(x) = 2e^{2x} \tag{2.15}$$

We try to use the restrictive Euler method for $h = 0.1, 0.01$ and 0.001

i) Firstly for h = 0.1:

$y_{Ex}(0.1) = 1.22140275816017$

$z_{Ex}(0.1) = 2.44280551632034$

The Euler method

$$y_{E_{n+1}} = y_{E_n} + h\ z_{E_n}$$
$$z_{E_{n+1}} = z_{E_n} + h[5z_{E_n} - 6\,y_{E\,n}] \tag{2.16}$$

$\therefore y_E(0.1) = 1.2$

$z_E(0.1) = 2.4$

The Restrictive Euler method

$$y_{R.E_{n+1}} = y_{R.E_n} + \varepsilon_1 h\ z_{R.E_n} \tag{2.17}$$

$\because y_{R.E}(0.1) = y_{Ex}(0.1)$

$\therefore \varepsilon_1 = 1.10701379080085$

$$z_{R.E_{n+1}} = z_{R.E_n} + \varepsilon_2 h[5z_{E_n} - 6\,y_{E\,n}] \tag{2.18}$$

$\because z_{R.E}(0.1) = z_{Ex}(0.1)$

$\therefore \varepsilon_2 = 1.10701379080085$

After which the solution and the absolute errors for Euler and restrictive Euler methods for x=0.1, 0.2 and 0.3 can be tabulated in table 2.4

y , z	x=0.1	x=0.2	x=0.3
y_{Exact}	1.22140275816017	1.49182469764127	1.82211880039051
z_{Exact}	2.44280551632034	2.98364939528254	3.64423760078102
y_{Euler}	1.2	1.44	1.728
z_{Euler}	2.4	2.88	3.456
$y_{R.Euler}$	ε_1=1.10701379080085	1.49182469764127	1.82211880039051
$z_{R.Euler}$	ε_2=1.10701379080085	2.98364939528254	3.64423760078102
$\|E\|(y_{Euler})$	———	5.182E-02	9.412E-02
$\|E\|(z_{Euler})$	———	1.036E-01	1.882E-01
$\|E\|(y_{R.Euler})$	———	0	0
$\|E\|(z_{R.Euler})$	———	0	0

Table 2.4 (Absolute errors for Euler and restrictive Euler methods for h=0.1)

ii) Secondly for h = 0.01 :

$$y_{Ex}(0.01) = 1.02020134\ 0026760$$
$$z_{Ex}(0.01) = 2.04040268\ 0053510$$

The Euler method

from equation (2.16) :

$$\therefore\ y_E(0.01) = 1.02$$
$$z_E(0.01) = 2.04$$

The Restrictive Euler method

from equation (2.17) :

$$\because\ y_{R.E}(0.01) = y_{Ex}(0.01)$$
$$\therefore \varepsilon_1 = 1.01006700\ 1338$$

from equation (2.18)

$$\because\ z_{R.E}(0.01) = z_{Ex}(0.01)$$
$$\therefore \varepsilon_2 = 1.01006700\ 133776$$

In the same way we can find the results in table 2.5 as:

y , z	x=0.01	x=0.02	x=0.03
y_{Exact}	1.020201340026760	1.04081077419239	1.06183654654536
z_{Exact}	2.040402680053510	2.08162154838478	2.12367309309072
y_{Euler}	1.02	1.0404	1.061208
z_{Euler}	2.04	2.0808	2.122416
$y_{R.Euler}$	ε_1=1.010067001338	1.0408107741924	1.06183654654537
$z_{R.Euler}$	ε_2=1.01006700133776	2.08162154838477	2.12367309309071
$\|E\|(y_{Euler})$	———	4.108E-04	6.285E-04
$\|E\|(z_{Euler})$	———	8.215E-04	1.257E-03
$\|E\|(y_{R.Euler})$	———	8.438E-15	1.266E-14
$\|E\|(z_{R.Euler})$	———	0	4.885E-15

