ENDPOINT-INFLATED DOUBLE TRUNCATED POISSON MODEL

H. A. Abd EL-Hady, A. T. El-Ghanam and A. E. Abu-Hussien
Statistics Department, Faculty of Commerce, AL-Azhar University (Girls' Branch), Cairo, Egypt

Abstract

This article is concerned with Endpoint-Inflated Double Truncated Poisson distribution which is developed for modeling count data with excessive zeros (left-endpoint) and excessive right-endpoint(m) compared with other observations of the data. This method for modeling such data is based on an assumption that the random variable is generated from a mixture distribution of three components. The probability when the value for the response variable is zero, the probability when the value for the response variable is m, and the other counts are defined by Double Truncated Poisson distribution. Some of its main properties are discussed. The maximum likelihood and moment methods of estimations are utilized to derive point estimators and confidence intervals for the parameters. Regression model based on the distribution is proposed and the corresponding computational procedures are introduced. A simulation study is conducted to evaluate the performance of the proposed methods. A real data set is analyzed to demonstrate how the methods can be applied in practice.

Keywords*: Count data; Truncated Poisson distribution; Zero-Inflated Poisson distribution; Endpoint-inflated Poisson distribution; Zero- one Inflated Poisson distribution; Maximum likelihood estimators; Moment estimators.*

1. Introduction

Many studies in different areas involve nonnegative integer values. The Poisson models are the most used tools for modeling count data. In practice, however, count data are often over dispersed, the variance can be greater than the average value. One frequent manifestation of over dispersion is that the incidence of zero counts is greater than expected for the Poisson distribution. Motivated by this fact, some studies have focused on inflated distributions for modeling count data with large frequencies of zeros that cannot be explained by models based on standard distributional assumptions. Such data are common in many fields including medicine, public health studies, epidemiology, ecology, sociology, psychology, econometrics, agriculture, engineering, manufacturing, and road safety. Inflated distributions can be thought as finite mixture distributions which involve a finite number of components to deal with the nature of the data. Mixture distributions arise when each distribution separately cannot describe the data. Some of these studies are interested in zero-inflated and others are interested in zero-and-one inflated families of models. zero-inflated distribution is based on an assumption that the random variable is generated by a mixture of two distributions, one is the discrete distribution and a degenerate distribution at zero [see Mullahy (1986) developed zero inflated family of models, Lampert (1992) extended zero-inflated Poisson (ZIP) distribution]. Many studies build regression models based on Zero-inflated distributions to clarify the relation between the covariates and the response variable. [see Lampert (1992) used a parametric ZIP regression model to study the effects of covariates with parameters of interest via appropriate link functions, Ridout *et al.* (2001) derived a score test for testing a ZIP regression model against zero-inflated negative binomial (ZINB) alternatives which the non-zero part of the count data is over dispersed and another distribution such as ZINB may be more appropriate than ZIP, Diop and Dupuy (2014) developed zero inflated Bernoulli (ZIBER) regression model to fit binary data that contain too many zeros. Fitriani *et al.* (2019) presented Simulation on the ZINB to model over dispersed Poisson distributed data, Diallo *et al.* (2019) presented estimation in zero-inflated binomial (ZIB) regression with missing covariates].

Zero-and-one inflated distributions have been developed to fit count data with excess zeros and ones simultaneously. There are many methods to build zero-and-one inflated distributions and one of these methods is based on an assumption that the random variable is generated by a mixture of three distributions, a degenerate distribution at zero, a degenerate distribution at one and a discrete distribution representing the other values [see Edwin (2014) considered zero-one inflated geometric (ZOIG) distribution in analysis of a real life. Alshkaki (2016) introduced zero-and-one inflated power series distributions, Poisson, binomial, negative binomial, geometric and logarithmic series distributions. Alshkaki (2016) discussed properties and parameters estimators of zero-and-one inflated Poisson (ZOIP) distribution. Alshkaki (2016) provided mathematical properties of zero-one inflated logarithmic series (ZOILS) distribution. Alshkaki (2016) provided mathematical properties of zero-one inflated negative binomial (ZOINB) and zero-one inflated binomial (ZOIB) distributions. Zhang *et al*. (2016) studied the likelihood based ZOIP model without covariates. Tang et al. (2017) studied the statistical inference for (ZOIP) distribution. Liu et al. (2018) derived the objective Bayesian estimation of ZOIP model, Alshkaki (2019) derived a combined estimation method to estimate the parameters of the (ZOINB) distribution, Tlhaloganyang *et al*. (2019) derived Structural properties of zero-one-inflated negative-binomial crack (ZOINBCR) distribution].

To investigate the relation between the covariates and the response variable, many studies built regression models based on zero-and-one inflated distributions [see Deng et al. (2015) introduced generalized endpoint-inflated binomial model, Liu et al. (2018) introduced zero-and-one inflated Poisson regression model].

