# The Right Truncated Topp -Leone Compound Pareto Type II-gamma $(\theta, \alpha, \beta, c)$ Distribution


**BY**

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## Abstract


This paper is concerned with a right truncated Topp-Leone compound Pareto Type II-gamma distribution RTTL-CPIIG (θ, α, β, c) with its mathematical forms of reliability function (rf), hazard rate function h(x), reversed hazard function (rhf), cumulative hazard function $H_{TL}(x)$, the approximate mean and variance, Mode, quantile and median are in this paper. Reliability and hazard rate function are obtained. The maximum likelihood method is used to estimate the parameters numerically. Simulation is used to represent the performance of the suggested distribution.



**Keywords**: *Topp-Leone compound Pareto Type II-gamma distribution; reliability function; hazard rate function; reversed hazard function; cumulative hazard function; the approximate mean; variance; mode; quartile; median; maximum likelihood estimation; Monte Carlo simulations.*


## 1. Introduction

Truncation in probability distributions may occur in many studies such as lifetime and reliability. Truncation arises because in many situations, failure of a unit is observed only if it fails before and / or after a certain period. In this paper, the right truncated Topp-Leone compound Pareto Type II-gamma distribution RTTL-CPIIG (θ, α, β, c) is described and some of its properties are investigated. A truncated distribution is formed from another distribution by cutting off the right side (i.e., right truncated) or from the left side (i.e., left truncated) or from both sides of the distribution (i.e., double truncated).

Many authors studied the truncated distribution and its applications. El-Din *et al.* (2010) derived the probability density function of mid truncated distribution and used Lindley distribution as illustrative example. Hattaway (2010) studied the parameter estimation and hypothesis testing for the truncated normal distribution with applications to introductory statistics grades. Abd El-Raheem *et al.* (2013) derive some statistical properties of the mid-truncated distributions such as moments, moment generating function, characteristic function, generating data, the maximum likelihood function of mid-truncated Weibull distribution. Zaninetti (2013) presented a right and left truncated gamma distribution with application to the stars that introduces an upper and a lower boundary to this distribution. Anithakumari *et al.* (2015) developed and analyzed a left truncated generalized Gaussian distribution. Abid (2016) considered a doubly truncated Fréchet random variable restricted by both a lower (c) and upper (d) truncation points provided with some statistical properties and different methods to estimate distribution parameters were evaluated. Abd El-Monem *et al.* (2020) provided a right truncated Fréchet Weibull distribution with its mathematical forms of density, cumulative, reliability and reverse hazard functions.

The cumulative distribution function (cdf) and the probability density function (pdf) for the TL-CPIIG $(\theta, \alpha, \beta, c)$ random variable X > 0 are given by:

$$F_{TL-CP}(x) = \left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}, \quad x \geq 0, \quad (\theta, \alpha, \beta, c \geq 0) \tag{1}$$

$$f_{TL-CP}(x) = 2\theta\alpha\beta^{2\alpha}(x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}\left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}, \quad x \geq 0, \quad (\theta, \alpha, \beta, c \geq 0), \tag{2}$$

For real value of θ, using following series representation of Prudnikov et al (1986):

$$(1+x)^{\theta} = \sum_{j=0}^{\infty} \frac{(1)^j \Gamma(\theta+1)}{j!\,\Gamma(\theta+1-j)} x^j = \sum_{j=0}^{\infty} (1)^j \binom{\theta}{j} x^j, \qquad \text{[see Abbas et al (2017)]}.$$

The cdf of TL-CPIIG $(\theta, \alpha, \beta, c)$ distribution given in (1) is expressed as infinite sum given as follows:

$$F_{TL-CP}(x) = \left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}$$

$$= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\theta+1)}{j!\,\Gamma(\theta+1-j)}\, \beta^{2\alpha j}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha j} \tag{3}$$

