Military Technical College
Kobry Elkobbah,
Cairo, Egypt
May 16-18,2006

3rd International Conference on
Engineering Mathematics and
Physics (ICMEP-3)

WARPED PRODUCT OF RIEMANNIAN MANIFOLDS

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ABSTRACT. The sectional curvature of the Riemannian warped product manifold is derived in terms of the original ones. The second fundamental form and totally geodesic submanifolds in warped product manifolds are introduced. We study the important example $\mathbb{R}\times_f\mathbb{R}$ (*warped plane*) as an application. The Livi-Civita connection on $\mathbb{R}\times_f\mathbb{R}$ is derived. Moreover, we discuss its godesics and Gauss curvature with specific forms of $f(x)$. The concepts of coloring and folding by curvature are introduced on the warped plane. Illustrating figures are given.

1. AN INTRODUCTION

Let M and N be two C^∞ Riemannian manifolds with dimensions m, n and Riemannian metrics g_1, g_2 respectively. The Cartesian product $M\times N$ gives a new and bigger C^∞ manifold of dimension $(m+n)$. Let M_a and N_b be the tangent spaces of M and N at $a\in M$ and $b\in N$ respectively.

Now, we define a new Riemannian product manifold of M and N which is called the Riemannian warped product ([1], [5], [6], [7] and [11]).

Definition 1. [1]*Let f be a positive, real-valued, and differentiable function defined on M. A warped Riemannian product manifold of two C^∞ Riemannian manifolds M and N, denoted by $M\times_f N$, is the C^∞ manifold $M\times N$ equipped with the Riemannian metric g defined by*

$$g((X_1,Y_1),(X_2,Y_2)) = g_1(X_1,X_2) + f^{\,2}(x)g_2(Y_1,Y_2)$$

where $X_1, X_2 \in M_a$ and $Y_1, Y_2 \in N_b$.

Thus $g = g_1 + f^{\,2}g_2$, and the function f is called the warping function of the warped product [5]. The warped manifold $M\times_f N$ is characterized by the fact that M is totally geodesic and N is totally umbilical submanifolds of $M\times_f N$ [2].

The following example is discussed in [4].

Example 1. *Consider the manifold $\mathbb{R}^n_0\times\mathbb{R}$ (where $\mathbb{R}^n_0 = \mathbb{R}^n \setminus \{0\}$) equipped with a Riemannian product of vectors $(X,\zeta),(Y,\eta)\in T_{(a,\alpha)}(\mathbb{R}^n_0\times\mathbb{R})$ defined by*

$$\langle (X,\zeta),(Y,\eta)\rangle = \langle X,Y\rangle + \|a\|^2\zeta\eta \tag{1.1}$$

i.e. the warped function is defined as

$$a\longmapsto \|a\| : \mathbb{R}^n_0 \longrightarrow \mathbb{R}$$

This manifold is called a central symmetric warped product manifold [4] *and corresponds to the potential function* $a\|x\|, a\succeq 0$[4].

Date: Augast 15, 2005.
2000 *Mathematics Subject Classification.* 53C20, 53C21.
Key words and phrases. Warped product, curvature, Riemannian metric, Riemannian connection, geodesics.

This paper is in final form and no version of it will be submitted for publication elsewhere.

[1]O'Neil and Bishop introduced warped products to construct Riemannian manifolds with negative sectional curvature. Later it was pointed out that many exact solutions to Einstien's field equation can be expressed in terms of wared products (*Bulent Unal*)

Let $h(x,y)$ be any real-valued function defined on the product $M \times_f N$, then we can take [12]

$$(1.2) \quad (X_1,Y_1)h(x,y) = X_1 h(x,y) + Y_1 h(x,y)$$

and to compute $X_1 h(x,y)$ we consider y as a constant and vise versa [3]. Now if D^1, D^2 are the Riemannian connections on M and N respectively, then the connection defined by [13]

$$(1.3) \quad D_{(X_1,Y_1)}(X_2,Y_2) = (D^1{}_{X_1}Y_1, D^2{}_{X_2}Y_2)$$

is Riemannian on the manifold $M \times N$ but not Riemannian on the manifold $M \times_f N$ for non-constant functions $f(x)$,for,

$$(X_1,Y_1)g((X_2,Y_2),(X_3,Y_3) = g(D_{(X_1,Y_1)}(X_2,Y_2),(X_3,Y_3))+ \\ g((X_2,Y_2),D_{(X_1,Y_1)}(X_3,Y_3)) + X_1(f^2)g_2(Y_1,Y_2)$$

i.e., $\left(D_{(X_1,Y_1)}g\right)\left((X_2,Y_2),(X_3,Y_3)\right) = X_1(f^2)g_2(Y_1,Y_2) \neq 0$ and hence D is not metric connection. Note that D is a torsion-free connection. If f is constant, the above connection becomes metric and hence Riemannian on the manifold $M \times_f N$.