In this article, endpoint-inflated model is developed to fit count data to handle variability from both excessive zeros and excessive right-endpoint m compared with other observations in the data assuming all zeros and m are from one structural source. The proposed model is an extension of zero- inflated models through addition of the right-endpoint inflation parameter. It provides alternative distributions for modeling count data that is found to be characterized by excessive zero and excessive right-endpoint counts. The model is based on an assumption that the random variable is generated from a mixture distribution of three components. The probability when the value for the response variable is zero, the probability when the value for the response variable is m, and the other counts are defined by Double Truncated Poisson distribution, so the model is said to be inflated since it allows for positive probability mass at some points, which assign higher probabilities to zero and m.

This article unfolds as follows; Section 2 presents the double truncated Poisson distribution and its mean and variance. The endpoint-inflated double truncated Poisson distribution is suggested in Section 3 and its main properties such as the mean and variance, moment generating function and the probability generating function are presented. Section 4 discusses the maximum likelihood estimators of the distribution and the elements of the Hessian matrix; the Fisher information matrix and the variance-covariance matrix of the maximum likelihood estimators are derived. The moment method is used to estimate the parameters in Section 5. Regression model based on endpoint-inflated double truncated Poisson distribution is suggested in Section 6. Section 7 discusses the maximum likelihood estimators of the model and the variance-covariance matrix of the maximum likelihood estimators is derived. In Section 8 a simulation study is conducted to evaluate the performance of the proposed methods and a real data set is analyzed to demonstrate how the methods can be applied in practice. Finally, some concluding remarks were given in section 9.

2. Double Truncated Poisson Distribution

This section is devoted to the description of the double truncated Poisson distribution. count data can be truncated where some values in a specific range cannot be observed. Count data in which zero and m counts cannot be observed are called double truncated count data. Double truncated Poisson data are a combination of the left truncated and right truncated Poisson data. Right truncation happens from loss of observations greater than some specified value. Left truncation happens from loss of observations smaller than some specified value.

Let Y be a discrete random variable has the pmf $f(y)$ given by

$$f(y) = \frac{e^{-\lambda}\lambda^{y}}{y!}, \quad y = 0,1,2,\ldots \quad , \quad \lambda > 0 \tag{1}$$

From Cohen (1954) the pmf for a discrete random variable has a double truncated Poisson distribution denoted by DTP(λ) is given by

$$P(Y = k|0 < k < m) = \frac{1}{P(0 < k < m)} \cdot f(y). \tag{2}$$

where f_1 is referred to as parent-process. the denominator gives a normalization that accounts for the truncation of f_1.

by substituting (1) in (2) then

$$P(Y = k|0 < k < m) = \frac{\lambda^{y}}{y!\left(\sum_{k=0}^{m}\frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}\right)} \quad , \quad 0 < k < m \tag{3}$$

The first two moments of the distribution are given by

$$E(Y) = \frac{\sum_{k=0}^{m}\frac{k\lambda^k}{k!}}{\sum_{k=0}^{m}\frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}}, \tag{4}$$

$$E(Y^2) = \frac{\sum_{k=0}^{m}\frac{k^2\lambda^k}{k!}}{\sum_{k=0}^{m}\frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}}, \tag{5}$$

The variance is given by

$$V(Y)=\frac{\sum_{k=0}^{m}\frac{k^2\lambda^k}{k!}}{\sum_{k=0}^{m}\frac{\lambda^k}{k!}-1-\frac{\lambda^m}{m!}}-\left(\frac{\sum_{k=0}^{m}\frac{k\lambda^k}{k!}}{\sum_{k=0}^{m}\frac{\lambda^k}{k!}-1-\frac{\lambda^m}{m!}}\right)^2. \quad (6)$$

While the regular Poisson distribution typically encounters difficulty due to the assumed equality of mean and variance, the mean and variance of the doubly truncated Poisson distribution are characteristic of under dispersion where the variance is less than the mean.

3. Endpoint-Inflated Double Truncated Poisson Distribution

This section is devoted to the description of the endpoint-inflated double truncated Poisson distribution. The proposed model is said to be inflated since it allows for positive probability mass at some points (zero and m). The distribution has been developed for count data with excessive zeros assuming all zeros are from one structural source and with excessive right endpoint m assuming all m are from one structural source. Thus, the random variable is generated by a mixture of the probability when the value for response variable are zeros and right-endpoint m, and the other counts are defined by $\text{DTP}(\lambda)$.

Such data are common in many fields including psychological, social, and public health related research. For example, many patients go to the cosmetology many times when others never visit; the number of working days in a week that individuals work may be zero due to unemployment as may have any value greater than zero. patients may be infected by the virus and have not received any doses of prescription medication for lake of detection while others have received multiple doses for early detection and the number of days people with psychiatric problems spent in hospitals exceeds months while others are not fully hospitalized.