Table 2.5 (Absolute errors for Euler and restrictive Euler methods for h=0.01)

iii) Thirdly for h = 0.001 :

$$y_{Ex}(0.001) = 1.002002001334000$$
$$z_{Ex}(0.001) = 2.004004002668000$$

The Euler method

from equation (2.16) :

$$\therefore y_E(0.001) = 1.002$$
$$z_E(0.001) = 2.004$$

The Restrictive Euler method

from equation (2.17) :

$$\because\ y_{R.E}(0.001) = y_{Ex}(0.001)$$

$$\therefore \varepsilon_1 = 1.00100066700015000$$

from equation (2.18) :

$$\because\ z_{R.E}(0.001) = z_{Ex}(0.001)$$

$$\therefore \varepsilon_2 = 1.00100066700015000$$

In the same way we can find the results in table 2.6 as:

y , z	x=0.001	x=0.002	x=0.003
y_{Exact}	1.002002001334	1.00400801067734	1.00601803605406
z_{Exact}	2.004004002668	2.00801602135468	2.01203607210813
y_{Euler}	1.002	1.004004	1.006012008
z_{Euler}	2.004	2.008008	2.012024016
$y_{R.Euler}$	ε_1=1.00100066700015	1.00400801067734	1.00601803605406
$z_{R.Euler}$	ε_2=1.00100066700015	2.00801602135468	2.01203607210813
$\lvert E\rvert(y_{Euler})$	———	4.011E-06	6.028E-06
$\lvert E\rvert(z_{Euler})$	———	8.021E-06	1.206E-05
$\lvert E\rvert(y_{R.Euler})$	———	0	0
$\lvert E\rvert(z_{R.Euler})$	———	0	0

Table 2.6 (Absolute errors for Euler and restrictive Euler methods for h=0.001**)**

Example 3

Considering the following I.V.P for O.D.E.with a given exact solution.

$$y' = e^{-x}\cos x - y \qquad ; \qquad y(0) = 0 \tag{2.19}$$

$$y_{Ex}(x) = e^{-x}\sin(x) \tag{2.20}$$

We try to use the restrictive Euler method for h = 0.1, 0.01 and 0.001

i) Firstly for h = 0.1:

$y_{Ex}(0.1) = 0.0903330109524242$

$$y_E(x+h) = y_E(x) + h\, y'_E(x) \tag{2.21}$$

$y_E(0.1) = 0.1$

$$y_{R.E}(x+h) = y_{R.E}(x) + \varepsilon\, h\, y'_{R.E}(x) \tag{2.22}$$

Where ε can be determined by using the exact solution at x = 0.1, i.e. :

$$y_{R.E}(0.1) = y_{Ex}(0.1) \tag{2.23}$$

$$\therefore \varepsilon = 0.903330109524242$$

After which the solution and the absolute errors for Euler and restrictive Euler methods for x=0.1, 0.2 and 0.3 can be tabulated in table 2.7

$y(x)$	x = 0.1	x = 0.2	x = 0.3
y_{Ex}	0.0903330109524242	0.162656690815339	0.21892675367434
y_E	0.1	0.180031699984519	0.242269594720319
$y_{R.E}$	ε=0.903330109524242	0.163501303492363	0.221215908247985
$\lvert E_{(y_E)}\rvert$	—	1.738E-02	2.3E-02
$\lvert E_{(y_{R.E})}\rvert$	—	8.45 E-04	2.3E-03

Table 2.7(Absolute errors for Euler and restrictive Euler methods for h=0.1**)**

ii) Secondly for h = 0.01 :

$y_{Ex}(0.01) = 0.0099003333\ 300111$

$y_E(0.01) = 0.01$

Using equations (2.22) and (2.23)

$$\therefore \varepsilon = 0.9900333330\ 0111$$

In the same way we can find the results in table 2.8 as:

$y(x)$	x = 0.01	x = 0.02	x = 0.03
y_{Ex}	0.0099003333300111	0.0196026665607091	0.0291089991980653
y_E	0.01	0.0198000033167	0.0294020296845996
$y_{R.E}$	ε=0.990033333300111	0.0196036500103203	0.0291119201420801
$\lvert E_{(y_E)}\rvert$	—	1.973E-04	2.93E-04
$\lvert E_{(y_{R.E})}\rvert$	—	9.834E-07	2.921E-06