The Riemannian connection defined on the warped product $M \times_f N$ is given in the following proposition [2]. We denote the Riemannian connection of the warped product metric tensor g by D.

Proposition 1. *(O'Neil) Let $M \times_f N$ be a warped Riemannian product manifold with the warping function $f > 0$ on . Then we have*

(1) $D_{(X_1,0)}(X_2,0) = D^1_{X_1}X_2$.
(2) $D_{(X_1,0)}(0,Y_1) = D_{(0,Y_1)}(X_1,0) = \frac{X_1(f)}{f}X_2$.
(3) *nor* $\left\{D_{(0,Y_1)}(0,Y_2)\right\} = -f < Y_1,Y_2 > \operatorname{grad} f$.
(4) *tan* $\left\{D_{(0,Y_1)}(0,Y_2)\right\} = D^2_{Y_1}Y_2$

for all $X_i \in \chi(M)$ and $Y_i \in \chi(N)$, where D^1 and D^2 are the Levi-Civita connections of Riemannian metric tensors g_1, g_2 respectively

Now, we discuss the geodesics of the warped product manifold $M \times_f N$. Let $\alpha(t)$ (*resp.* $\beta(t)$) be a geodesic in M.(*resp.* N) with tangent field T_1 (*resp.* T_2). Hence, the diagonal product $\gamma(t) = (\alpha(t), \beta(t))$ of $\alpha(t)$ and $\beta(t)$ is a curve in $M \times_f N$ with tangent field $T = (T_1, T_2)$. Using proposition (1) we get

$$\begin{aligned} D_T T &= D_{T_1}T_1 + D_{T_1}T_2 + D_{T_2}T_1 + D_{T_2}T_2 \\ &= D^1_{T_1}T_1 + \frac{2}{f}T_1(f)T_2 + D^2_{T_2}T_2 - f\langle T_2,T_2\rangle \operatorname{grad} f \\ (1.4) \qquad &= \frac{2}{f}T_1(f)T_2 - f\langle T_2,T_2\rangle \operatorname{grad} f \end{aligned}$$

Then the diagonal product of two geodesics is a geodesic in $M \times_f N$ if

- $T_2 = \mathbf{0}$ i.e., $\beta(t)$ is a constant curve, or
- $\operatorname{grad} f = \mathbf{0}$ i.e., f is a constant function which is the case of the Cartesian product.

2. Curvature of the warped product

In this section we discuss the curvature tensor R of the warped product $M \times_f N$ and its relation with the curvature tensors R_1 and R_2 of M and N respectively. Also the sectional curvature is derived. Some special cases of the warped product is considered. The following proposition determines the curvature tensor[2].

Proposition 2. *Let $M \times_f N$ be a warped Riemannian product manifold with the warping function $f \succ 0$ and Riemannian curvature tensor R. Then we have*

(1) $R(X_1, X_2)X_3 = R_1(X_1, X_2)X_3 \in \chi(M)$.
(2) $R(Y_1, X_1)X_2 = \frac{1}{f}H^f(X_1, X_2)Y_1$
(3) $R(X_1, X_2)Y_1 = R(Y_1, Y_2)X_1 = 0$
(4) $R(Y_1, Y_2)Y_3 = R_2(Y_1, Y_2)Y_3 - \langle \operatorname{grad} f, \operatorname{grad} f\rangle \{\langle Y_1, Y_3\rangle Y_2 - \langle Y_2, Y_3\rangle X_1\}$
(5) $R(X_1, Y_1)Y_2 = f < Y_1, Y_2 > D^1_{X_1} \operatorname{grad} f$.

for all $X_i \in \chi(M)$, and $Y_i \in \chi(N)$, where R_1, and R_2 denote the Riemannian curvature tensors of M and N respectively, and H^f is the Hessian form of the warping function f.

Remark 1. *In the above two propositions, if we put $f = 1$ we get the case of the usual Cartesian product with $D = (D^1, D^2)$, and $R = (R_1, R_2)$.*

Now, we can derive the sectional curvature $K(X, Y)$ of the 2−subspace generated by the vectors $X = (X_1, X_2), Y = (Y_1, Y_2)$ in $\chi(M \times_f N)$. We denote the sectional curvature of M and N by $K_1(X_1, X_2)$, $K_2(Y_1, Y_2)$ respectively, where $X_i \in \chi(M)$ and $Y_i \in \chi(N)$.

Theorem 1. *Let $X = (X_1, X_2), Y = (Y_1, Y_2)$ be in $M_a \times N_a$. Then the curvature $K(P)$ of the section P generated by X, Y is given by*

$$K(1+f^2)^2 = K_1 + K_2 f^2 + f H^f(X_1, Y_1) + f\langle D_{X_1} \operatorname{grad} f, X_1\rangle + f^2 \|\operatorname{grad} f\|^2$$

where K_1, and K_2 are the curvatures of the sections generated by the orthonormal sets $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ in M_a and N_b respectively.