A density function of (2) TL-CPIIG $(\theta, \alpha, \beta, c)$ distribution can be written as follows:

$$f_{TL-CP}(x) = \sum_{j=0}^{\infty} \frac{(-1)^j 2\alpha\theta\Gamma(\theta)}{j!\,\Gamma(\theta-j)}\, \beta^{2\alpha(j+1)}(x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha(j+1)-1} \tag{4}$$

This paper is organized as follows: In Section 2 right truncated Topp-Leone compound Pareto Type II-gamma distribution, reliability function (rf), hazard rate function $h(x)$, reversed hazard function (rhf) and cumulative hazard function $H_{TL}(x)$ are obtained. The approximate mean and variance, also some properties of the distribution are contained in Section 3. Estimation of right truncated Topp-Leone compound Pareto Type II-gamma (θ, α, β, c) distribution's parameters by maximum likelihood method is obtained in Section 4. Numerical study is introduced in Section 5. Finally, concluding remarks are presented to illustrate the theoretical results derived for ML estimation in Section 6.

## 2. Right truncated Topp-Leone compound Pareto Type II-gamma (θ, α, β, c) distribution.

A random variable X is said to have RT TL-CPIIG with parameters (θ, α, β, c) where its cumulative distribution function (CDF) for $0 < x < b$ is defined as:

- $$F_{Rt}(x) = \frac{F(x)}{F(b)} = \frac{\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}}{(A(b))^{\theta}} \quad ; \qquad 0 < x < b. \tag{5}$$

where $A(b) = \left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{b+c}\right)\right)^{-2\alpha}\right)$ and its probability density function (PDF) is can be written as:

- $$f_{Rt}(x) = \frac{f(x)}{F(b)} = \frac{2\theta\alpha\beta^{2\alpha}(x+c)^{-1}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}}{(A(b))^{\theta}} \quad ; \quad 0 < x < b. \tag{6}$$

where $(\theta, \alpha)$ are shape parameters and $(\beta, c)$ are a scale parameters.

The RT TL-CPIIG density function for $(\theta, \alpha, \beta, c)$ is provided in figure 1.

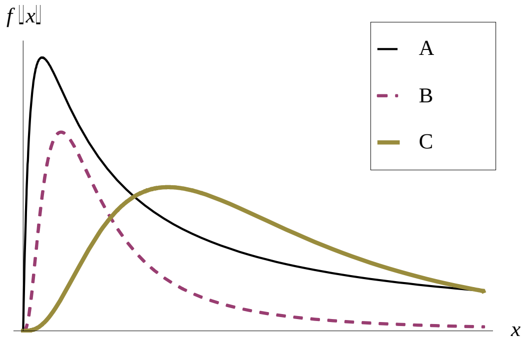


**Figure 1: Probability density function for the RT TL-CPIIG distribution for different parameter values**

**(A)** $(\theta = 2.06,\ \alpha = 0.02,\ \beta = 2.12,\ c = 0.43,\ b = 0.45)$

**(B)** $(\theta = 4.12,\ \alpha = 3,\ \beta = 1.85,\ c = 1.81,\ b = 0.83)$

**(C)** $(\theta = 4.18,\ \alpha = 7,\ \beta = 3.29,\ c = 9,\ b = 1.97)$

From Figure 1, it is noticed that the pdf curve is right skewed in (A, B). While, the pdf curve is oblate flat kurtosis in (C).

The reliability function (rf) of RT TL-CPIIG $(\theta, \alpha, \beta, c)$ is given by:

- $$R_{Rt}(x) = 1 - F_{Rt}(x) = \frac{(A(b))^{\theta} - \left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}}{(A(b))^{\theta}}; \quad 0 < x < b. \qquad (7)$$

The hazard rate function (hrf) of RT TL-CPIIG $(\theta, \alpha, \beta, c)$ is given by

- $$h_{Rt}(x) = \frac{f_{Rt}(x)}{1 - F_{Rt}(x)} = \frac{2\theta\alpha\beta^{2\alpha}(x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}\left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}}{(A(b))^{\theta} - \left(1 - \beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}}; 0 < x < b. \qquad (8)$$

The (hrf) of RT TL-CPIIG for different parameter values are presented in figure 2.