Let Y be a discrete random variable has an endpoint-inflated double truncated Poisson distribution, denoted by $\text{EIDTP}(\varphi_0,\varphi_1,\lambda)$.Suppose that $\mathcal{g}_o(0)$ is the probability when the value for response variable is zero, $\mathcal{g}_1(m)$ is the probability when the value for response variable is m and that $\mathcal{g}_2(k), k=1,2,\ldots,m-1$ is a probability function when the response variable is another positive integer. Therefore, the probability function of the $\text{EIDTP}(\varphi_0,\varphi_1,\lambda)$ is given by:

$$P(Y=k;\varphi_0,\varphi_1,\lambda)=\begin{cases}\mathcal{g}_o(0) & for\ \ k=0,\\ \mathcal{g}_1(m) & for\ \ k=m,\\ (1-\mathcal{g}_o(0)-\mathcal{g}_1(m))\mathcal{g}_2(k), & for\ \ 0<k<m\,.\end{cases}$$

All parts of the distribution are based on probability functions for nonnegative integers [see Mullahy (1986)].In terms of the general model above, let $g_o(0) = \varphi_o, g_1(m) = \varphi_1$ and $g_2(k)$ is the pmf of $\mathrm{DTP}(\lambda)$ in (3).

The $\mathrm{EIDTP}(\varphi_0, \varphi_1, \lambda)$ can be expressed as a mixture of three components as follows:

$$f(y; \varphi_0, \varphi_1, \lambda) = \begin{cases} \varphi_o & if \;\; y = 0, \\ \varphi_1 & if \;\; y = m, \\ \varphi_2 \dfrac{\lambda^y}{y!\left(\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}\right)}, & if \;\; 0 < y < m. \end{cases} \quad (7)$$

$\varphi_o \in [0,1]$, $\varphi_1 \in [0,1]$ denote the probability values when the values for response variable are zero and right-endpoint m, respectively and $\varphi_2 = 1 - \varphi_0 - \varphi_1 \in [0,1]$,assuming that all zeros and extra right-endpoint m are from one structural source rather than two sources.

Figure 1 shows some different $\mathrm{EIDTP}(\varphi_0, \varphi_1, \lambda)$ probability mass functions along with the corresponding values of $(\varphi_0, \varphi_1, \lambda)$. It is noteworthy that the probability functions can display different shapes depending on the values of the three parameters. In particular, when $\varphi_0 = \varphi_1 = 0$, the $\mathrm{EIDTP}(\varphi_0, \varphi_1, \lambda)$ in (7) becomes the $\mathrm{DTP}(\lambda)$ in (3).

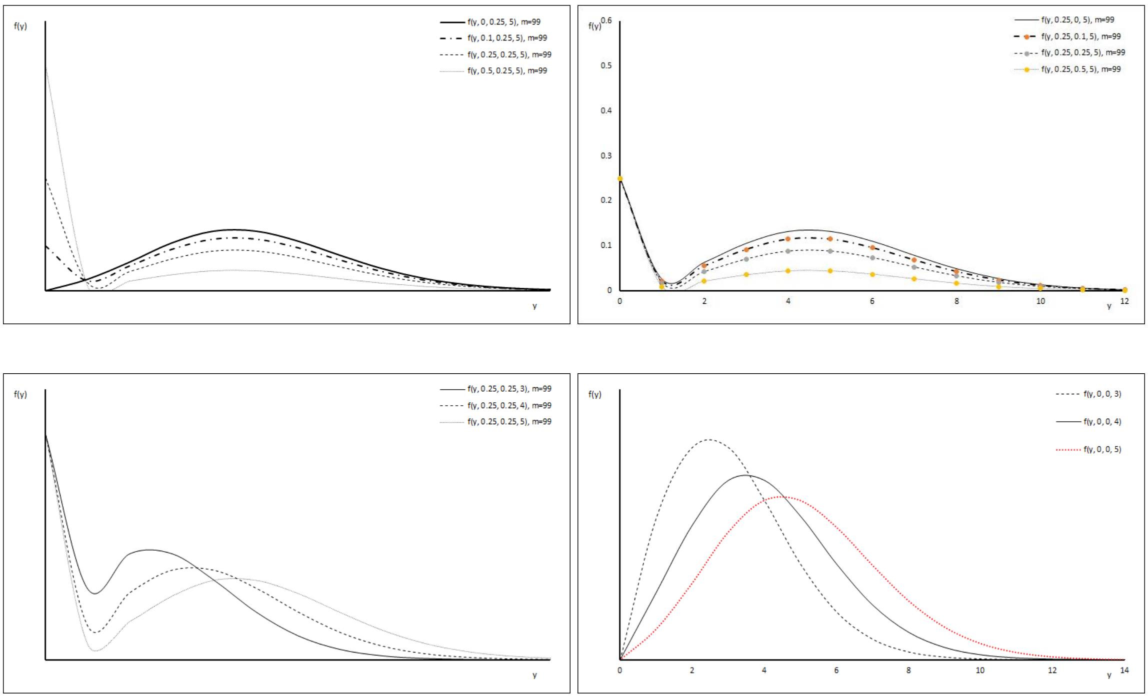

Figure 1 Endpoint-inflated Double Truncated Poisson Probability Mass Functions for different combinations of $(\varphi_0, \varphi_1, \lambda)$ and m