Proof. The sectional curvature of $M \times_f N$ is given by

$$\begin{aligned}
K(X,Y)A &\neq \langle R(X,Y)Y, X\rangle \\
&= g\{R_1(X_1,Y_1)Y_1 - \frac{1}{f}H^f(X_1,Y_1)Y_2 + f\langle Y_2, Y_2\rangle D_{X_1}\operatorname{grad} f + \\
&\quad \frac{1}{f}H^f(Y_1,Y_1)X_2 - f\langle X_2, Y_2\rangle D_{Y_1}\operatorname{grad} f + R_2(X_2,Y_2)Y_2 - \\
&\quad \langle \operatorname{grad} f, \operatorname{grad} f\rangle \{\langle X_2, Y_2\rangle Y_2 - \langle Y_2, Y_2\rangle X_2\}, (X_1, X_2)\} \\
&= K_1 A_1 + f^2 K_2 A_2 - f H^f(X_1,Y_1)\langle X_2, Y_2\rangle + f H^f(Y_1,Y_1)\langle X_2, X_2\rangle + \\
&\quad f\langle Y_2, Y_2\rangle\langle D_{X_1}\operatorname{grad} f, X_1\rangle - f\langle X_2, Y_2\rangle\langle D_{Y_1}\operatorname{grad} f, X\rangle + \\
&\quad f^2 A_2 \|\operatorname{grad} f\|^2
\end{aligned}$$

If X_1, Y_1 are orthonormal and X_2, Y_2 are also orthonormal, then $A_1 = A_2 = 1$ and $A = (1+f^2)^2$. Hence the result. □

If we put $f = 1$ in the above result we get

$$K = \frac{1}{4}\{K_1 + K_2\}$$

which is the case of the usual Cartesian product.

Now, we generalize the concept of the second fundamental form on the warped product manifold $\bar{M} \times_f \bar{N}$. Let $\bar{D}^1$ $(resp.\bar{D}^2)$ be the Riemannian connection defined on $\bar{M}$ $(resp.\bar{N})$. Let D^1 $(resp.D^2)$ be the induced connection defined on a submanifold M $(resp.N)$ of $\bar{M}$ $(resp.\bar{N})$ with codimension r $(resp.s)$. Let $\zeta_i, i = 1, ..., r$ $(resp.\eta_i, i = 1, ..., s)$ be a set of orthonormal vector fields on M $(resp.N)$. The Gauss decomposition equations of M and N are given by

$$\bar{D}^1_{X_1} X_2 = D^1_{X_1} X_2 + \alpha(X_1, X_2), \; X_i \in \chi(M) \tag{2.1}$$

$$\bar{D}^2_{Y_1} Y_2 = D^2_{Y_1} Y_2 + \beta(Y_1, Y_2), \; Y_i \in \chi(N) \tag{2.2}$$

where $\alpha(X_1, X_2)$ and $\beta(Y_1, Y_2)$ are the second fundamental forms of M and N respectively which can be written as follows

$$\alpha(X_1, X_2) = \sum_{i=1}^{r} \alpha_i(X_1, X_2)\zeta_i, \ \ \beta(Y_1, Y_2) = \sum_{i=1}^{s} \beta_i(Y_1, Y_2)\eta_i$$

The Cartesian product $M \times N$ is a submanifold of $\bar{M} \times_f \bar{N}$, then

$$\bar{D}_{Z_1} Z_2 = D_{Z_1} Z_2 + h(Z_1, Z_2), \ Z_i \in \chi(M \times N) \tag{2.3}$$

where $h(Z_1, Z_2)$ is the second fundamental form of $M \times N$. $h(Z_1, Z_2)$ can be written as follows

$$h(Z_1, Z_2) = \sum_{i=1}^{r+s} h_i(Z_1, Z_2)\xi_i$$

where $\xi_i, i = 1, ..., r+s$ are a set of orthonormal vector fields on $M \times N$, say,

$$\xi_1 = (\zeta_1, 0), ..., \xi_r = (\zeta_r, 0), \xi_{r+1} = \left(0, \frac{\eta_1}{f}\right), ..., \xi_{r+s} = \left(0, \frac{\eta_s}{f}\right)$$

Hence,

$$h(Z_1, Z_2) = \sum_{i=1}^{r} h_i(Z_1, Z_2)(\zeta_i, 0) + \sum_{i=1}^{s} h_{i+r}(Z_1, Z_2)\left(0, \frac{\eta_i}{f}\right)$$

The components of $h(Z_1, Z_2)$ are determined by the following equations

$$\begin{aligned} h_i(Z_1, Z_2) &= \langle h(Z_1, Z_2), (\zeta_i, 0)\rangle \\ &= \langle \bar{D}_{Z_1} Z_2, (\zeta_i, 0)\rangle \\ &= \langle \bar{D}^1_{X_1} X_2 - f\langle Y_1, Y_2\rangle \ \operatorname{grad} f, \zeta_i\rangle \\ &= \alpha_i(X_1, X_2) - f\langle Y_1, Y_2\rangle \ \langle \operatorname{grad} f, \zeta_i\rangle, i \leq r \end{aligned} \tag{2.4}$$