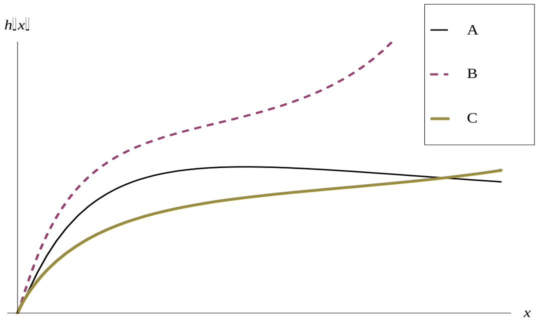


**Figure 2: The hazard of RT TL-CPIIG distribution for different parameter values**

**(A)** $(\theta = 2.19,\ \alpha = 5.62,\ \beta = 1.6,\ c = 4,\ b = 6)$

**(B)** $(\theta = 2.14,\ \alpha = 4.43,\ \beta = 1.99,\ c = 2.38,\ b = 2.26)$

**(C)** $(\theta = 1.9,\ \alpha = 6.15,\ \beta = 1.56,\ c = 7.05,\ b = 3.71)$

From Figure 2, it is noticed (hrf) of the RT TL-CPIIG distribution has monotonic increasing.

The reverse hazard rate function (rhrf) of RT TL-CPIIG $(\theta, \alpha, \beta, c)$ is given by:

- $$rh_{Rt}(x) = \frac{f_{Rt}(x)}{F_{Rt}(x)} = \frac{2\theta\alpha\beta^{2\alpha}(x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}}{\left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)} \ ; \qquad 0 < x < b. \qquad (9)$$

The cumulative hazard function of RT TL-CPIIG $(\theta, \alpha, \beta, c)$ is:

- $$H_{RT}(x) = -\ln\left(1 - F_{Rt}(x)\right) = -\ln\left(\frac{(A(b))^{\theta} - \left(1-\beta^{2\alpha}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}}{(A(b))^{\theta}}\right); \ 0 < x < b. \qquad (10)$$

## 3. Some Properties of The Distribution

In this section we discuss important and useful statistical characteristics of the proposed distribution.

- **The approximate Mean and Variance:**

The r th moment about zero of the random variable x; mode and quartile can be obtained as follows:

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$$

$$= 2\theta\alpha\beta^{2\alpha}\left(A(b)\right)^{\theta} \int_0^{\infty} x^r (x+c)^{-1}\left(\beta - \ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha+1)}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1} dx \quad (11)$$

It is difficult to find it in a closed form but the approximate mean and variance of the RT TL-CPIIG ($\theta$, $\alpha$, $\beta$, c) distribution can be obtained as follows:

If y is a random variable distributed as an exponential distribution with parameter $\theta$ [$Y \sim \mathrm{Exp}(\theta)$] with $\mu = E(y) = \left(\frac{1}{\theta}\right)$ and $\sigma^2 = \mathrm{Var}(y) = \frac{1}{\theta^2}$, then the variable $T = g(y) = c\left(e^{-\beta\left(1-(1-A(b)e^{-y})^{\frac{-1}{2\alpha}}\right)} - 1\right) \sim \mathrm{RTTL-CPIIG}\,(\theta,\alpha,\beta,c)$ . This relation will be used to find the approximate mean and variance of $\mathrm{RTTL-CPIIG}\,(\theta,\alpha,\beta,c)$. The approximate mean and variance of g(y), based on the method of statistical differentials [See El-Sayad (1993)], are given by:

$$E(g(Y)) \cong g(\mu) + \frac{1}{2}\sigma^2 g''(\mu) \quad (11a)$$

and

$$\mathrm{Var}(g(Y)) \cong \sigma^2\left(g'(\mu)\right)^2 \quad (11b)$$

The function of $\mu$ is given by

$$g(\mu) = c\left(e^{-\beta\left(1-(1-A(b)e^{-\mu})^{\frac{-1}{2\alpha}}\right)} - 1\right) \quad (11c)$$

