Military Technical College
Kobry Elkobbah,
Cairo, Egypt
May 16-18,2006

3rd International
Conference on Engineering
Mathematics and Physics
(ICMEP-3)

Restrictive Taylor's Approximation for Higher Dimension IBV Problem for Parabolic PDE

By:
Hassan N. A. Ismail*, Ahmed R. A. Shafay**[1]

Abstract

The Restrictive Taylor's Approximation for one-dimension initial boundary value problem for parabolic partial differential equation is treated by Hassan N. A. Ismail and others in 1999 and 2003. Also two-dimensional problem is treated by the author and others in 2004 and 2006. In this work we try to solve the three-dimensional case. We prove that the local truncation error are exactly zero and found that the numerical results are satisfied .

1. Introduction

Many of the applications of parabolic partial differential equation are arise as macroscopic description of processes, which is essentially probabilistic. The heat flow is related to the random motion of electrons and atoms. Also the unsteady heat conduction equation is parabolic. It is also used in economic models to give a macroeconomics of market behavior.

The general form for parabolic partial differential equation in three -dimensional space is

$$\frac{\partial u}{\partial t} = \gamma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad ,\gamma > 0 \tag{1}$$

in the rectangular domain bounded by $0 \le x \le x_0$, $0 \le y \le y_0$, $0 \le z \le z_0$ and $t \ge 0$ with the boundary and initial conditions:

$u(0, y, z, t)=f_1(y, z, t)$, where $0 \le y \le y_0$, $0 \le z \le z_0$ and $t \ge 0$,

$u(x_0, y, z, t)=f_2(y, z, t)$, where $0 \le y \le y_0$, $0 \le z \le z_0$ and $t \ge 0$,

$u(x, 0, z, t)=f_3(x, z, t)$, where $0 \le x \le x_0$, $0 \le z \le z_0$ and $t \ge 0$,

$u(x, y_0, z, t)=f_4(x, z, t)$, where $0 \le x \le x_0$, $0 \le z \le z_0$ and $t \ge 0$,

$u(x, y, 0, t)=f_5(x, y, t)$, where $0 \le x \le x_0$, $0 \le y \le y_0$ and $t \ge 0$,

$u(x, y, z_0, t)=f_6(x, y, t)$, where $0 \le x \le x_0$, $0 \le y \le y_0$ and $t \ge 0$,

$u(x, y, z, 0)=f_7(x, y, z)$, where $0 \le x \le x_0$, $0 \le y \le y_0$ and $0 \le z \le z_0$.

[1] * Banha Higher Institute of Technology
** Alfayoum University, Faculty of Science, Mathematics Department.

The region to be examined in (x, y, z, t) space is covered by a rectilinear grid with sides parallel to axes with h and k the grid spacing in the distance and time directions respectively. The grid point (x, y, z, t) are given by (ℓh, mh, nh, qk) where ℓ, m, n=0(1)M and q=0(1)∞.The function satisfying the difference equation at the grid point is $u^{q}_{\mathrm{l,m,n}}$.

Several differential and finite difference relations and forms are derived in Michell [6].

The Taylor's expansion of $u^{q+1}_{\mathrm{l,m,n}}$ is given by

$$u^{q+1}_{\mathrm{l,m,n}} = Exp(k\frac{\partial}{\partial t})u^{q}_{\mathrm{l,m,n}} \ , \tag{2}$$

From Eq. (1) we get

$$u^{q+1}_{\mathrm{l,m,n}} = Exp(k\gamma(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2})u^{q}_{\mathrm{l,m,n}} \tag{3}$$

Where the difference replacement of $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$ and $\frac{\partial^2}{\partial z^2}$ which can be used to give various finite difference methods is given by

$$\frac{\partial^2}{\partial x^2} = \frac{1}{h^2}(\delta_x^2 - \frac{1}{12}\delta_x^4 + \frac{1}{90}\delta_x^6 + ...) ,$$

$$\frac{\partial^2}{\partial y^2} = \frac{1}{h^2}(\delta_y^2 - \frac{1}{12}\delta_y^4 + \frac{1}{90}\delta_y^6 + ...)$$

and

$$\frac{\partial^2}{\partial z^2} = \frac{1}{h^2}(\delta_z^2 - \frac{1}{12}\delta_z^4 + \frac{1}{90}\delta_z^6 + ...) .$$

Many finite difference approximations can be used to represent the derivatives in equation (3). The resulting finite difference methods can be classified into two broad categories, explicit or implicit methods.

The famous implicit finite difference methods for Eq.(1) are Crank-Nicolson formula:

$$(1-0.5r\delta_x^2)(1-0.5r\delta_y^2)(1-0.5r\delta_z^2)u^{q+1}_{\mathrm{l,m,n}} = (1+0.5r\delta_x^2)(1+0.5r\delta_y^2)(1+0.5r\delta_z^2)u^{q}_{\mathrm{l,m,n}} \tag{4}$$

where $r=\frac{k}{h^2}$ and $\gamma=1$

and Douglas and Rachford formula:

$$(1-\frac{1}{2}(r-\frac{1}{6})\delta_x^2)(1-\frac{1}{2}(r-\frac{1}{6})\delta_y^2)(1-\frac{1}{2}(r-\frac{1}{6})\delta_z^2)u^{q+1}_{\mathrm{l,m,n}}$$
$$= (1+\frac{1}{2}(r+\frac{1}{6})\delta_x^2)(1+\frac{1}{2}(r+\frac{1}{6})\delta_y^2)(1+\frac{1}{2}(r+\frac{1}{6})\delta_z^2)u^{q}_{\mathrm{l,m,n}} \tag{5}$$

where $r=\frac{k}{h^2}$ and $\gamma=1$,

both these methods in Eq.(4) and (5) are unconditionally stable.

The famous explicit finite difference methods for Eq.(1) are

$$u^{q+1}_{l,m,n} = (1 + r(\delta^2_x + \delta^2_y + \delta^2_z))u^q_{l,m,n}, \text{ where } r = \frac{k}{h^2} \text{ and } \gamma = 1$$

with stability condition $0 < r \le \frac{1}{6}$

and

$$u^{q+1}_{l,m,n} = (1 + r\delta^2_x)(1 + r\delta^2_y)(1 + r\delta^2_z)u^q_{l,m,n}, \text{ where } r = \frac{k}{h^2} \text{ and } \gamma = 1$$

with stability condition $0 < r \le \frac{1}{2}$.