Also,

$$\begin{aligned} h_{i+r}(Z_1, Z_2) &= \left\langle h(Z_1, Z_2), \left(0, \frac{\eta_i}{f}\right)\right\rangle \\ &= \left\langle \bar{D}_{Z_1} Z_2, \frac{\eta_i}{f}\right\rangle \\ &= \frac{1}{f}.f^2\langle \bar{D}^2_{Y_1} Y_2, \eta_i\rangle \\ &= f\beta_i(Y_1, Y_2) \end{aligned} \tag{2.5}$$

From equations (2.4) and (2.5) we get

$$h(Z_1, Z_2) = \alpha(X_1, X_2) + \beta(Y_1, Y_2) - f\langle Y_1, Y_2\rangle \mathbf{F}$$

where $\mathbf{F} = \sum_{i=1}^{r} \zeta_i(f)$.

Theorem 2. *The Cartesian product of two totally geodesic submanifolds M and N of $\bar{M}$ and $\bar{N}$ respectively is a totally geodesic submanifold of $\bar{M} \times_f \bar{N}$ if $\mathbf{F} = 0$.*

Theorem 3. *The submanifold $\bar{M}$ of $\bar{M} \times \bar{N}$ is totally geodesic. The submanifold $\bar{N}$ of $\bar{M} \times \bar{N}$ is not generally totally geodesic.*

We now discuss the concept of Lie derivative on the warped product. Let $Z = (X, Y)$, $Z_1 = (X_1, Y_1)$ and $Z_2 = (X_2, Y_2)$ be arbitrary elements in $\chi(M \times_f N)$,

$$(3.1) \qquad (g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & f^2(x) \end{pmatrix}$$

The inverse of this matrix is given by

$$(3.2) \qquad (g^{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & f^{-2}(x) \end{pmatrix}$$

Using this Riemannian metric we derive the Riemannian connection on $\mathbb{R} \times_f \mathbb{R}$ through finding its coefficients Γ^i_{jk} using the relation [8]

$$(3.3) \qquad \Gamma^i_{jk} = \frac{1}{2}\sum g^{ir}(\frac{\partial g_{jr}}{\partial x_k} + \frac{\partial g_{rk}}{\partial x_j} - \frac{\partial g_{jk}}{\partial x_r})$$

where $1 \leq i, j, k, r \leq 2$, and $x_1 = x, x_2 = y$.Simple computations give

$$\Gamma^1_{11} = 0, \qquad \Gamma^1_{12} = 0$$

$$\Gamma^1_{21} = 0, \qquad \Gamma^1_{22} = -f(x)f'(x)$$

$$\Gamma^2_{11} = 0, \qquad \Gamma^2_{12} = \frac{f'(x)}{f(x)}$$

$$\Gamma^2_{21} = \frac{f'(x)}{f(x)}, \qquad \Gamma^2_{22} = 0$$

Now, we completely defined the Riemannian connection D on $\mathbb{R} \times_f \mathbb{R}$ and consequently we may write

$$D_X X = 0, \ D_X Y = \frac{f'(x)}{f\ (x)} Y$$

$$D_Y X = \frac{f'(x)}{f(x)} Y, \ D_Y Y = -f(x)f'(x)X$$

The curvature tensor of the connection D is given by

$$R(X,Y)Z = D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z$$

and hence we find, for $X = (\frac{\partial}{\partial x}, 0)$ and $Y = (0, \frac{\partial}{\partial y})$, that

$$R(X,X)X = 0, \qquad R(X,X)Y = 0,$$
$$R(X,Y)X = -R(Y,X)X = \frac{f''(x) - 2f'^2(x)}{f^2(x)} Y,$$
$$R(X,Y)Y = -R(Y,X)Y = -f(x)f''(x)X,$$
$$R(Y,Y)X = 0, \qquad R(Y,Y)Y = 0$$

Let $K(\sigma)$ be the curvature(Gauss curvature) of the two dimensional space σ spanned by $X = (\frac{\partial}{\partial x}, 0)$ and $Y = (0, \frac{\partial}{\partial y})$, then

$$(3.4) \qquad K(\sigma) = \frac{\langle X, R(X,Y)Y \rangle}{A(X,Y)} = -f(x)f''(x)$$

Remark 3. • *As $K(\sigma)$ depends only on x (not on y), then all of the points of the straight lines $x = a, a \in R$, are of the same curvature.*

- *A point $(x, y) \in \mathbb{R} \times_f \mathbb{R}$ is planar i.e. $K = 0$ if and only if $f''(x) = 0$.*
- *If all the points of $\mathbb{R} \times_f \mathbb{R}$ are planar,i.e.$K(x) \equiv 0$, then $f''(x) = 0$,for all x ,i.e. ,$f(x) = cx + d$, wherec, d are constants. As $f(x) \succ 0$,we get $c = 0$, and $d \succ 0$ and hence $f(x)$ is constant (the converse is also true).*
- *A point $(x, y) \in \mathbb{R} \times_f \mathbb{R}$ is hyperbolic(elliptic) iff $f'' \succ 0(f'' \prec 0)$.*
- *If ff'' is a positive constant, we obtain a hyperbolic plane.*
- *For $f'' \succ 0$, $\mathbb{R} \times_f \mathbb{R}$ is a complete simply connected C^∞ Riemannian manifold without focal points and hence without conjugate points.*

The last remark is very useful in constructing Riemannian manifolds of dimension 2 without focal (or conjugate) points.