The first and second derivatives are given by:

$$g'(\mu) = -\frac{c\beta}{2\alpha} A(b)(1-A(b)e^{-\mu})^{-\left(\frac{1}{2\alpha}+1\right)} e^{-\left(\mu+\beta\left(1-(1-A(b)e^{-\mu})^{\frac{-1}{2\alpha}}\right)\right)}, \quad (11d)$$

$$g''(\mu) = \frac{c\beta}{2\alpha} A(b)(1-A(b)e^{-\mu})^{-\left(\frac{1}{2\alpha}+2\right)} e^{-\left(2\mu+\beta\left(1-(1-A(b)e^{-\mu})^{\frac{-1}{2\alpha}}\right)\right)}$$
$$\times\left[e^{\mu} + \frac{A(b)}{2\alpha}\left(1-\beta(1-A(b)e^{-\mu})^{-\frac{1}{2\alpha}}\right)\right] \quad (11e)$$

By substituting (11c) and (11e) in (11a) we obtain:

$$E\big(g(Y)\big) \cong c\left(e^{-\beta\left(1-(1-A(b)e^{-\mu})^{\frac{-1}{2\alpha}}\right)}-1\right)+\frac{c\beta}{4\alpha\theta^2}A(b)(1-A(b)e^{-\mu})^{-\left(\frac{1}{2\alpha}+2\right)}e^{-\left(2\mu+\beta\left(1-(1-A(b)e^{-\mu})^{\frac{-1}{2\alpha}}\right)\right)}$$
$$\times\left[e^{\mu}+\frac{A(b)}{2\alpha}\left(1-\beta(1-A(b)e^{-\mu})^{-\frac{1}{2\alpha}}\right)\right] \tag{12}$$

and by substituting (11d) in (11b) we obtain

$$\mathrm{Var}\big(g(Y)\big) \cong \frac{c^2\beta^2}{4\alpha^2\theta^2}\big(A(b)\big)^2\big(1-A(b)e^{-\mu}\big)^{-2\left(\frac{1}{2\alpha}+1\right)}e^{-2\left(\mu+\beta\left(1-(1-A(b)e^{-\mu})^{\frac{-1}{2\alpha}}\right)\right)} \tag{13}$$

- **The mode:**

The mode of the RT TL-CPIIG (θ, α, β, c) distribution (6) can be obtained by differentiating the pdf with respect to x and equating the resulting equation to zero as $\grave{f}_{Rt}(x)=0$ as follows:

$$\grave{f}_{Rt}(x)=2\theta\alpha\beta^{2\alpha}\big(A(b)\big)^{\theta}\Bigg[2\alpha\beta^{2\alpha}(\theta-1)(x+c)^{-2}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-(4\alpha-2)}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-2}$$
$$-(2\alpha+1)(x+c)^{-2}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha-2)}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}$$
$$-\ (x+c)^{-2}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha-1)}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}\Bigg]=0$$

$$=2\theta\alpha\beta^{2\alpha}(x+c)^{-2}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha-1)}\big(A(b)\big)^{\theta}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta-1}$$
$$\times\Bigg[2\alpha\beta^{2\alpha}(\theta-1)\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha-1)}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{-1}$$
$$-(2\alpha+1)\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)-1\Bigg]=0$$

$$=2\alpha\beta^{2\alpha}(\theta-1)\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-(2\alpha-1)}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{-1}$$
$$-(2\alpha+1)\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)-1=0 \tag{14}$$

It is difficult to find it in a closed form but we can study numerical.