Table 2.8(Absolute errors for Euler and restrictive Euler methods for h=0.01)

iii)Thirdl y for h = 0.001

$y_{Ex}(0.001) = 0.0009990003\ 333333$

$y_E(0.001) = 0.001$

Using equations (2.22) and (2.23)

$$\therefore \varepsilon = 0.9990003333\ 333$$

In the same way we can find the results in table 2.9 as

$y(x)$	x = 0.001	x = 0.002	x = 0.003
y_{Ex}	0.0009990003333333	0.0019960026666656	0.00299100899999191
y_E	0.001	0.00199800000033317	0.00299400200299683
$y_{R.E}$	ε=0.9990003333333	0.0019960036650001	0.00299101199200140
$\lvert E_{(y_E)}\rvert$	—	1.99E-06	2.993E-06
$\lvert E_{(y_{R.E})}\rvert$	—	9.984E-10	2.992E-09

Table 2.9(Absolute errors for Euler and restrictive Euler methods for h=0.001)

Since our computations are up to sixteen decimals, the absolute error for restrictive Euler methods in examples 1 and 2 for tables (2.1)-(2.6) varies from 0 to 10^{-14} i.e. it is only rounding – off error. It is because of the solution e^{3x} and e^{2x} , each belongs to the semi-group of linear operator. It us easy to say that:

- $e^0 = 1$
- $e^{2x_1+2x_2} = e^{2x_1} e^{2x_2}$.
- $e^{3x_1+3x_2} = e^{3x_1} e^{3x_2}$

While example 3 can not satisfy the semi – group of linear operator properties. Then it is not useful to use the restrictive Euler method. i.e. the coefficient of various power of h in equation (2.8) is sufficiently small but not equal to zero.

3.Stability condition for the Restrictive Euler method

The stability of the restrictive Euler method can be determined by the same way in the Euler method [2]
So that, the stability condition for the restrictive Euler method is:

$$\varepsilon h \prec \frac{2}{|f_y|} \qquad (3.1)$$

3.1Comparison between the Euler and Restrictive Euler methods for permissible values of h:

For the following example:

$$y' = \frac{1}{1+t} e^{-5x} - 5y \qquad ; \qquad y(0) = 0$$

$$\text{of} \qquad y_{Ex}(x) = e^{-5x} \ln(1+x).$$

• *For the Euler method :*

$$h_E < \frac{2}{|f_y|}$$

$$\therefore \quad h_E < 0.4 \quad \text{is the stability condition of the Euler method.} \qquad (3.2)$$

• *For the re strictive Euler meth od:*

$$\varepsilon \ h_{R.E} < \frac{2}{|f_y|}$$

$$\therefore \quad \varepsilon \ h_{R.E} < 0.4 \ \text{is the stability condition of the restrictiv e Euler method.} \qquad (3.3)$$

$$\text{For } h = 0.1 \qquad \Rightarrow \qquad \varepsilon = 0.57808546\ 2340469 \ < 1$$

$$\text{from (3.2) and (3.3)} \qquad \Rightarrow \qquad h_{R.E} > h_E$$

Which we can say that the mesh size of the restrictiv e Euler method is greater than that for Euler method.

4. Restrictive Explicit Euler For higher Order IVP for ODE:

We can give an example for second order case to show that the explicit Euler method is not A–Stable while the restrictive Euler method is A–Stable.