4. Geodesics of the warped plane

Let $\alpha(t) = (x(t), y(t))$ be a geodesic of $\mathbb{R} \times_f \mathbb{R}$. .The existence of geodesics in any C^∞ manifold M of dimension n is guaranteed by the following theorem [9]

Theorem 5. *Let x be a point in $M, X \in M_x$ .Then for any real number b there exists a real number $r \succ 0$,and a unique curve $\alpha(t)$ defined on $[b - r, b + r]$ such that $\alpha(b) = x, \alpha^{\cdot}(b) = X$, and α is a geodesic.*

This theorem gives rise a second order differential system

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j} \Gamma_{ij}^k \frac{dx_i}{dt} \cdot \frac{dx_j}{dt} = 0, where 1 \leq i, j, k \leq 2$$

with initial conditions $x_1(0) = x_0, x_2(0) = y_0, x_1^{\cdot}(0) = u, and\ x_2^{\cdot}(0) = v$ which determines the geodesics of $\mathbb{R} \times_f \mathbb{R}$.Now we try to solve this system for $k = 1, 2$(*put* $x_1 = x$ *and* $x_2 = y$), we get

$$x^{\cdot\cdot} - f(x)f'(x)y^{\cdot 2} = 0 \tag{4.1}$$

$$y^{\cdot\cdot} + \frac{2f'(x)}{f(x)} x^{\cdot} y^{\cdot} = 0 \tag{4.2}$$

with the conditions $x(0) = x_0, y(0) = y_0, x^{\cdot}(0) = u, and\ y^{\cdot}(0) = v$ .Put $x^{\cdot} = z, y^{\cdot} = w$, then we get

$$z^{\cdot} - f(x)f'(x)w^2 = 0 \tag{4.3}$$

$$w^{\cdot} + \frac{2f'(x)}{f(x)} wz = 0 \tag{4.4}$$

Equation(4.4) implies that

$$\begin{aligned} w^{\cdot} &= \frac{-2f'(x)}{f(x)} wz \\ \int \frac{dw}{w} &= -2 \int \frac{f'(x)}{f(x)} dx, given\ that\ w = y^{\cdot} \neq 0 \end{aligned}$$

$$y^{\cdot} = \frac{A}{f^2(x)}, A \text{ is constant} \tag{4.5}$$

At $t = 0$ we have $A = vf^2(x_0)$. Note that, without loss of generality, we take the positive sign of $y^{\cdot}$.Using equation(4.3) we get

$$x^{\cdot\cdot} - f'(x)\frac{A^2}{f^3(x)} = 0 \rightarrow x^{\cdot 2} = B - \frac{A^2}{f^2(x)}$$

At $t = 0$ we get $B = u^2 + v^2 f^2(x_0)$.Without loss of generality we consider the positive sign of $x^{\cdot}$, so

$$x^{\cdot} = \sqrt{B - \frac{A^2}{f^2(x)}} \rightarrow t = \int \frac{dx}{\sqrt{B - \frac{A^2}{f^2(x)}}} + c, \textit{given that } x^{\cdot} \neq 0 \tag{4.6}$$

(the cases $x^{\cdot} = 0, y^{\cdot} = 0$ will be discussed later).From equation(4.6) we have

$$x^{\cdot}x^{\cdot} = B - Ay^{\cdot} \rightarrow \int \frac{dx}{dt}dx = \int (B - Ay^{\cdot})dt$$

and hence we find

$$y = \frac{\sqrt{B}}{A}\left(\int \frac{f(x)}{\sqrt{f^2(x) - \frac{A^2}{B}}}dx - \int \frac{\sqrt{f^2(x) - \frac{A^2}{B}}}{f(x)}dx\right) + D \tag{4.7}$$

Equation(4.7) determines the geodesics of the warped plane.

Remark 4.

- *If $f(x) = 1$, the warped plane becomes the Euclidean plane and the equation(4.7) becomes*

  $y = y_0 + \frac{v}{u}(x - x_0)$

  which is an equation of a straight line.
- *If $y^{\cdot} = 0$, then equation(4.1) implies that*

  $x = x_0 + ut, y = y_0$

  which are the parametric equations of a straight line parallel to $x-$axis.
- *If $x^{\cdot} = 0$, then equation(4.2) implies that*

  $x = x_0, y = y_0 + vt$

  which are the parametric equations of a straight line parallel to $y-$axis.