The *cumulative distribution function* (cdf) of the $\mathrm{EIDTP}(\varphi_0, \varphi_1, \lambda)$ is given as:

$$F(y;\varphi_0,\varphi_1,\lambda)=P(Y\le y)=\sum_{Y\le y} f(y;\varphi_0,\varphi_1,\lambda)$$

$$=[\varphi_0]I(0\le y<1)+\left[\varphi_0+\varphi_2\sum_{i=0}^{y}\frac{\lambda^i}{i!\left(\sum_{k=0}^{m}\frac{\lambda^k}{k!}-1-\frac{\lambda^m}{m!}\right)}\right]I(1\le y<m)+I(y\ge m). \quad (8)$$

Some Properties of the Endpoint-Inflated Double Truncated Poisson Distribution

- The r^th^ moment about the origin of the random variable Y can be obtained as follows:

$$E(Y^r)=\sum_{y=0}^{m} y^r f(y;\varphi_0,\varphi_1,\lambda)=\varphi_1 m^r+\varphi_2\sum_{k=1}^{m-1} k^r\frac{\lambda^k}{k!\left(\sum_{k=0}^{m}\frac{\lambda^k}{k!}-1-\frac{\lambda^m}{m!}\right)},\quad r=1,2,\dots \quad (9)$$

- The mean and variance respectively, are given by:

$$E(Y)=\varphi_1 m+\varphi_2\,\frac{\sum_{k=1}^{m-1}\frac{k\lambda^k}{k!}}{\sum_{k=0}^{m}\frac{\lambda^k}{k!}-1-\frac{\lambda^m}{m!}}, \quad (10)$$

$$V(Y)=\left[\varphi_1 m^2+\varphi_2\frac{\sum_{k=1}^{m-1}\frac{k^2\lambda^k}{k!}}{\sum_{k=0}^{m}\frac{\lambda^k}{k!}-1-\frac{\lambda^m}{m!}}\right]-\left[\varphi_1 m+\varphi_2\frac{\sum_{k=1}^{m-1}\frac{k\lambda^k}{k!}}{\sum_{k=0}^{m}\frac{\lambda^k}{k!}-1-\frac{\lambda^m}{m!}}\right]^2. \quad (11)$$

- The moment generating function and the probability generating function, respectively, are given by

$$M_y(t)=\varphi_o+\varphi_1 e^{mt}+\varphi_2\left(\frac{\sum_{k=1}^{m-1}\frac{(e^t\lambda)^k}{k!}}{\sum_{k=0}^{m}\frac{\lambda^k}{k!}-1-\frac{\lambda^m}{m!}}\right), \quad (12)$$

$$G_y(t)=\varphi_o+\varphi_1 t^m+\varphi_2\left(\frac{\sum_{k=1}^{m-1}\frac{(t\lambda)^k}{k!}}{\sum_{k=0}^{m}\frac{\lambda^k}{k!}-1-\frac{\lambda^m}{m!}}\right). \quad (13)$$

By substituting $\varphi_o=0$, $\varphi_1=0$, the main properties of the DTP(λ) can be obtained.

4. The Maximum Likelihood Estimators

The *maximum likelihood estimation* (MLE) method is used to estimate the parameters of the EIDTP($\varphi_0,\varphi_1,\lambda$).

Let $y_1,\dots,y_n$ be a random sample of size n drown from the pmf in (7). The likelihood function of the observed sample is given by:

$$L(\underline{\theta};\underline{y})=\prod_{i=1}^{n} f(y_i;\varphi_0,\varphi_1,\lambda), \quad (14)$$

where $\underline{\theta} = (\varphi_0, \varphi_1, \lambda)$.

The likelihood function of $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$ is derived by substituting (7), in (10).

$$L\left(\underline{\theta}; \underline{y}\right) = [\varphi_o]^{I_o} [\varphi_1]^{I_1} \frac{[\varphi_2]^{I_2}}{\left(\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}\right)^{I_2}} \prod_{i=I_o+1}^{n-I_1} \left(\frac{\lambda^{y_i}}{y_i\,!}\right). \tag{15}$$

where

$I_o = I_o(y) = \# \{i{:}\ y_i = 0\}$,

$I_1 = I_1(y) = \# \{i{:}\ y_i = m\}$,

and

$I_2 = n - I_o - I_1$.