- **Quantile:**

Putting $F_{Rt}(x) = q$ to obtain the quantile as follows:

$$F_{Rt}(x) = \int_0^x f_{Rt}(x)\, dx \; = q$$

$$\frac{\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x+c}\right)\right)^{-2\alpha}\right)^{\theta}}{\left(A(b)\right)^{\theta}} = q$$

which yields:

$$x_q = c\left(e^{-\beta\left(1-\left(1-q^{\frac{1}{\theta}}A(b)\right)^{-\frac{1}{2\alpha}}\right)} - 1\right), \tag{15}$$

The median is obtained at $q = \frac{1}{2}$ as follows:

$$x_{\frac{1}{2}} = c\left(e^{-\beta\left(1-\left(1-2^{-\frac{1}{\theta}}A(b)\right)^{-\frac{1}{2\alpha}}\right)} - 1\right), \tag{16}$$

## 4. Parameter Estimation

For estimating the parameters of RT TL-CPIIG (θ, α, β, c) maximum likelihood method is used. Let $X = (x_1, x, \dots, x_n)$ be i.i.d random sample having probability density function (2), and then the likelihood function is given by:

$$L(x) = 2^n\theta^n\alpha^n\beta^{2n\alpha}\left(A(b)\right)^{-n\theta}\prod_{i=1}^{n}\left(\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right)^{\theta-1}\right)$$
$$\prod_{i=1}^{n}\left((x_i+c)^{-1}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-(2\alpha+1)}\right) \tag{17}$$

The natural logarithm of the likelihood function is:

$$\ell = \ln L(x) = n\left(\ln 2 + \ln\theta + \ln\alpha + 2\alpha\ln\beta - \theta\ln\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{-2\alpha}\right)\right)$$
$$+(\theta-1)\sum_{i=1}^{n}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right)$$
$$-\sum_{i=1}^{n}(x_i+c) - (2\alpha+1)\sum_{i=1}^{n}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right) \tag{18}$$

Differentiating partially with respect to the parameters (θ, α, β, c) and equating the resulting derivatives to zero, we get the following maximum likelihood equations:

$$\frac{\partial \ell}{\partial \theta}=\frac{n}{\theta}-n\ln\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{-2\alpha}\right)+\sum_{i=1}^{n}\left(1-\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right)=0, \tag{19}$$

$$\frac{\partial \ell}{\partial \alpha}=\frac{n}{\alpha}+2n\ln\beta-2\ln\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)+\frac{2(\theta-1)\ln\left(\beta^{2\alpha+1}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha-1}\right)}{\left(\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right)-1}$$

$$-\frac{2n\theta\ln\left(\beta^{2\alpha+1}\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{-2\alpha-1}\right)}{\left(\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{-2\alpha}\right)-1}=0, \tag{20}$$

$$\frac{\partial \ell}{\partial \beta}=\frac{2n\alpha}{\beta}-\frac{2\alpha+1}{\beta-\ln\left(\frac{c}{x_i+c}\right)}$$

$$-\frac{2\alpha(\theta-1)\left(\beta^{2\alpha-1}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{1-2\alpha}\right)\left(\beta\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{2}-\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-1}\right)}{\left(\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right)-1}$$

$$+\frac{2n\alpha\theta\left(\beta^{2\alpha-1}\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{1-2\alpha}\right)\left(\beta\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{2}-\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{-1}\right)}{\left(\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{-2\alpha}\right)-1}=0, \tag{21}$$

$$\frac{\partial \ell}{\partial c}=-(\theta-1)\left[\frac{2\alpha\beta\left(\beta^{2\alpha-1}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{1-2\alpha}\right)(x_i+c)(c(x_i+c)^2-(x_i+c)^{-1})}{c\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{2}\left(\left(\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right)-1\right)}\right.$$

$$\left.-\frac{2n\alpha\beta\theta\left(\beta^{2\alpha-1}\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{1-2\alpha}\right)(b+c)(c(b+c)^2-(x_i+c)^{-1})}{c\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{2}\left(\left(\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{-2\alpha}\right)-1\right)}\right]$$

$$-(x_i+c)^{-1}-\frac{(2\alpha+1)(x_i+c)[c(x_i+c)^2-(x_i+c)^{-1}]}{c\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)}=0, \tag{22}$$

$$\frac{\partial \ell}{\partial b}=\frac{2n\alpha\beta\theta\left(\beta^{2\alpha-1}\left(\beta-\ln\left(\frac{c}{b+c}\right)\right)^{1-2\alpha}\right)}{(b+c)\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{2}\left(\left(\beta^{2\alpha}\left(\beta-\ln\left(\frac{c}{x_i+c}\right)\right)^{-2\alpha}\right)-1\right)}=0. \tag{23}$$