5. Applications

In this section we discuss the geodesics and curvature on the warped plane for some specific forms of $f(x)$.

Case 1. $f(x) = \exp x$

In this case the above equations(4.1),(4.2) become

$$x^{\cdot\cdot} - \exp(2x)\, y^{\cdot 2} = 0 \tag{5.1}$$

$$y^{\cdot\cdot} + 2x^{\cdot}y^{\cdot} = 0 \tag{5.2}$$

The solution of this system is given by equation(4.7) where $f(x) = \exp x$ as follows

$$y = \frac{\sqrt{B}}{A}\left(\int \frac{\exp x}{\sqrt{\exp 2x - \frac{A^2}{B}}}dx - \int \frac{\sqrt{\exp 2x - \frac{A^2}{B}}}{\exp x}dx\right) + D \tag{5.3}$$

Putting $\mu = \exp x$, then $dx = \frac{1}{\mu}d\mu$, and hence

$$\begin{aligned} y &= \frac{\sqrt{B}}{A}\left(\int \frac{1}{\sqrt{\mu^2-\frac{A^2}{B}}}d\mu - \int \frac{\sqrt{\mu^2-\frac{A^2}{B}}}{\mu^2}d\mu\right) + D \\ &= \frac{\sqrt{B}}{A}\ln\left(\mu+\sqrt{\mu^2-\frac{A^2}{B}}\right) - \\ &\quad \frac{\sqrt{B}}{A}\left\{-\frac{\sqrt{\mu^2-\frac{A^2}{B}}}{\mu} + \ln\left(\mu+\sqrt{\mu^2-\frac{A^2}{B}}\right)\right\} + D \\ &= \sqrt{\frac{B}{A^2}-\exp(-2x)} + D. \end{aligned}$$

At $t=0$ we get $D = y_0 - \sqrt{\frac{B}{A^2}-\exp(-2x_0)}$ and so

$$y = y_0 + \sqrt{\frac{B}{A^2}-\exp(-2x)} - \sqrt{\frac{B}{A^2}-\exp(-2x_0)} \tag{5.4}$$

The geodesics emanating from the points (0,0) and (0,1) with initial velocity $(u,v) = (1,1)$ are represented by $y=\sqrt{2-\exp(-2x)}-1$ and $y=\sqrt{2-\exp(-2x)}$ respectively. These equations are represented graphically in 1.

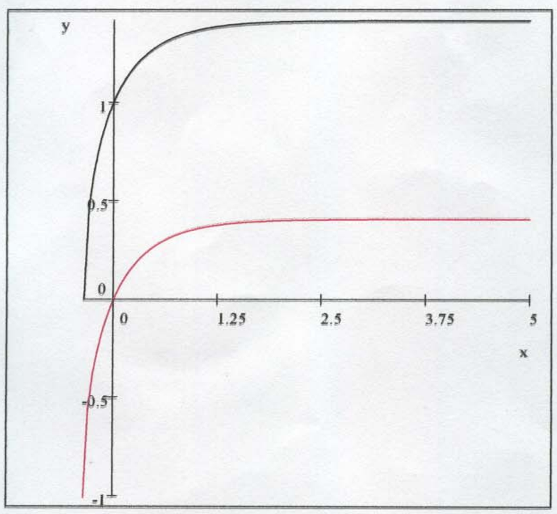


FIGURE 1. Sample geodesics

Now we consider the curvature of the warped plane .From equation (3.4) we find that

$$K = -\exp 2x$$

Remark 5. • *The curvature depends only on x ,and hence all points on the same vertical line are of the same curvature. The following figure draws, for every point (x,y),a point $(x,y,-\exp(2x))$ to show the behavior of the* *curvature on this warped plane. The lines parallel to $y-$axis are lines of the same curvature see 2.*

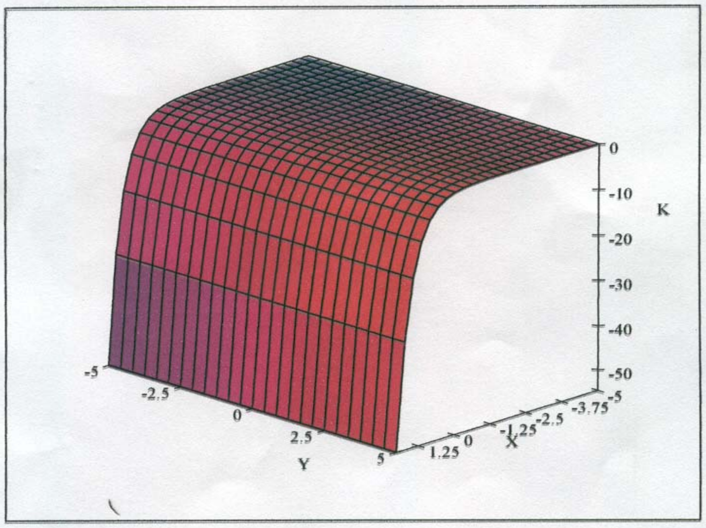


FIGURE 2. Equicurvature lines

- *All points (x,y) in this warped plane are hyperbolic i.e. $K \prec 0$.*
- *Curvature decreases as x increases and it vanishes as x tends to $-\infty$.Also K tends to $-\infty$ as x tends to ∞. 3 is a cross section of 2 by $zx-$plane, representing curvature decreasing along $x-$axis.*