Here $\#\,\mathbb{x}$ is used to denote the number of elements of the set $\mathbb{x}$. The natural logarithm of (15) can be obtained as follows:

$$\ell(\underline{\theta}) = ln\, L\left(\underline{\theta}; \underline{y}\right) = I_o\, ln(\varphi_o) + I_1 ln(\varphi_1)$$

$$+ I_2 ln(\varphi_2) + \sum_{i=I_o+1}^{n-I_1} y_i\ ln(\lambda) - \sum_{i=I_o+1}^{n-I_1} ln\, y_i! - I_2\, ln\left(\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}\right). \tag{16}$$

The elements of the score vector for φ_0, φ_1 and λ can be obtained by taking the first partial derivatives of the log likelihood function (16) with respect to the unknown parameters $\underline{\theta} = (\varphi_0, \varphi_1, \lambda)$, as follows:

$$\frac{\partial \ell(\underline{\theta})}{\partial \varphi_0} = \frac{I_o}{\varphi_o} - \frac{I_2}{\varphi_2}, \tag{17}$$

$$\frac{\partial \ell(\underline{\theta})}{\partial \varphi_1} = \frac{I_1}{\varphi_1} - \frac{I_2}{\varphi_2}, \tag{18}$$

and

$$\frac{\partial \ell(\underline{\theta})}{\partial \lambda} = \frac{\sum_{i=I_o+1}^{n-I_2} y_i}{\lambda} - \frac{\sum_{k=0}^{m} \frac{k\lambda^{k-1}}{k!} - \frac{m\lambda^{m-1}}{m!}}{\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}} I_2. \tag{19}$$

From (17), (18) and (19), the score vector for φ_0, φ_1 and λ can be written as follows:

$$U(\underline{\theta}) = \left(\frac{\partial \ell(\underline{\theta})}{\partial \varphi_0}, \frac{\partial \ell(\underline{\theta})}{\partial \varphi_1}, \frac{\partial \ell(\underline{\theta})}{\partial \lambda}\right)^T.$$

The ML estimators of φ_0, φ_1 *and* λ can be obtained as the solution of the nonlinear system.

$$\left(\frac{\partial \ell(\underline{\theta})}{\partial \varphi_0}, \frac{\partial \ell(\underline{\theta})}{\partial \varphi_1}, \frac{\partial \ell(\underline{\theta})}{\partial \lambda}\right)^T = \underline{0}.$$

φ_o and φ_1 can be estimated respectively as follows:

$$\hat{\varphi}_o = \frac{I_o}{n}, \quad (20)$$

and

$$\hat{\varphi}_1 = \frac{I_1}{n}. \quad (21)$$

The ML estimator of λ cannot be obtained in closed form, hence estimation must be accomplished numerically using methods such as Newton-Raphson.

The variance-covariance matrix

The variance-covariance matrix of the ML estimators of the EIDTP$(\varphi_0, \varphi_1, \lambda)$, is the inverse of Fisher information matrix, the elements of the Fisher information matrix can be obtained by taking the negative expectation of the Hessian matrix.

The elements of the Hessian matrix of the ML estimators of the EIDTP$(\varphi_0, \varphi_1, \lambda)$, are obtained by taking the second derivatives of the natural logarithm of the likelihood function, $\ell(\underline{\vartheta})$ in (16) with respect to the unknown parameters, $\underline{\theta} = (\varphi_0, \varphi_1, \lambda)$, as follows:

$$[J(\underline{\theta})]_{i,j} = \left[\frac{\partial^2}{\partial\theta_i \partial\theta_j} \ell(\underline{\theta})\right]\Bigg|_{\hat{\theta}_i, \hat{\theta}_j,}, i, j = 1,2,3.$$

the Hessian matrix can be written as follows:

$$J(\underline{\theta}) = \begin{pmatrix} J_{\varphi_o\varphi_o} & J_{\varphi_o\varphi_1} & 0 \\ J_{\varphi_1\varphi_o} & J_{\varphi_1\varphi_1} & 0 \\ 0 & 0 & J_{\lambda\lambda} \end{pmatrix}. \quad (22)$$

where

$$J_{\varphi_o\varphi_o} = \frac{\partial^2 \ell(\underline{\theta})}{\partial {\varphi_0}^2} = \frac{-I_o}{{\varphi_o}^2} - \frac{I_2}{{\varphi_2}^2},$$

$$J_{\varphi_1\varphi_1} = \frac{\partial^2 \ell(\underline{\theta})}{\partial {\varphi_1}^2} = \frac{-I_1}{{\varphi_1}^2} - \frac{I_2}{{\varphi_2}^2},$$

$$J_{\lambda\lambda} = \frac{\partial^2 \ell(\underline{\theta})}{\partial \lambda^2} = -\frac{\sum_{i=I_o+1}^{n-I_1} y_i}{\lambda^2} - \frac{\left(\sum_{k=0}^{m} \frac{k(k-1)\lambda^{k-2}}{k!} - \frac{m(m-1)\lambda^{m-2}}{m!}\right)}{\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}} I_2$$