1.1: Restrictive Taylor's Approximation of the function f(x)

Consider a function f(x) defined in a neighborhood of the point a, and it has derivatives up to order (n+1) in this neighborhood. We use this derivatives to construct the function [3]

$$RT_{n,f(x)}(x,a) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{\varepsilon f^{(n)}(a)}{n!}(x-a)^n$$

Now $RT_{n,f(x)}(x)$ is called The restrictive Taylor's approximation for f(x) at the point a. The parameter ε is to be determent, such that $RT_{n,f(x)}(x_0) = f(x_0)$. It mean that the considered approximation is exact at two points a and x_0 .Let us up,

$$f(\mathrm{x}) = RT_{n,f(x)}(x) + \Re_{n+1}(x) \ ,$$

where $\Re_{n+1}(x)$ is the remainder term of restrictive Taylor's series.

Some published papers [1]-[7] applying this approach in the parabolic and hyperbolic problems also some application in fluid dynamics as convection-diffusion equation in one-dimensional and non-linear type as KdV are investigated.

1.2. The Restrictive Taylor's Approximation for Solving Parabolic PDE in one-Dimension Space

Ismail, Elbabary [3] suggested the following finite difference approximation .

The parabolic partial differential equation in one-dimension space is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad , \gamma = 1 \tag{6}$$

Eq.(6) can be written as:

$$u_{l,m+1} = Exp(r\delta^2_x)u_{l,m}. \tag{7}$$

where $r = \frac{k}{h^2}$

They use the Restrictive Taylor's Approximation of the exponential form $Exp(r\delta_x^2)$ hence

$$u_{l,m+1} = (1 + r\varepsilon_l \delta_x^2) u_{l,m}$$

then

$$u_{l,m+1} = u_{l,m} + r\varepsilon_l (u_{l+1,m} + u_{l-1,m}) \tag{8}$$

and they prove the stability condition $0 < r\varepsilon_l \le \frac{1}{2}$, where ℓ =1(1)M-1.

1.3. The first Approach for Restrictive Taylor's Approximation for Solving Parabolic PDE in Two-Dimensional Space

Ismail, Elbabary and Ghada [4] suggested the following approach for the using finite difference approximation of the considered exponential form.

The parabolic partial differential equation in two-dimensional space is

$$\frac{\partial u}{\partial t} = \gamma(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \quad ,\gamma > 0 \tag{9}$$

Eq.(9) can be written as:

$$u_{l,m}^{n+1} = Exp(r\gamma(\delta_x^2 + \delta_y^2)) u_{l,m}^n . \quad \text{where } r = \frac{k}{h^2} \tag{10}$$

They use the Restrictive Taylor's Approximation of the exponential form $Exp(r\gamma(\delta_x^2 + \delta_y^2))$ hence

$$u_{l,m}^{n+1} = (1 + r\gamma\varepsilon_{l,m} (\delta_x^2 + \delta_y^2)) u_{l,m}^n ,$$

then

$$u_{l,m}^{n+1} = u_{l,m}^n + r\gamma\varepsilon_{l,m} (u_{l+1,m}^n + u_{l-1,m}^n + u_{l,m+1}^n + u_{l,m-1}^n - 4u_{l,m}^n) \tag{11}$$

and they prove the stability condition $0 < r\gamma\varepsilon_{l,m} \le \frac{1}{4}$, where ℓ , m=1(1)M-1.

1.4. Another Approach for Restrictive Taylor's Approximation for Solving Parabolic PDE in Two-Dimensional space

Hassan N. A. Ismail and Ahmed R. A. Shafay [2] suggested the following approach for the using finite difference approximation of the considered exponential form.

The parabolic partial differential equation in two-dimensional space is

$$\frac{\partial u}{\partial t} = \gamma(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \quad ,\gamma > 0 \tag{12}$$

Eq.(12) can be written as:

$$u_{\mathrm{l,m}}^{n+1} = Exp(r\gamma\delta_x^2)Exp(r\gamma\delta_y^2)u_{\mathrm{l,m}}^{n}, \quad \text{where } r = \frac{k}{h^2}. \tag{13}$$

They use the Restrictive Taylor's Approximation of the exponential form $Exp(r\gamma\delta_x^2)$ and $Exp(r\gamma\delta_y^2)$ we get

$$u_{l,m}^{n+1} = (1 + r\gamma\varepsilon_{l,m}\delta_y^2)(1 + r\gamma\varepsilon_{l,m}\delta_x^2)u_{l,m}^{n}.$$

Then

$$\begin{aligned} u_{l,m}^{n+1} &= u_{l,m}^{n} + r\gamma\varepsilon_{l,m}[u_{l+1,m}^{n} + u_{l-1,m}^{n} + u_{l,m+1}^{n} + u_{l,m-1}^{n} - 4u_{l,m}^{n}] \\ &+ r^2\gamma^2\varepsilon_{l,m}^2[u_{l+1,m+1}^{n} + u_{l+1,m-1}^{n} + u_{l-1,m+1}^{n} + u_{l-1,m-1}^{n} - 2u_{l+1,m}^{n} \\ &- 2u_{l-1,m}^{n} - 2u_{l,m+1}^{n} - 2u_{l,m-1}^{n} + 4u_{l,m}^{n}] \end{aligned} \tag{14}$$

and they prove the stability condition $0 < r\gamma\varepsilon_{l,m} \le \frac{1}{2}$,, where ℓ , m=1(1)M-1.

2. The first Approach for Restrictive Taylor's Approximation for Solving Parabolic PDE in Three-Dimensional Space

In this section we introduce finite difference equation for parabolic partial differential equation in two dimension space by using the restrictive Taylor's approximation.