In Section (4), we can obtain the estimates of unknown parameters by setting the last equations equal zero, but solving these equations simultaneously to get the unknown parameters ($\hat{\theta}$, $\hat{\alpha}$, $\hat{\beta}$,

$\hat{c}$) in explicit form is mathematically complicated, so these estimates will be obtained numerically.

According to the invariance property of ML estimation, the MLE of any function $\xi(\vartheta)$ of $\theta$ is the function $\xi(\hat{\vartheta})$ of the MLE $\underline{\hat{\vartheta}}$ of $\underline{\vartheta}$. Since the equations (19-23) haven't closed form solution, they solved numerical simultaneously. So, the MLE's of the $r(x), h(x)$, (rhf) and $H_{TL}(x)$ are obtained by replacing the parameters $\theta$, $\alpha$, $\beta$, c and b (7), (8), (9) and (10) by the corresponding MLE's. Hence, for given value of x, the MLE of $r(x)$ is given by:

$$\hat{R}_{Rt}(x) = \frac{\left(A(\hat{b})\right)^{\hat{\theta}} - \left(1 - \hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x+\hat{c}}\right)\right)^{-2\hat{\alpha}}\right)^{\hat{\theta}}}{\left(A(\hat{b})\right)^{\hat{\theta}}}; \quad 0 < x < \hat{b}, \tag{24}$$

where $A(\hat{b}) = \left(1 - \hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{\hat{b}+\hat{c}}\right)\right)^{-2\hat{\alpha}}\right)$ for given value of x, the MLE of $h(x)$ is given by:

$$\hat{h}_{Rt}(x) = \frac{2\hat{\theta}\hat{\alpha}\hat{\beta}^{2\hat{\alpha}}(x+\hat{c})^{-1}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x+\hat{c}}\right)\right)^{-(2\hat{\alpha}+1)}\left(1 - \hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x+\hat{c}}\right)\right)^{-2\hat{\alpha}}\right)^{\hat{\theta}-1}}{\left(A(\hat{b})\right)^{\hat{\theta}} - \left(1 - \hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x+\hat{c}}\right)\right)^{-2\hat{\alpha}}\right)^{\hat{\theta}}}; 0 < x < \hat{b}. \tag{25}$$

for given value of x, the MLE of $rh(x)$ is given by:

$$\widehat{rh}_{Rt}(x) = \frac{2\hat{\theta}\hat{\alpha}\hat{\beta}^{2\hat{\alpha}}(x+\hat{c})^{-1}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x+\hat{c}}\right)\right)^{-(2\hat{\alpha}+1)}}{\left(1 - \hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x+\hat{c}}\right)\right)^{-2\hat{\alpha}}\right)}; \quad 0 < x < \hat{b}. \tag{26}$$

and for given value of x, the MLE of $H(x)$ is given by:

$$\hat{H}_{RT}(x) = -\ln\left(\frac{\left(A(\hat{b})\right)^{\hat{\theta}} - \left(1 - \hat{\beta}^{2\hat{\alpha}}\left(\hat{\beta} - \ln\left(\frac{\hat{c}}{x+\hat{c}}\right)\right)^{-2\hat{\alpha}}\right)^{\hat{\theta}}}{\left(A(\hat{b})\right)^{\hat{\theta}}}\right), \quad 0 < x < \hat{b}. \tag{27}$$

where $\hat{\theta}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{c}$ and $\hat{b}$ are the MLE of $\theta$, $\alpha$, $\beta$, c and b.