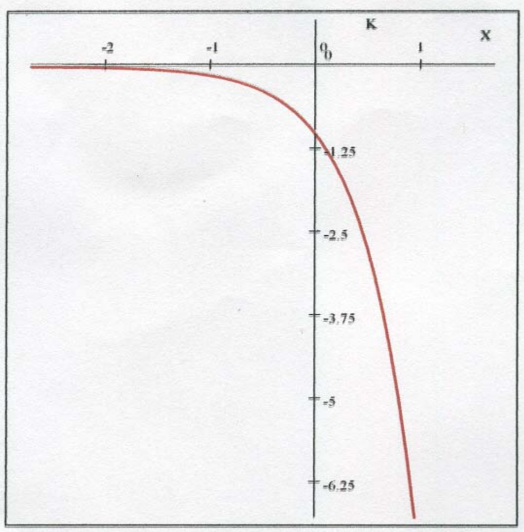


FIGURE 3. Curvature decreasing

Case 2. $f(x) = \frac{1}{\sqrt{x^2+1}}$

In this case the above equations (4.1)(4.2) become

$$x^{\cdot\cdot}+\frac{x}{\sqrt{(x^2+1)^3}}y^{\cdot 2} = 0$$

$$y^{\cdot\cdot}-\frac{x}{(x^2+1)^2}x^{\cdot}y^{\cdot} = 0 \tag{5.5}$$

The solution of this system is given by equation (4.7) as follows

$$\begin{aligned} y &= \frac{\sqrt{B}}{A}\left(\int\frac{1}{\sqrt{x^2+1}\sqrt{\frac{1}{x^2+1}-\frac{A^2}{B}}}dx-\int\sqrt{x^2+1}\sqrt{\frac{1}{x^2+1}-\frac{A^2}{B}}dx\right)+D \\ &= \frac{B}{A^2}\int\frac{1}{\sqrt{(\frac{B}{A^2}-1)-x^2}}dx-\int\sqrt{\left(\frac{B}{A^2}-1\right)-x^2dx}+D \end{aligned}$$

$$(5.6) = \left(\frac{B+A^2}{2A^2}\right)\sin^{-1}\left(\frac{x}{\sqrt{\frac{B}{A^2}-1}}\right)-\frac{x}{2}\sqrt{\left(\frac{B}{A^2}-1\right)-x^2}+D$$

At $t=0$, we find that

$$D=y_0-\left(\frac{B+A^2}{2A^2}\right)\sin^{-1}\left(\frac{x_0}{\sqrt{\frac{B}{A^2}-1}}\right)+\frac{x_0}{2}\sqrt{\left(\frac{B}{A^2}-1\right)-x_0^2} \tag{5.7}$$

$$\begin{aligned} y &= y_{0+}\left(\frac{B+A^2}{2A^2}\right)\left(\sin^{-1}\left(\frac{x}{\sqrt{\frac{B}{A^2}-1}}\right)-\sin^{-1}\left(\frac{x_0}{\sqrt{\frac{B}{A^2}-1}}\right)\right)- \\ & \frac{x}{2}\sqrt{\left(\frac{B}{A^2}-1\right)-x^2}+\frac{x_0}{2}\sqrt{\left(\frac{B}{A^2}-1\right)-x_0^2} \end{aligned} \tag{5.8}$$

The geodesics emanating from the point (0,0) are given by

$$y=\left(\frac{u^2}{2v^2}+1\right)\left(\sin^{-1}\left(\frac{v^2x}{u^2}\right)-\frac{\pi}{2}\right)+\frac{x}{2}\sqrt{\frac{u^2}{v^2}-x^2} \tag{5.9}$$

For $(u,v)=(1,1)$, we get $y=\frac{3}{2}\sin^{-1}(x)+\frac{x}{2}\sqrt{1-x^2}$, which is represented in 4

The curvature is given by

$$K=-f(x)f''(x)=\frac{1-2x^2}{(1+x^2)^3} \tag{5.10}$$

In physical and chemical applications, many concepts as temperature may be encoded by color. Now, for differential geometry, curvature is one of the most important concepts, so we introduce a new process which is called "Coloring by Curvature".In this process, color encodes curvature. This may be done in many choices. The following choice is one of them, which consists of two steps.