The general form for parabolic partial differential equation in three -dimensional space is

$$\frac{\partial u}{\partial t} = \gamma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad ,\gamma > 0 \tag{15}$$

The previous equation can be written as

$$u_{\mathrm{l,m,n}}^{q+1} = Exp\left(k\gamma\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\right)u_{\mathrm{l,m,n}}^{q}$$

If we use the Approximation of $\frac{\partial^2}{\partial x^2}$, $\frac{\partial^2}{\partial y^2}$ and $\frac{\partial^2}{\partial z^2}$ as

$\frac{\partial^2}{\partial x^2} = \frac{1}{h^2}\delta_x^2$, $\frac{\partial^2}{\partial y^2} = \frac{1}{h^2}\delta_y^2$ and $\frac{\partial^2}{\partial z^2} = \frac{1}{h^2}\delta_z^2$ we get

$$u_{\mathrm{l,m,n}}^{q+1} = Exp(r\gamma(\delta_x^2 + \delta_y^2 + \delta_z^2))u_{\mathrm{l,m,n}}^{q}. \quad \text{where } r = \frac{k}{h^2} \tag{16}$$

If we use the Restrictive Taylor's Approximation of the exponential form $Exp(r\gamma(\delta_x^2 + \delta_y^2 + \delta_z^2))$ we get

$$u_{\mathrm{l,m,n}}^{q+1} = (1 + r\gamma\varepsilon_{l,m,n}(\delta_x^2 + \delta_y^2 + \delta_z^2))u_{\mathrm{l,m,n}}^{q}$$

then

(17)

$$u^{q+1}_{\text{l,m,n}} = u^{q}_{\text{l,m,n}} + r\gamma\varepsilon_{l,m,n}(u^{q}_{\text{l+1,m,n}} + u^{q}_{\text{l-1,m,n}} + u^{q}_{\text{l,m+1,n}} + u^{q}_{\text{l,m-1,n}} + u^{q}_{\text{l,m,n+1}} + u^{q}_{\text{l,m,n-1}} - 6u^{q}_{\text{l,m,n}})$$

Where the constants $\varepsilon_{l,m,n}$ are to be determined. Then we must know an additional condition, u(x, y, z, k) to be given , i.e., $\varepsilon_{l,m,n}$ must be given such that the truncation error at certain r is zero, after which, we should use the Eq.(17) for another level for calculations .

2.1. Stability Analysis of The First Approach for Restrictive Taylor's Approximation

Using Von-Neumann method, let

$$u^{q}_{l,m,n} = \xi^{n} e^{i\eta hl} e^{i\lambda hm} e^{i\sigma hm}$$

where $i = \sqrt{-1}$, ξ^{n} is the amplitude at time level n, and η , λ and σ are the wave numbers in the x and y directions respectively. If the phase angle $\theta = \eta h$, $\beta = \lambda h$ and $\varphi = \sigma h$ are defined, then

$$u^{q}_{l,m,n} = \xi^{n} e^{i\theta l} e^{i\beta m} e^{i\varphi m} \; . \tag{18}$$

Substituting Eq.(18) into Eq.(17), we get

$$\xi = 1 - 4r\gamma\varepsilon_{l,m,n}(\sin^{2}\frac{\theta}{2} + \sin^{2}\frac{\beta}{2} + \sin^{2}\frac{\varphi}{2}) \; ,$$

then the method suggested by Eq.(17) will be stable when $0 < r\gamma\varepsilon_{l,m,n} \le \frac{1}{6}$, where ℓ, m, n = 1(1)M-1.

2.2. The Almost Zero Local Truncation Error of The First Approach for Restrictive Taylor's Approximation

As done by Ismail and Ghada in [5] we can derive form for the local truncation error of the first approach for Restrictive Taylor's Approximation for Parabolic PDE in two-dimensional space. By using Taylor's expansion for the points $u^{q}_{\text{r,s,t,}}$ where r=ℓ-1, ℓ, ℓ+1, s=m-1, m, m+1and t=n-1, n, n+1 in Eq. (17), the local truncation error formula of the finite difference equation (17) can take the form

$$kT^{q}_{l,m,n} = \delta^{q}_{l,m,n} - r\varepsilon_{l,m,n}\beta^{q}_{l,m,n} \tag{19}$$

where

$$\delta^{q}_{l,m,n} = [ku_{t} + \frac{k^{2}}{2}u_{t^{2}} + \frac{k^{3}}{6}u_{t^{3}} + \frac{k^{4}}{24}u_{t^{4}} + \ldots]^{q}_{l,m,n},$$

and

$$\beta^{q}_{l,m,n} = [h^2(u_{x^2} + u_{y^2} + u_{z^2}) + \frac{1}{12}h^4(u_{x^4} + u_{y^4} + u_{z^4}) + \frac{h^6}{360}(u_{x^6} + u_{y^6} + u_{z^6}) + ...]^{q}_{l,m,n}$$

Similarly

$$kT^{0}_{l,m,n} = \delta^{0}_{l,m,n} - r\varepsilon_{l,m,n}\beta^{0}_{l,m,n} \tag{20}$$

where

$$\delta^{0}_{l,m,n} = [ku_t + \frac{k^2}{2}u_{t^2} + \frac{k^3}{6}u_{t^3} + \frac{k^4}{24}u_{t^4} + ...]^{0}_{l,m,n},$$

and

$$\beta^{0}_{l,m,n} = [h^2(u_{x^2} + u_{y^2} + u_{z^2}) + \frac{1}{12}h^4(u_{x^4} + u_{y^4} + u_{z^4}) + \frac{h^6}{360}(u_{x^6} + u_{y^6} + u_{z^6}) + ...]^{0}_{l,m,n}$$

The main idea of the Restrictive Taylor's method is to put $T^{0}_{l,m,n} = 0$ to get $\varepsilon_{l,m,n}$, then

$$\delta^{0}_{l,m,n} - r\varepsilon_{l,m,n}\beta^{0}_{l,m,n} = 0. \tag{21}$$

Substitute from Eq. (21) into Eq (19) we get

$$k\beta^{0}_{l,m,n}T^{q}_{l,m,n} = \delta^{q}_{l,m,n}\beta^{0}_{l,m,n} - \delta^{0}_{l,m,n}\beta^{q}_{l,m,n}.$$