## 5. Numerical Study

In this section, a simulation is conducted to illustrate the performance of the presented ML estimates based on complete samples the estimation of the parameters. The ML of the estimates of the parameters. Moreover, confidence intervals (CIs) of the parameters. Mathcad (14) is used for illustrating the obtained results.

**The steps of the procedure are as follows:**

- For a given vector $\underline{\vartheta} = (\theta, \alpha, \beta, c, b)$, generate random samples of different sizes (n = 30, 50, 100 and 150) from a RT TL-CPIIG (θ, α, β, c, b), using the following equation:

$$X = c\left(e^{-\beta\left(1-\left(1-u^{\frac{1}{\theta}}A(b)\right)^{\frac{-1}{2\alpha}}\right)} - 1\right) \sim \text{RT TL} - \text{CPIIG}\ (\theta, \alpha, \beta, c, b).$$

[see Niyogi (2003)].

- Obtain numerically the MLE of θ, α, β, c, b by solving the equations $(19), (20), (21), (22)$ and $(23)$.
- The number of repetitions is m = 1000.
- Evaluating the performance of the estimates is considered through some measurements of accuracy. To study the precision and variation of the estimates, it is convenient to use the mean square error (MSE).
- $\text{MSE} = (\text{Bias}^2) + \text{variance}$.
- The estimates, biase ($\text{Bias}^2$), variance (Var), mean square error (MSE), interval estimation, upper, lower and length of the ML estimates where x = 0.9 are calculated. The computational results are displayed in following table, the initial values are ($\theta = 0.06,\ \alpha = 0.86, \beta = 0.03, c = 0.3, b = 0.01$).

The following table shows the ML of the estimates (ML), biases ($\text{Bias}^2$), variance (Var), mean squared errors (MSEs), interval estimation, upper, lower and length of the ML estimates of the parameters for each sample size. It is easy to notice that estimates are close to their actual values with small enough MSE.

**Table : ML, Var, $Bias^2$ , MSE and 95% confidence intervals of the MLE's of the RT TL-CPIIG (θ, α, β, c) for Different Sample Sizes n, Repetitions m =1000 and Initial values ($\theta = 0.06,\ \alpha = 0.86, \beta = 0.03, c = 0.3, b = 0.01$)**

| N | Parameters | ML | Var | $Bias^2$ | MSE | UL | LL | length |
|---|---|---|---|---|---|---|---|---|
| 30 | θ | 2.438 | 0.576 | 0.192 | 0.767 | 3.942 | 0.953 | 2.989 |
| | α | 6.686 | 0.607 | 0.099 | 0.705 | 8.213 | 5.16 | 3.053 |
| | β | 0.124 | 0.001 | 0.001 | 0.002 | 0.188 | 0.061 | 0.127 |
| | c | 1.789 | 0.164 | 0.506 | 0.669 | 2.582 | 0.997 | 1.585 |
| | b | 0.331 | 0.002 | 0.029 | 0.031 | 0.419 | 0.244 | 0.175 |
| 50 | θ | 1.951 | 0.124 | 0.002 | 0.126 | 2.64 | 1.262 | 1.378 |
| | α | 6.74 | 0.722 | 0.067 | 0.789 | 8.405 | 5.075 | 3.33 |
| | β | 0.14 | 0.001 | 0.002 | 0.003 | 0.208 | 0.072 | 0.136 |
| | c | 1.842 | 0.092 | 0.433 | 0.524 | 2.439 | 1.249 | 1.19 |
| | b | 0.391 | 0.009 | 0.012 | 0.021 | 0.571 | 0.21 | 0.361 |
| 100 | θ | 1.944 | 0.143 | 0.003 | 0.146 | 2.685 | 1.203 | 1.482 |
| | α | 6.466 | 0.707 | 0.285 | 0.991 | 8.114 | 4.819 | 3.295 |
| | β | 0.15 | 0.001 | 0.003 | 0.004 | 0.215 | 0.085 | 0.13 |
| | c | 1.638 | 0.035 | 0.743 | 0.778 | 2.004 | 1.272 | 0.732 |
| | b | 0.395 | 0.006 | 0.011 | 0.017 | 0.55 | 0.24 | 0.31 |
| 150 | θ | 2.139 | 0.109 | 0.019 | 0.128 | 2.785 | 1.493 | 1.292 |
| | α | 6.359 | 0.27 | 0.411 | 0.681 | 7.378 | 5.34 | 2.038 |
| | β | 0.144 | 0.001 | 0.002 | 0.003 | 0.192 | 0.096 | 0.096 |
| | c | 1.563 | 0.016 | 0.878 | 0.894 | 1.815 | 1.312 | 0.503 |
| | b | 0.328 | 0.006 | 0.029 | 0.035 | 0.375 | 0.282 | 0.093 |