(1) Draw for each point (x,y) in the warped plane a point $(x,y,K(x))$ in the space R^3

(2) Tinge all points $(x,y,K(x)), K(x)\succeq 0$, using different intensities of red at a rate matching its curvature to encode positive curvature. Tinge all points $(x,y,K(x)), K(x)\prec 0$, using different intensities of blue at a rate matching its curvature to encode negative curvature

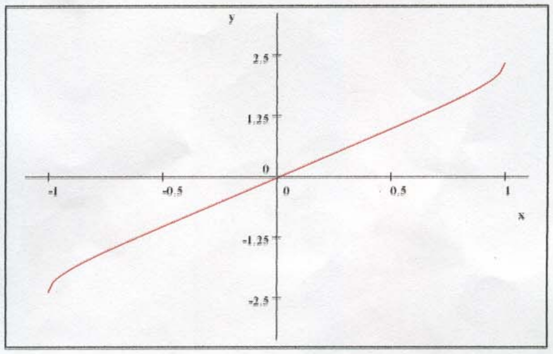
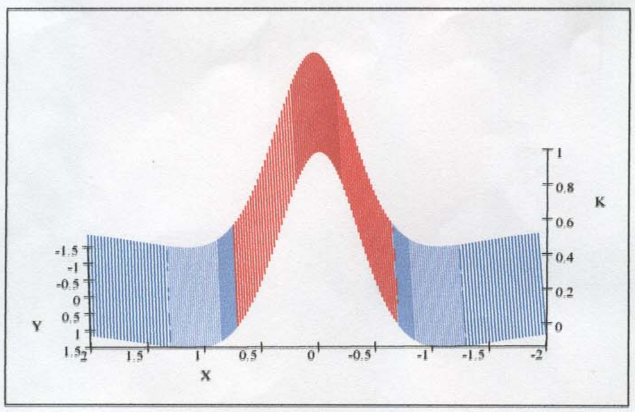


FIGURE 4. Sample geodesic



Coloring by Curvature

From equation (5.10) we have the following cases

(1) $K \succ 0$ i.e. $1 - 2x^2 \succ 0 \rightarrow \frac{-1}{\sqrt{2}} \prec x \prec \frac{1}{\sqrt{2}}$,
(2) $K = 0$ i.e. $1 - 2x^2 = 0 \rightarrow x = \pm\frac{1}{\sqrt{2}}$,
(3) $K \prec 0$ i.e. $1 - 2x^2 \prec 0 \rightarrow x \prec \frac{-1}{\sqrt{2}}, x \succ \frac{1}{\sqrt{2}}$.

The following figure shows the behavior of the curvature as a function of x. It is clear that K is bounded, and vanishes at $x = \pm\frac{1}{\sqrt{2}}$,and at $x \rightarrow \pm\infty$.as illustrated in fig (5), which is a cross section of the above figure by $zx-$plane.

Remark 6. *In this warped plane, the curvature is a function $K(x, y)$ satisfies $K(-x, y) = K(x, y)$, and hence symmetric about y-axis. Now, the idea of "isocurvature folding" [10] is introduced. Define the folding $u : \mathbb{R} \times_f \mathbb{R} \rightarrow \mathbb{R} \times_f \mathbb{R}$ by $u(x, y) = (|x|, y)$.By this folding, the plane is folded in half towards y-axis, and the image is a manifold of dimension two with boundary. The point and its image are of the same curvature, so the folding is isocurvature. The following folding is isocurvature. Let $V = u(\mathbb{R} \times_f \mathbb{R})$, define the folding $v : V \rightarrow V$ by $v(|x|, y) = (|x|, |y|)$. This second folding is at right angle to the first, and its image is again a manifold of dimension two with boundary.*

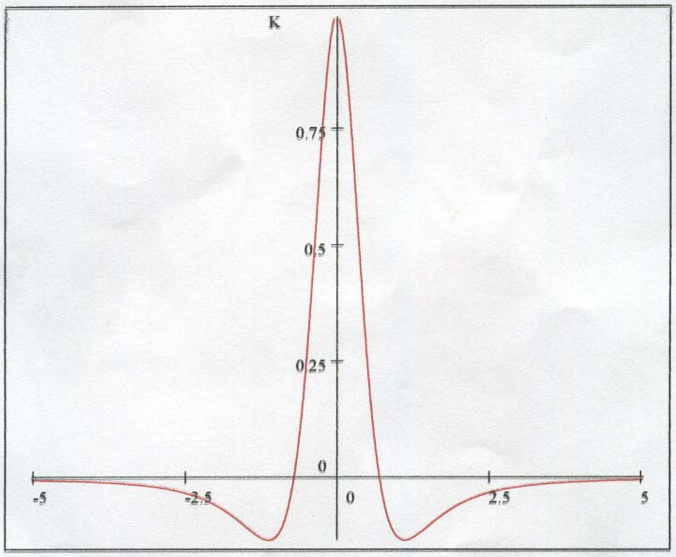


FIGURE 5. Curvature along $x-$axis

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