Hence

$$\begin{aligned}
k\beta^{0}_{l,m,n}T^{q}_{l,m,n} &= kh^2[(u_t)^{q}_{l,m,n}(u_{x^2} + u_{y^2} + u_{z^2})^{0}_{l,m,n} - (u_t)^{0}_{l,m,n}(u_{x^2} + u_{y^2} + u_{z^2})^{q}_{l,m,n}] \\
&+ \frac{k^2h^2}{2}[(u_{t^2})^{q}_{l,m,n}(u_{x^2} + u_{y^2} + u_{z^2})^{0}_{l,m,n} - (u_{t^2})^{0}_{l,m,n}(u_{x^2} + u_{y^2} + u_{z^2})^{q}_{l,m,n}] \\
&+ \frac{kh^4}{12}[(u_t)^{q}_{l,m,n}(u_{x^4} + u_{y^4} + u_{z^4})^{0}_{l,m,n} - (u_t)^{0}_{l,m,n}(u_{x^4} + u_{y^4} + u_{z^4})^{q}_{l,m,n}] \\
&+ \frac{k^3h^2}{6}[(u_{t^3})^{q}_{l,m,n}(u_{x^2} + u_{y^2} + u_{z^2})^{0}_{l,m,n} - (u_{t^3})^{0}_{l,m,n}(u_{x^2} + u_{y^2} + u_{z^2})^{q}_{l,m,n}] \\
&+ \frac{k^2h^4}{2}[(u_{t^2})^{q}_{l,m,n}(u_{x^4} + u_{y^4} + u_{z^4})^{0}_{l,m,n} - (u_{t^2})^{0}_{l,m,n}(u_{x^4} + u_{y^4} + u_{z^4})^{q}_{l,m,n}] \\
&+ \dots
\end{aligned}$$

Any term in the right hand side of the previous equation can be written as

$$A_{i,j} = (u_{t^i})^{q}_{l,m,n}(u_{x^{2j}} + u_{y^{2j}} + u_{z^{2j}})^{0}_{l,m,n} - (u_{t^i})^{0}_{l,m,n}(u_{x^{2j}} + u_{y^{2j}} + u_{z^{2j}})^{q}_{l,m,n}$$

where i =1(1)∞ and j =1(1) ∞.

Since the exact solution has the closed form $u(x, y, z, t) = e^{-3t}\sin x \sin y \sin z$, where $x = \ell h$, $y = mh$, $z = nh$ and $t = qk$ then

$$A_{i,j} = (-3)^{i+j}\big[(e^{-3t}\sin x \sin y \sin z)^q_{l,m,n}(e^{-3t}\sin x \sin y \sin z)^0_{l,m,n} - (e^{-3t}\sin x \sin y \sin z)^0_{l,m,n}(e^{-3t}\sin x \sin y \sin z)^q_{l,m,n}\big]$$

Hence $A_{i,j} = 0$ where i =1(1)∞ and j =1(1)∞.

Then we can derive that $k\beta^0_{l,m,n}T^q_{l,m,n}$ is almost zero ,but since in general $\beta^0_{l,m,n} \neq 0$ then $T^q_{l,m,n}$ is almost zero, i.e. the first approach for Restrictive Taylor's method has zero local truncation error.

3. The Second Approach for Restrictive Taylor's Approximation for Solving Parabolic PDE in Three-Dimensional Space

In this section we introduce another finite difference equation for parabolic partial differential equation in two dimension space by using the restrictive Taylor's approximation.

The general form for parabolic partial differential equation in three -dimensional space is

$$\frac{\partial u}{\partial t} = \gamma\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad ,\gamma > 0 \tag{22}$$

The previous equation can be written as

$$u^{q+1}_{\text{l,m,n}} = Exp(r\gamma\delta^2_x)Exp(r\gamma\delta^2_y)Exp(r\gamma\delta^2_z)u^q_{\text{l,m,n}}. \quad \text{where } r = \frac{k}{h^2} \tag{23}$$

if we use the Restrictive Taylor's Approximation of the exponential form $Exp(r\gamma\delta^2_x)$, $Exp(r\gamma\delta^2_y)$ and $Exp(r\gamma\delta^2_z)$ we get

$$u^{q+1}_{\text{l,m,n}} = (1 + r\gamma\varepsilon_{l,m,n}\delta^2_x)(1 + r\gamma\varepsilon_{l,m,n}\delta^2_y)(1 + r\gamma\varepsilon_{l,m,n}\delta^2_z)u^q_{\text{l,m,n}}$$

then

$$\begin{aligned}
u^{q+1}_{\text{l,m,n}} &= u^q_{\text{l,m,n}} + r\gamma\varepsilon_{l,m,n}(u^q_{\text{l+1,m,n}} + u^q_{\text{l-1,m,n}} + u^q_{\text{l,m+1,n}} + u^q_{\text{l,m-1,n}} + u^q_{\text{l,m,n+1}} + u^q_{\text{l,m,n-1}} - 6u^q_{\text{l,m,n}}) \\
&+ r^2\gamma^2\varepsilon^2_{\ell,m,n}(u^q_{\text{l+1,m+1,n}} + u^q_{\text{l+1,m-1,n}} + u^q_{\text{l-1,m+1,n}} + u^q_{\text{l-1,m-1,n}} + u^q_{\text{l,m+1,n+1}} + u^q_{\text{l,m+1,n-1}} \\
&+ u^q_{\text{l,m-1,n+1}} + u^q_{\text{l,m-1,n-1}} + u^q_{\text{l+1,m,n+1}} + u^q_{\text{l+1,m,n-1}} + u^q_{\text{l-1,m,n+1}} + u^q_{\text{l-1,m,n-1}} - 2u^q_{\text{l+1,m,n}} \\
&- 2u^q_{\text{l-1,m,n}} - 2u^q_{\text{l,m+1,n}} - 2u^q_{\text{l,m-1,n}} - 2u^q_{\text{l,m,n+1}} - 2u^q_{\text{l,m,n-1}} + 12u^q_{\text{l,m,n}}) + r^3\gamma^3\varepsilon^3_{\ell,m,n}(u^q_{\text{l+1,m+1,n+1}} \\
&+ u^q_{\text{l+1,m+1,n-1}} + u^q_{\text{l+1,m-1,n+1}} + u^q_{\text{l+1,m-1,n-1}} + u^q_{\text{l-1,m+1,n+1}} + u^q_{\text{l-1,m+1,n-1}} + u^q_{\text{l-1,m-1,n+1}} \\
&+ u^q_{\text{l-1,m-1,n-1}} - 2u^q_{\text{l+1,m+1,n}} - 2u^q_{\text{l+1,m-1,n}} - 2u^q_{\text{l-1,m+1,n}} - 2u^q_{\text{l-1,m-1,n}} - 2u^q_{\text{l,m+1,n+1}} \\
&- 2u^q_{\text{l,m+1,n-1}} - 2u^q_{\text{l,m-1,n+1}} - 2u^q_{\text{l,m-1,n-1}} - 2u^q_{\text{l+1,m,n+1}} - 2u^q_{\text{l+1,m,n-1}} - 2u^q_{\text{l-1,m,n+1}} - 2u^q_{\text{l-1,m,n-1}} \\
&+ 4u^q_{\text{l+1,m,n}} + 4u^q_{\text{l-1,m,n}} + 4u^q_{\text{l,m+1,n}} + 4u^q_{\text{l,m-1,n}} + 4u^q_{\text{l,m,n+1}} + 4u^q_{\text{l,m,n-1}} - 8u^q_{\text{l,m,n}})
\end{aligned} \tag{24}$$