## 5. Concluding Remarks

From this table it is noticed that as the sample size is increased the MSE is decreased.

**References**


1. **Abbas S. , Taqi S. A., Mustafa F., Murtaza M., and Shahbaz M. Q. (2017).** "Topp-Leone Inverse Weibull Distribution: Theory and Application" ***European Journal of Pure and Applied Mathematics,*** Vol. **10,** No. **5,** pp. 1005-1022.
2. **Abd El-Raheem A.M., Mohie El-Din M.M. and Amein M.M. (2013)**. **"**On Mid - Truncated Distributions and its Applications**"** ***Journal of Advanced Research in Applied Mathematics,*** Vol**.** **5**, No. **2,** pp. 20-38.
3. **Abid S.H. (2016)**. **"**Properties of Doubly Truncated Fréchet Distribution**"**. ***American Journal of Applied Mathematics and Statistics***, Vol**. 4**, No. **1,** pp. 9-15.
4. **Anithakumari K., Srinivas Rao K. and PRS Reddy (2015)**. **"**On A Left Truncated Generalized Gaussian Distribution**"** ***Journal of Advance Computing,*** Vol**. 4**, No. **1,** pp. 1-21.
5. **El-Din M., Teamah A. A. M., Abd El-Bar A.M.T., and Salem A. (2010).** **"**Random Sum of Mid Truncated Lindley Distribution**".** ***J.Adv. Res. Stat. Probab*, **Vol. **2**, No. **1,** pp. 27-36.
6. **El-Sayad, G. M. (1993***).*"Statistical Inference",* Dar El-Mariekh, Al-Riyad, Kingdom of Saudia Arabia.
7. **Hattaway J.T. (2010).** "Parameter Estimation and Hypothesis Testing for The Truncated Normal Distribution with Applications to Introductory Statistics Grades*",* ***Brigham Young University-Provo***.
8. **Niyogi. P. (2003).** *"Numerical Analysis and Algorithms"*, Tata Mc. Graw - Hill publishing company limited, New York.
9. **Prudnikov A.P., Brychkov Y.A. and Marichev O.I. (1986). "**Integrals and Series**"**, ***New York: Gordon and Breach Science Publishers,*** Vol**. 3.**
10. **Teamah A. A. M., Elbanna A. A. and Gemeay A.M. (2020). "**Right Truncated Fréchet-Weibull Distribution: Statistical Properties and Application", ***Delta Journal of Science*** Vol. **41,** pp.20-29.
11. **Topp, C., W. and Leone, F., C. (1955). "**A family of J-Shaped Frequency Functions**"**, ***Journal of the American Statistical Association***, Vol. **50**, No. **269,** pp. 209-219.
12. **Youssef, E. A. (2006).** "*A Study On Compound Pareto Distribution"*, M. Sc. Thesis, Department of Statistics, Faculty of Commerce, Al- Azhar University.
13. **Zaninetti L. (2013).** "A Right and Left Truncated Gamma Distribution with Application to the Stars". ***Advanced Studies in Theoretical Physics,*** Vol. **23,** pp. 1139-1147.