$$+ \frac{\left(\sum_{k=0}^{m} \frac{k\lambda^{k-1}}{k!} - \frac{m\lambda^{m-1}}{m!}\right)^2}{\left(\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}\right)^2} I_2,$$

$$J_{\varphi_o\varphi_1} = J_{\varphi_1\varphi_0} = \frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_0 \partial\varphi_1} = \frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_1 \partial\varphi_o} = -\frac{I_2}{{\varphi_2}^2},$$

$$J_{\varphi_o\lambda} = J_{\lambda\varphi_0} = \frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_0 \partial\lambda} = \frac{\partial^2 \ell(\underline{\theta})}{\partial\lambda \partial\varphi_0} = 0,$$

and

$$J_{\varphi_1\lambda} = J_{\lambda\varphi_1} = \frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_1 \partial\lambda} = \frac{\partial^2 \ell(\underline{\theta})}{\partial\lambda \partial\varphi_1} = 0.$$

The elements of the Fisher information matrix of the ML estimators of the EIDTP$(\varphi_0, \varphi_1, \lambda)$,are obtained by taking the negative expectation of the Hessian matrix (22) as follows:

$$[\mathrm{K}(\underline{\theta})]_{i,j} = -E\left[\frac{\partial^2}{\partial\theta_i \partial\theta_j}\ell(\underline{\theta})\right]\Bigg|_{\hat{\vartheta}_i,\hat{\vartheta}_j,}, i,j = 1,2,3.$$

Note that $\mathbb{E}(I_o) = \varphi_o, \mathbb{E}(I_1) = \varphi_1$ and $\mathbb{E}(I_2) = \varphi_2 \frac{\lambda^y}{y!\left(\sum_{k=0}^{m}\frac{\lambda^k}{k!}-1-\frac{\lambda^m}{m!}\right)}$,[see Deng et.al. (2015)].

The Fisher information matrix can be written as follows:

$$\mathrm{K}(\underline{\theta}) = \begin{pmatrix} \mathrm{K}_{\varphi_o\varphi_o} & \mathrm{K}_{\varphi_o\varphi_1} & 0 \\ \mathrm{K}_{\varphi_1\varphi_o} & \mathrm{K}_{\varphi_1\varphi_1} & 0 \\ 0 & 0 & \mathrm{K}_{\lambda\lambda} \end{pmatrix}. \quad (23)$$

where

$$\mathrm{K}_{\varphi_o\varphi_o} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_o^2}\right) = \frac{1}{\varphi_o} + \frac{1}{\varphi_2},$$

$$\mathrm{K}_{\varphi_1\varphi_1} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial{\varphi_1}^2}\right) = \frac{1}{\varphi_1} + \frac{1}{\varphi_2},$$

$$\mathrm{K}_{\lambda\lambda} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\lambda^2}\right)$$

$$\mathrm{K}_{\lambda\lambda} = \frac{\sum_{i=I_o+1}^{n-I_1} y_i}{\lambda^2} + \varphi_2\left[\frac{\left(\sum_{k=0}^{m}\frac{k(k-1)\lambda^{k-2}}{k!} - \frac{m(m-1)\lambda^{m-2}}{m!}\right)}{\sum_{k=0}^{m}\frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}} - \frac{\left(\sum_{k=0}^{m}\frac{k\lambda^{k-1}}{k!} - \frac{m\lambda^{m-1}}{m!}\right)^2}{\left(\sum_{k=0}^{m}\frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}\right)^2}\right],$$

$$\mathrm{K}_{\varphi_o\varphi_1} = \mathrm{K}_{\varphi_1\varphi_0} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_0 \partial\varphi_1}\right) = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_1 \partial\varphi_o}\right) = \frac{1}{\varphi_2},$$

$$\mathrm{K}_{\varphi_o\lambda} = \mathrm{K}_{\lambda\varphi_0} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_0 \partial\lambda}\right) = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\lambda \partial\varphi_0}\right) = 0,$$

and

$$K_{\varphi_1\lambda} = K_{\lambda\varphi_1} = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\varphi_1\partial\lambda}\right) = -\mathbb{E}\left(\frac{\partial^2 \ell(\underline{\theta})}{\partial\lambda\partial\varphi_1}\right) = 0.$$