Where the constants $\varepsilon_{l,m,n}$ are to be determined. Then we must know an additional condition, u(x, y, z, k) to be given , i.e., $\varepsilon_{l,m,n}$ must be given such that the truncation error at certain r is zero, after which, we should use the Eq.(24) for another level for calculations.

3.1. Stability Analysis for The Second Approach for Restrictive Taylor's Approximation

Using Von-Neumann method, let

$$u^q_{l,m,n} = \xi^n e^{i\eta hl} e^{i\lambda hm} e^{i\sigma hm}$$

where $i = \sqrt{-1}$, ξ^n is the amplitude at time level n, and η , λ and σ are the wave numbers in the x and y directions respectively. If the phase angle $\theta = \eta h$, $\beta = \lambda h$ and $\varphi = \sigma h$ are defined, then

$$u^q_{l,m,n} = \xi^n e^{i\theta l} e^{i\beta m} e^{i\varphi m} \quad . \tag{25}$$

Substituting Eq.(25) into Eq.(24), we get

$$\xi = (1 - 4r\gamma\varepsilon_{l,m,n} \sin^2\frac{\theta}{2})(1 - 4r\gamma\varepsilon_{l,m,n} \sin^2\frac{\beta}{2})(1 - 4r\gamma\varepsilon_{l,m,n} \sin^2\frac{\varphi}{2}) \ ,$$

then the method suggested by Eq.(24) will be stable when $0 < r\gamma\varepsilon_{l,m,n} \le \frac{1}{2}$, where ℓ, m, n = 1(1)M-1.

3.2. The Almost Zero Local Truncation Error for The Second Approach for Restrictive Taylor's Approximation

We derive the local truncation error of algorithm in Eq. (24) corresponding in similar way as in section (2.2). By using Taylor's expansion for the points $u^q_{r,s,t}$, where r=ℓ-1, ℓ, ℓ+1, s=m-1, m, m+1and t=n-1, n, n+1 in Eq. (24), the local truncation error formula of the finite difference equation (24) can take the form

$$kT^q_{l,m,n} = \delta^q_{l,m,n} - r\varepsilon_{l,m,n}\beta^q_{l,m,n} - r^2\varepsilon^2_{l,m,n}\sigma^q_{l,m,n} - r^3\varepsilon^3_{l,m,n}\psi^q_{l,m,n} \tag{26}$$

where

$$\delta^q_{l,m,n} = [ku_t + \frac{k^2}{2}u_{t^2} + \frac{k^3}{6}u_{t^3} + \frac{k^4}{24}u_{t^4} + \ldots]^q_{l,m,n} ,$$

$$\beta^q_{l,m,n} = [h^2(u_{x^2} + u_{y^2} + u_{z^2}) + \frac{1}{12}h^4(u_{x^4} + u_{y^4} + u_{z^4}) + \frac{h^6}{360}(u_{x^6} + u_{y^6} + u_{z^6}) + ...]^q_{l,m,n}$$

$$\sigma^q_{l,m,n} = [h^4(u_{x^2y^2} + u_{x^2z^2} + u_{y^2z^2}) + \frac{h^6}{12}(u_{x^4y^2} + u_{x^2y^4} + u_{x^4z^2} + u_{x^2z^4} + u_{y^4z^2} + u_{y^2z^4}) + \frac{h^8}{360}(u_{x^6y^2} + u_{x^2y^6} + u_{x^6z^2} + u_{x^2z^6} + u_{y^6z^2} + u_{y^2z^6}) + \frac{h^8}{144}(u_{x^4y^4} + u_{x^4z^4} + u_{y^4z^4}) + ...]^q_{l,m,n}$$

and

$$\psi^q_{l,m,n} = [h^6(u_{x^2y^2z^2}) + u_{x^2z^2} + u_{y^2z^2}) + \frac{h^8}{12}(u_{x^4y^2z^2} + u_{x^2y^4z^2} + u_{x^2y^2z^4}) + \frac{h^8}{360}(u_{x^6y^2} + u_{x^2y^6} + u_{x^6z^2} + u_{x^2z^6} + u_{y^6z^2} + u_{y^2z^6}) + \frac{h^{10}}{144}(u_{x^4y^4z^2} + u_{x^4y^2z^4} + u_{x^2y^4z^4}) + ...]^q_{l,m,n}$$