Example 4:

$$y'' - (1-\alpha)\, y' - \alpha\, y = 0 \quad ; \quad y(0) = 0 \quad , \quad y'(0) = 1 \tag{4.1}$$

we usually let

$$\begin{aligned} y' &= z \quad , \quad y(0) = 0 \\ z' &= (1-\alpha)\, z + \alpha\, y \quad , \quad z(0) = 1 \end{aligned} \tag{4.2}$$

i) Firstly it is already known that the Explicit Euler method is :

$$\begin{aligned} y_{n+1} &= y_n + h\, z_n \quad , \quad y(0) = 0 \\ z_{n+1} &= z_n + h\,[\alpha\, y_n + (1-\alpha)\, z_n] \quad , \quad z(0) = 1 \end{aligned} \tag{4.3}$$

$$\Rightarrow (E-1)\, y_n - h\, z_n = 0$$

$$[E - 1 - h\,(1-\alpha)] z_n - h\, \alpha\, y_n = 0$$

Eliminating z_n, then :

$$((E - 1 - h(1-\alpha))(E-1) + h^2\alpha) y_n = 0$$

its characteristic equation is :

$$m^2 - [2 + h(1-\alpha)]\, m + [1 + h(1-\alpha) + h^2\alpha] = 0$$

$$\text{i.e :} \quad [m - (1+h)]\ [m - (1-\alpha h)] = 0$$

$$\therefore y_n = A(1+h)^n + B(1-\alpha h)^n$$

$$z_n = A(1+h)^n - \alpha B(1-\alpha h)^n$$

And by using the initial conditions we get :

$$A = -B = \frac{1}{1+\alpha}$$

$$\therefore \quad y_n = \frac{1}{1+\alpha}[(1+h)^n - (1-\alpha h)^n]$$

For $\alpha = 10^6$, $h = 0.001$

$$\therefore \quad y_{1000} = \frac{1}{1+10^6}[(1+0.001)^{1000} - (1-1000)^{1000}] \quad \to \infty \quad (\text{divergent})$$

i.e. : $y \to \infty$ as $n \to \infty$

$\therefore$ The explicit Euler method is not A - Stable.

ii) Secondly for the restrictive explicit Euler :

$$y_{n+1} = y_n + \varepsilon_1 h\ z_n \qquad , y(0) = 0$$
$$z_{n+1} = z_n + \varepsilon_2 h[\alpha y_n + (1-\alpha) z_n] \ , \ z(0) = 1 \qquad (4.4)$$
$$\Rightarrow \ (E-1) y_n - \varepsilon_1 h\ z_n = 0$$
$$[E - 1 - \varepsilon_2 h(1-\alpha)]z_n - \varepsilon_2 h\ \alpha y_n = 0$$

Eliminating z_n, then :

$$[E^2 + (-2 - \varepsilon_2 h(1-\alpha))E + (1 + \varepsilon_2 h(1-\alpha) - \varepsilon_1\varepsilon_2 h^2\alpha)]\, y_n = 0$$

The characteristic equation :

$$m^2 + [-2 - \varepsilon_2 h(1-\alpha)]\, m + [1 + \varepsilon_2 h(1-\alpha) - \varepsilon_1\varepsilon_2 h^2\alpha] = 0 \qquad (4.5)$$

After which we get ε_1 and ε_2, we can calculate m_1 and m_2

$$\therefore \ y_n = A(m_1)^n + B(m_2)^n$$
$$z_n = A(m_1)^n - \alpha B(m_2)^n$$

And by using the initial conditions we get

$$A = -B = \frac{1}{1+\alpha}$$
$$\therefore y_n = \frac{1}{1+\alpha}[(m_1)^n - (m_2)^n] \qquad (4.6)$$

For specific values of α to be $\alpha = 10^6$, the equation (4.1) will take the form

$$y'' - (1 - 10^6)y' - 10^6 y = 0 \qquad , \quad y(0) = 0 \ , \ y'(0) = 1$$

* Its exact solution will take the form :

$$y_{Ex}(x) = \frac{1}{1+10^6}[e^x - e^{-10^6 x}]$$
$$z_{Ex}(x) = \frac{1}{1+10^6}[e^x + 10^6 e^{-10^6 x}] \qquad (4.7)$$
$$y_{Ex}(0.001) = 1.000999499(10)^{-6}$$
$$z_{Ex}(0.001) = 1.000999499(10)^{-6}$$