The variance-covariance matrix of the ML estimators of the EIDTP$(\varphi_0, \varphi_1, \lambda)$, is the inverse of Fisher information matrix (23), can be obtained as follows:

$$K(\underline{\theta})^{-1} = \frac{1}{|K(\underline{\theta})|} adj\ K(\underline{\theta}), \tag{24}$$

where

$|K(\underline{\theta})|$ is the determinant of $K(\underline{\theta})$, can be obtained as follows:

$$|K(\underline{\theta})| = \begin{vmatrix} K_{\varphi_o\varphi_o} & K_{\varphi_o\varphi_1} & 0 \\ K_{\varphi_1\varphi_0} & K_{\varphi_1\varphi_1} & 0 \\ 0 & 0 & K_{\lambda\lambda} \end{vmatrix} = K_{\lambda\lambda}\left[K_{\varphi_o\varphi_o}K_{\varphi_1\varphi_1} - K_{\varphi_o\varphi_1}K_{\varphi_1\varphi_0}\right]. \tag{25}$$

and

$adj\ K(\underline{\theta})$ is the adjoint of $K(\underline{\theta})$,can be obtained as follows:

cof $K(\underline{\theta})$ =

$$\begin{pmatrix} K_{\varphi_1\varphi_1} K_{\lambda\lambda} & -(K_{\varphi_1\varphi_0} K_{\lambda\lambda}) & 0 \\ -(K_{\varphi_o\varphi_1} K_{\lambda\lambda}) & K_{\varphi_o\varphi_o} K_{\lambda\lambda} & 0 \\ 0 & 0 & K_{\varphi_o\varphi_o}K_{\varphi_1\varphi_1} - K_{\varphi_o\varphi_1} K_{\varphi_1\varphi_0} \end{pmatrix}. \tag{26}$$

The transpose of (26) can be obtained as follows:

$adj\ K(\underline{\theta})$

$$= \begin{pmatrix} K_{\varphi_1\varphi_1} K_{\lambda\lambda} & -(K_{\varphi_o\varphi_1} K_{\lambda\lambda}) & 0 \\ -(K_{\varphi_1\varphi_0} K_{\lambda\lambda}) & K_{\varphi_o\varphi_o} K_{\lambda\lambda} & 0 \\ 0 & 0 & K_{\varphi_o\varphi_o}K_{\varphi_1\varphi_1} - K_{\varphi_o\varphi_1} K_{\varphi_1\varphi_0} \end{pmatrix}, \tag{27}$$

By substituting (25) and (27) in (24), then the variance-covariance can be written as follows:

$k(\theta)^{-1}$

$$= \begin{pmatrix} K^{\varphi_o\varphi_o} & K^{\varphi_o\varphi_1} & 0 \\ K^{\varphi_1\varphi_o} & K^{\varphi_1\varphi_1} & 0 \\ 0 & 0 & K^{\lambda\lambda} \end{pmatrix}, \tag{28}$$

where

$$K^{\varphi_o\varphi_o} = \frac{K_{\varphi_1\varphi_1}}{K_{\varphi_o\varphi_o}K_{\varphi_1\varphi_1} - K_{\varphi_o\varphi_1}K_{\varphi_1\varphi_0}},$$

$$K^{\varphi_1\varphi_1} = \frac{K_{\varphi_o\varphi_o}}{K_{\varphi_o\varphi_o}K_{\varphi_1\varphi_1} - K_{\varphi_o\varphi_1}K_{\varphi_1\varphi_0}},$$

$$K^{\lambda\lambda} = \frac{1}{K_{\lambda\lambda}},$$

and

$$K^{\varphi_o\varphi_1} = K^{\varphi_1\varphi_o} = \frac{1}{K_{\varphi_o\varphi_1}} = \frac{1}{K_{\varphi_1\varphi_o}}.$$

The diagonal elements, k^{u} of the variance-covariance matrix, $k(\theta)^{-1}$ in (28) are the variance of the ML estimators, $(\hat{\varphi}_0, \hat{\varphi}_1, \hat{\lambda})$ and the square roots of the diagonal elements of the variance-covariance matrix, are the standard errors of the ML estimators, $(\hat{\varphi}_0, \hat{\varphi}_1, \hat{\lambda})$.

Thus, $(1-\delta)100\%$ asymptotic confidence intervals (CIs) of $\hat{\varphi}_0, \hat{\varphi}_1$ and $\hat{\lambda}$ can be obtained as follows:

$$\hat{\theta} \pm z_{\frac{\delta}{2}}(k^{u})^{\frac{1}{2}}, \tag{29}$$

where

$\hat{\theta} = (\hat{\varphi}_0, \hat{\varphi}_1, \hat{\lambda})$, and z_{δ} represent the δ^{th} quantile of the $N(0,1)$ distribution.