Similarly

$$kT^0_{l,m,n} = \delta^0_{l,m,n} - r\varepsilon_{l,m,n}\beta^0_{l,m,n} - r^2\varepsilon^2_{l,m,n}\sigma^0_{l,m,n} - r^3\varepsilon^3_{l,m,n}\psi^0_{l,m,n} \tag{27}$$

where

$$\delta^0_{l,m,n} = [ku_t + \frac{k^2}{2}u_{t^2} + \frac{k^3}{6}u_{t^3} + \frac{k^4}{24}u_{t^4} + ...]^0_{l,m,n},$$

$$\beta^0_{l,m,n} = [h^2(u_{x^2} + u_{y^2} + u_{z^2}) + \frac{1}{12}h^4(u_{x^4} + u_{y^4} + u_{z^4}) + \frac{h^6}{360}(u_{x^6} + u_{y^6} + u_{z^6}) + ...]^0_{l,m,n}$$

$$\sigma^0_{l,m,n} = [h^4(u_{x^2y^2} + u_{x^2z^2} + u_{y^2z^2}) + \frac{h^6}{12}(u_{x^4y^2} + u_{x^2y^4} + u_{x^4z^2} + u_{x^2z^4} + u_{y^4z^2} + u_{y^2z^4}) + \frac{h^8}{360}(u_{x^6y^2} + u_{x^2y^6} + u_{x^6z^2} + u_{x^2z^6} + u_{y^6z^2} + u_{y^2z^6}) + \frac{h^8}{144}(u_{x^4y^4} + u_{x^4z^4} + u_{y^4z^4}) + ...]^0_{l,m,n}$$

and

$$\psi^0_{l,m,n} = [h^6(u_{x^2y^2z^2}) + u_{x^2z^2} + u_{y^2z^2}) + \frac{h^8}{12}(u_{x^4y^2z^2} + u_{x^2y^4z^2} + u_{x^2y^2z^4}) + \frac{h^8}{360}(u_{x^6y^2} + u_{x^2y^6} + u_{x^6z^2} + u_{x^2z^6} + u_{y^6z^2} + u_{y^2z^6}) + \frac{h^{10}}{144}(u_{x^4y^4z^2} + u_{x^4y^2z^4} + u_{x^2y^4z^4}) + ...]^0_{l,m,n}$$

The main idea of the Restrictive Taylor's method is to put $T^{0}_{l,m,n}=0$ to get $\varepsilon_{l,m,n}$, then

$$\delta^{0}_{l,m,n}-r\varepsilon_{l,m,n}\beta^{0}_{l,m,n}-r^{2}\varepsilon^{2}_{l,m,n}\sigma^{0}_{l,m,n}-r^{3}\varepsilon^{3}_{l,m,n}\psi^{0}_{l,m,n}=0. \quad (28)$$

Substitute from Eq. (28) into Eq (26) we get

$$k\psi^{0}_{l,m,n}T^{q}_{l,m,n}=\delta^{q}_{l,m,n}\psi^{0}_{l,m,n}-\delta^{0}_{l,m,n}\psi^{q}_{l,m,n}-r\varepsilon_{l,m,n}(\beta^{q}_{l,m,n}\psi^{0}_{l,m,n}-\beta^{0}_{l,m,n}\psi^{q}_{l,m,n})$$
$$-r^{2}\varepsilon^{2}_{l,m,n}(\sigma^{q}_{l,m,n}\psi^{0}_{l,m,n}-\sigma^{0}_{l,m,n}\psi^{q}_{l,m,n}).$$

Hence

$$\begin{aligned}
k\psi^{0}_{l,m,n}T^{q}_{l,m,n}=\{&kh^{6}[(u_t)^{q}_{l,m,n}(u_{x^2y^2z^2})^{0}_{l,m,n}-(u_t)^{0}_{l,m,n}(u_{x^2y^2z^2})^{q}_{l,m,n}]\\
&+\frac{k^2h^6}{2}[(u_{t^2})^{q}_{l,m,n}(u_{x^2y^2z^2})^{0}_{l,m,n}-(u_{t^2})^{0}_{l,m,n}(u_{x^2y^2z^2})^{q}_{l,m,n}]\\
&+\frac{kh^8}{12}[(u_t)^{q}_{l,m,n}(u_{x^4y^2z^2}+u_{x^2y^4z^2}+u_{x^2y^2z^4})^{0}_{l,m,n}\\
&-(u_t)^{0}_{l,m,n}(u_{x^4y^2z^2}+u_{x^2y^4z^2}+u_{x^2y^2z^4})^{q}_{l,m,n}]\\
&+\frac{k^3h^6}{6}[(u_{t^3})^{q}_{l,m,n}(u_{x^2y^2z^2})^{0}_{l,m,n}-(u_{t^3})^{0}_{l,m,n}(u_{x^2y^2z^2})^{q}_{l,m,n}]\\
&+\ldots\}
\end{aligned}$$

$$\begin{aligned}
&-r\varepsilon_{l,m,n}\{h^{8}[(u_{x^2}+u_{y^2}+u_{z^2})^{q}_{l,m,n}(u_{x^2y^2z^2})^{0}_{l,m,n}\\
&-(u_{x^2}+u_{y^2}+u_{z^2})^{0}_{l,m,n}(u_{x^2y^2z^2})^{q}_{l,m,n}]\\
&+\frac{h^{10}}{12}[(u_{x^2}+u_{y^2}+u_{z^2})^{q}_{l,m,n}(u_{x^4y^2z^2}+u_{x^2y^4z^2}+u_{x^2y^2z^4})^{0}_{l,m,n}\\
&-(u_{x^2}+u_{y^2}+u_{z^2})^{0}_{l,m,n}(u_{x^4y^2z^2}+u_{x^2y^4z^2}+u_{x^2y^2z^4})^{q}_{l,m,n}\\
&+(u_{x^4}+u_{y^4}+u_{z^4})^{q}_{l,m,n}(u_{x^2y^2z^2})^{0}_{l,m,n}-(u_{x^4}+u_{y^4}+u_{z^4})^{0}_{l,m,n}(u_{x^2y^2z^2})^{q}_{l,m,n}]\\
&+\ldots\}
\end{aligned}$$

$$-r^2\varepsilon^2_{l,m,n}\{h^{10}[(u_{x^2y^2}+u_{x^2z^2}+u_{y^2z^2})^q_{l,m,n}(u_{x^2y^2z^2})^0_{l,m,n}$$
$$-(u_{x^2y^2}+u_{x^2z^2}+u_{y^2z^2})^0_{l,m,n}(u_{x^2y^2z^2})^q_{l,m,n}]$$
$$+\frac{h^{12}}{12}[(u_{x^2y^2}+u_{x^2z^2}+u_{y^2z^2})^q_{l,m,n}(u_{x^4y^2z^2}+u_{x^2y^4z^2}+u_{x^2y^2z^4})^0_{l,m,n}$$
$$-(u_{x^2y^2}+u_{x^2z^2}+u_{y^2z^2})^0_{l,m,n}(u_{x^4y^2z^2}+u_{x^2y^4z^2}+u_{x^2y^2z^4})^q_{l,m,n}$$
$$+(u_{x^4y^2}+u_{x^2y^4}+u_{x^4z^2}+u_{x^2z^4}+u_{y^4z^2}+u_{y^2z^4})^q_{l,m,n}(u_{x^2y^2z^2})^0_{l,m,n}$$
$$-(u_{x^4}+u_{y^4}+u_{z^4})^0_{l,m,n}(u_{x^2y^2z^2})^q_{l,m,n}]+\ldots\}$$