For $h = 0.001$ we can compare the explicit Euler and restrictive explicit Euler as follow :

* The Explicit Euler method (4.3) :

$$y_{E_{n+1}} = y_{E_n} + h\ z_{E_n}$$
$$z_{E_{n+1}} = z_{E_n} + h[\alpha y_{E\,n} + (1-\alpha) z_{E_n}]$$
$$\therefore \qquad y_E(0.001) = 0.001$$
$$z_E(0.001) = 998.999$$

* The Restrictiv e Explicit Euler method (4.4) :

$$y_{R.E_{n+1}} = y_{R.E_n} + \varepsilon_1 h\ z_{R.E_n}$$

$$\because y_{R.E}(0.001) = y_{Ex}(0.001)$$

$$\therefore \varepsilon_1 = 1.00099949\,9(10)^{-3} \quad (4.8)$$

$$z_{R.E_{n+1}} = z_{R.E_n} + \varepsilon_2 h [\alpha\, y_{R.E_n} + (1-\alpha)\, z_{R.E_n}]$$

$$\because z_{R.E}(0.001) = z_{Ex}(0.001)$$

$$\therefore \varepsilon_2 = 9.99999(10)^{4} \quad (4.9)$$

After which the solution and the absolute errors for Euler and restrictive Euler methods for x=0.001, 0.002 and 0.003 can be tabulated in table 4.1.

y , z	x=0.001	x=0.002	x=0.003
y_{Exact}	$1.000999499(10)^{-6}$	$1.002000999\ (10)^{-6}$	$1.003003502\ (10)^{-6}$
z_{Exact}	$1.000999499(10)^{-6}$	$1.002000999\ (10)^{-6}$	$1.003003502\ (10)^{-6}$
y_{Euler}	0.001	-0.997999	997.002003
z_{Euler}	-998.999	$9.98(10)^{5}$	$-9.97002(10)^{8}$
$y_{R.Euler}$	$\varepsilon_1=1.000999499(10)^{-3}$	$1.001000501(10)^{-6}$	$1.001001503(10)^{-6}$
$z_{R.Euler}$	$\varepsilon_2=9.99999(10)^{-4}$	$1.0010005\ (10)^{-6}$	$1.001001502(10)^{-6}$
$\|E\|(y_{Euler})$	———	0.998	997.002
$\|E\|(z_{Euler})$	———	$-9.98(10)^{5}$	$9.97(10)^{8}$
$\|E\|(y_{R.Euler})$	———	$1.00049(10)^{-9}$	$2.00199(10)^{-9}$
$\|E\|(z_{R.Euler})$	———	$1.000499(10)^{-9}$	$2.002(10)^{-9}$

Table (4.1) Solutions and Errors for restrictive explicit Euler method

Using the values of ε_1 and ε_2 from equation (4.8) and (4.9)into the equation (4.5) and (4.6) we get

$$m^2 -(1.000002)\, m +(9.99001502E(-7)) =0$$

i.e. : $m_1=1.000001001$, $m_2=9.99(10)^{-7}$

then after 1000 steps to 10^6 steps i.e. $n = 1000$ to $n = 10^6$:

$$y_{1000} = \frac{1}{1+10^6}[(1.000001\,001)^{1000} - (9.99(10)^{-7})^{1000}] \approx 0$$

$\therefore y_{R.E}$ is sufficiently small and less than one up to for $n< 13(10)^6$

So that the restrictive explicit Euler is seems to be A–Stable but it is not A–Stable.

5. CONCLUSION AND RESULTS:

From section 2., 3., and 4., it is clear that the restrictive approximation for solving first and second order I.V.P. for ODE by the explicit Euler method have the following advantages:

* The restrictive explicit Euler is more accurate than that of the explicit Euler method.

* The permissible values of the step length h for the restrictive Euler method are greater than that for explicit Euler method. i.e. the restrictive Euler method can save times and calculations.

* When the solution belongs to a semi - group of linear operator, then the error of the restrictive Euler approximation is equal to zero i.e. the absolute error is only rounding – off error.

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