Any term in the right hand side of the previous equation can be written as

$$A^h_{i,j,k}=(u_{t^i})^q_{l,m,n}(u_{x^{2j}y^{2k}z^{2h}})^0_{l,m,n}-(u_t)^0_{l,m,n}(u_{x^{2j}y^{2k}z^{2h}})^q_{l,m,n}$$

where i, j, k, h =1(1) ∞.

$$B^h_{i,j,k}=(u_{x^{2i}}+u_{y^{2i}}+u_{z^{2i}})^q_{l,m,n}(u_{x^{2j}y^{2k}z^{2h}})^0_{l,m,n}-(u_{x^{2i}}+u_{y^{2i}}+u_{z^{2i}})^0_{l,m,n}(u_{x^{2j}y^{2k}z^{2h}})^q_{l,m,n}$$

where i, j, k, h =1(1) ∞.

and

$$c^g_{i,j,k,h}=(u_{x^{2i}y^{2j}}+u_{x^{2i}z^{2j}}+u_{y^{2i}z^{2j}})^q_{l,m,n}(u_{x^{2k}y^{2h}z^{2g}})^0_{l,m,n}$$
$$-(u_{x^{2i}y^{2j}}+u_{x^{2i}z^{2j}}+u_{y^{2i}z^{2j}})^0_{l,m,n}(u_{x^{2k}y^{2h}z^{2g}})^q_{l,m,n}$$

where i, j, k, h, g =1(1) ∞.

Since the exact solution has the closed form $u(x,y,z,t)=e^{-3t}\sin x\sin y\sin z$, where $x=\ell h,\ y=mh$, $z=nh$ and $t=qk$ then

$$A^h_{i,j,k}=3(-1)^{i+j+k+h}[(e^{-3t}\sin x\sin y\sin z)^q_{l,m,n}(e^{-3t}\sin x\sin y\sin z)^0_{l,m,n}$$
$$-(e^{-3t}\sin x\sin y\sin z)^0_{l,m,n}(e^{-3t}\sin x\sin y\sin z)^q_{l,m,n}]$$

$$B^h_{i,j,k}=3(-1)^{i+j+k+h}[(e^{-3t}\sin x\sin y\sin z)^q_{l,m,n}(e^{-3t}\sin x\sin y\sin z)^0_{l,m,n}$$
$$-(e^{-3t}\sin x\sin y\sin z)^0_{l,m,n}(e^{-3t}\sin x\sin y\sin z)^q_{l,m,n}]$$

$$c^g_{i,j,k,h}=3(-1)^{i+j+k+h+g}[(e^{-3t}\sin x\sin y\sin z)^q_{l,m,n}(e^{-3t}\sin x\sin y\sin z)^0_{l,m,n}$$
$$-(e^{-3t}\sin x\sin y\sin z)^0_{l,m,n}(e^{-3t}\sin x\sin y\sin z)^q_{l,m,n}]$$

Hence $A^h_{i,j,k}=0$, $B^h_{i,j,k}=0$ and $c^g_{i,j,k,h}=0$ where i, j, k, h, g =1(1) ∞.

Hence we can derive that $k\psi^{0}_{l,m,n}T^{q}_{l,m,n}$ is almost zero ,but since in general $\psi^{0}_{l,m,n} \neq 0$ then $T^{q}_{l,m,n}$ is almost zero, i.e. the first approach for Restrictive Taylor's method has zero local truncation error.

6. Numerical Examples and Comparative Results

Example 1:

To provide some indication of the accuracy of our method, we consider the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \ , \quad 0 \le x, y, z \le \pi \ , \ t \ge 0$$

where the exact solution is given by, u(x, y,z, t) $= e^{-3t} \sin x \sin y \sin z$.

The initial and boundary conditions are defined as so to agree with the exact solution. The accuracy of the first approach (RT1) and the second approach (RT2) are compared in Table (1) for various values of time t. Table (1) give the absolute error along (x, y, z) = (0.1π, 0.1π, 0.1π), (0.5π, 0.5π0, .5π) and (0.9π, 0.9π, 0.9π) as this is the end and the middle points of the domain, where h= 0.1π and k = 0.01

t	(x, y, z) = (0.1π,0.1π, 0.1π)		(x, y, z) = (0.5π, 0.5π, 0.5π)		(x, y, z) = (0.9π, 0.9π, 0.9π)	
	RT1	RT2	RT1	RT2	RT1	RT2
0.1	1.387×10^{-17}	1.387×10^{-17}	6.661×10^{-16}	1.110×10^{-16}	1.734×10^{-17}	1.040×10^{-17}
0.2	2.775×10^{-17}	1.040×10^{-17}	1.221×10^{-15}	3.330×10^{-16}	2.775×10^{-17}	1.040×10^{-17}
0.3	3.122×10^{-17}	1.387×10^{-17}	1.165×10^{-15}	2.220×10^{-16}	2.949×10^{-17}	1.387×10^{-17}
0.4	3.295×10^{-17}	1.040×10^{-17}	1.165×10^{-15}	2.775×10^{-16}	3.295×10^{-17}	1.040×10^{-17}
0.5	3.122×10^{-17}	8.673×10^{-18}	1.082×10^{-15}	3.330×10^{-16}	3.035×10^{-17}	1.040×10^{-17}
1	1.366×10^{-17}	4.336×10^{-18}	4.787×10^{-16}	1.457×10^{-16}	1.366×10^{-17}	4.336×10^{-18}

Example 2:

To provide some indication of the accuracy of our method, we consider the heat conduction equation

$$\frac{\partial u}{\partial t} = 0.02\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \ , \quad 0 \le x, y, z \le \pi \ , \ t \ge 0$$

where the exact solution is given by , u(x, y,z, t) $= e^{-0.06t} \sin x \sin y \sin z$.

The initial and boundary conditions are defined as so to agree with the exact solution. The accuracy of the first approach (RT1) and the second approach (RT2) are compared in Table (1) for various values of time t. Table (1) give the absolute error along (x, y, z) = (0.1π, 0.1π, 0.1π), (0.5π, 0.5π0, .5π) and (0.9π, 0.9π, 0.9π) as this is the end and the middle points of the domain, where h= 0.1π and k = 0.5

t	(x, y, z) = (0.1π,0.1π, 0.1π)	(x, y, z) = (0.5π, 0.5π, 0.5π)	(x, y, z) = (0.9π, 0.9π, 0.9π)

	RT1	RT2	RT1	RT2	RT1	RT2
5	2.428×10^{-17}	1.734×10^{-17}	9.992×10^{-16}	7.771×10^{-16}	2.428×10^{-17}	2.081×10^{-17}
10	3.816×10^{-17}	2.428×10^{-17}	1.332×10^{-15}	8.881×10^{-16}	3.816×10^{-17}	2.428×10^{-17}
15	4.510×10^{-17}	2.775×10^{-17}	1.665×10^{-15}	1.054×10^{-15}	4.510×10^{-17}	2.775×10^{-17}
20	4.510×10^{-17}	2.949×10^{-17}	1.554×10^{-15}	1.054×10^{-15}	4.510×10^{-17}	2.949×10^{-17}
25	4.336×10^{-17}	3.122×10^{-17}	1.526×10^{-15}	1.054×10^{-15}	4.336×10^{-17}	3.122×10^{-17}
50	1.994×10^{-17}	1.334×10^{-17}	6.730×10^{-16}	4.718×10^{-16}	1.973×10^{-17}	1.301×10^{-17}

Example 3:

To provide some indication of the accuracy of our method, we consider the heat conduction equation

$$\frac{\partial u}{\partial t}=\frac{1}{3}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)\ ,\qquad 0\le x,y,z\le\pi\ ,\ \ t\ge 0$$

where the exact solution is given by , u(x, y,z, t) $=e^{-t}\sin x\sin y\sin z$.

The initial and boundary conditions are defined as so to agree with the exact solution. The accuracy of the first approach (RT1) and the second approach (RT2) are compared in Table (1) for various values of time t. Table (1) give the absolute error along (x, y, z) = (0.1π, 0.1π, 0.1π), (0.5π, 0.5π0, .5π) and (0.9π, 0.9π, 0.9π) as this is the end and the middle points of the domain, where h= 0.1π and k = 0.01

t	(x, y, z) = (0.1π,0.1π, 0.1π)		(x, y, z) = (0.5π, 0.5π, 0.5π)		(x, y, z) = (0.9π, 0.9π, 0.9π)	
	RT1	RT2	RT1	RT2	RT1	RT2
0.1	2.428×10^{-17}	3.122×10^{-17}	1.221×10^{-15}	1.554×10^{-15}	2.428×10^{-17}	3.469×10^{-17}
0.2	5.551×10^{-17}	7.285×10^{-17}	2.220×10^{-15}	2.664×10^{-15}	5.551×10^{-17}	7.285×10^{-17}
0.3	7.632×10^{-17}	1.006×10^{-16}	2.886×10^{-15}	3.552×10^{-15}	7.632×10^{-17}	1.006×10^{-16}
0.4	9.029×10^{-17}	1.179×10^{-16}	3.330×10^{-15}	4.218×10^{-15}	9.029×10^{-17}	1.179×10^{-16}
0.5	1.006×10^{-16}	1.353×10^{-16}	3.774×10^{-15}	4.662×10^{-15}	1.006×10^{-16}	1.353×10^{-16}
1	1.283×10^{-16}	1.665×10^{-16}	4.496×10^{-15}	5.662×10^{-15}	1.283×10^{-16}	1.665×10^{-16}

Conclusion

1. The Restrictive Taylor's Approximation gives the solution for IBVP for three-dimensional Parabolic PDE with zero local truncation error.
2. Changing the approach of the finite difference chosen equation associated with the Restrictive Taylor's Approximation gives almost the same accuracy.
3. The error are referred to the Rounding-off error of the computational algorithms.

Reference:

[1] Davod Khojasteh Salkuyeh " On the Finite Difference Approximation to the Convection-Diffusion Equation" Appl. Math. & Computation, Article in Press, (2005).

[2] Hassan N. A. Ismail and Ahmed R. A. Shafay. "Explicit Restrictive Taylor's Methods of Almost Exact Solution for IBVP for Two-Dimensional Parabolic PDE ", The 31st International conference for Statistics, Computer science and its Application, 2, pp 49-59, (2006).

[3] Hassan N. A. Ismail and Elsayed M. E. Elbarbary. "Restrictive Taylor's Approximation and Parabolic Partial Differential Equations", Int J. Computer Math., 78, pp 73-82, (2001).

[4] Hassan N. A. Ismail, Elsayed M. E. Elbarbary and Ghada S. E. Salem. "Restrictive Taylor's Approximation for Two Dimensions Initial Boundary Value Problem for Parabolic PDE", Appl. Math. & Computation, 147, 607-615, (2004).

[5] Hassan N. A. Ismail and Ghada S. E. Salem. "On the Convergence of the Restrictive Taylor's Approximation to the Exact Solution of IBVP for Parabolic, Hyperbolic, Convection Diffusion and KdV Equations", The 29th International conference for Statistics, Computer science and its Application. 2 pp, 11-22, (2003).

[6] Mitchell, A. R. "Computational Methods in Partial Differential Equation", John Wiley & Sons Ltd., London, (1969).

[7] Mustafa Gulsu and Turgut Ozis " Numerical Solution of Burger's Equation with Restrictive Taylor's Approximation" Appl. Math. & Computation, Article in Press, (2005).