5. The Moment Estimators

The moment estimation (ME) method is used to estimate the parameters of the $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$. The first three distribution moments about the origin for the $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$ can be found to be,

$$\mu_1' = \varphi_1 m + \varphi_2 \frac{\sum_{k=1}^{m-1} \frac{k\lambda^k}{k!}}{\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}}, \tag{30}$$

$$\mu_2' = \varphi_1 m^2 + \varphi_2 \frac{\sum_{k=1}^{m-1} \frac{k^2\lambda^k}{k!}}{\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}}, \tag{31}$$

and

$$\mu_3' = \varphi_1 m^3 + \varphi_2 \frac{\sum_{k=1}^{m-1} \frac{k^3\lambda^k}{k!}}{\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}}. \tag{32}$$

Let $y_1, y_2,,,,, y_n$ be a random sample from $f(y; \varphi_0, \varphi_1, \lambda)$ in (7), and let,

$$M_{\mathrm{r}}' = \frac{\sum_{i=1}^{n} y^r \varkappa_i}{\sum_{i=1}^{n} \varkappa_i}, \qquad \mathrm{r} = 1,2,3.$$

be their sample moments about the origin, then solving the following simultaneous equations:

$$M_1' = \varphi_1 m + \varphi_2 \frac{\sum_{k=1}^{m-1} \frac{k\lambda^k}{k!}}{\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}}, \tag{33}$$

$$M_2' = \varphi_1 m^2 + \varphi_2 \frac{\sum_{k=1}^{m-1} \frac{k^2\lambda^k}{k!}}{\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}}, \tag{34}$$

and

$$M_3' = \varphi_1 m^3 + \varphi_2 \frac{\sum_{k=1}^{m-1} \frac{k^3 \lambda^k}{k!}}{\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}}. \tag{35}$$

For given m and M_r', the ME estimators of φ_0, φ_1 and λ can be obtained numerically.

6. Endpoint-Inflated Double Truncated Poisson Regression Model

In this section the endpoint-inflated double truncated Poisson regression (EIDTPR) model is proposed to investigate the dependence of the response variable of count data containing both extra zeros (left-endpoints) and extra right-endpoint, on a set of explanatory variables. The regression model is based on the assumption that the response variable has the EIDTP$(\varphi_0, \varphi_1, \lambda)$. However, the modeling procedures are proposed similar to those for (GLMs), the parameters of the response distribution are related to linear predictors through the link functions, the linear predictors involve covariates and unknown regression parameters. The regression parameters are interpretable in terms of the parameters of the response distribution.

Let Y_i is the response variable of the ith individual, such that Y_i for $i = 1,2, \dots, n$ has the pmf in (7) with parameters $\varphi_0 = \varphi_{oi}, \varphi_1 = \varphi_{1i}$, and $\lambda = \lambda_i$, which satisfy the following functional relations:

$$log\left(\frac{\varphi_{oi}}{\varphi_{2i}}\right) = a_i' \, \alpha = \eta_{1i} , \tag{36}$$

$$log\left(\frac{\varphi_{1i}}{\varphi_{2i}}\right) = b_i' \, \beta = \eta_{2i} , \tag{37}$$

and

$$log(\lambda_i) = c_i' \, \gamma = \eta_{3i}. \tag{38}$$

where

$\alpha = (\alpha_1, \dots, \alpha_p)'$, $\beta = (\beta_1, \dots, \beta_r)'$ and $\gamma = (\gamma_1, \dots, \gamma_q)'$ are vectors of unknown regression parameters; $(p + r + q < n)$, $\eta_{1i} = (\eta_{11}, \dots, \eta_{1n})'$ $\eta_{2i} = (\eta_{21}, \dots, \eta_{2n})'$ and $\eta_{3i} = (\eta_{31}, \dots, \eta_{3n})'$ are predictor vectors. $a_i = (a_{1i}, \dots, a_{pi})$, $b_i = (b_{1i}, \dots, b_{ri})$, and $c_i = (c_{1i}, \dots, c_{qi})$ are observations on $p' + r' + q'$ known covariates, then

$$\varphi_{oi} = \frac{e^{\eta_{1i}}}{1 + e^{\eta_{1i}} + e^{\eta_{2i}}} , \tag{39}$$

$$\varphi_{1i} = \frac{e^{\eta_{2i}}}{1 + e^{\eta_{1i}} + e^{\eta_{2i}}} , \tag{40}$$

$$\varphi_{2i} = \frac{1}{1 + e^{\eta_{1i}} + e^{\eta_{2i}}} ,$$

and

$$\lambda_i = e^{\eta_{3i}}. \tag{41}$$

7. Model Estimation

In this section, the maximum likelihood estimation method is used to estimate the parameters of the EIDTPR model.

Let $y_1, \dots, y_n$ be a random sample of size n drown from $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$, with parameters $\varphi_0 = \varphi_{oi}, \varphi_1 = \varphi_{1i}$ and $\lambda = \lambda_i$, which satisfy the functional relations (39),(40) and (41).

The likelihood function of the observed sample is:

$$L\left(\underline{\vartheta}; \underline{y}\right) = \prod_{i=1}^{n} f(y_i;\ \varphi_{oi}, \varphi_{1i}, \lambda_i), \tag{42}$$

where

$\underline{\vartheta} = (\alpha', \beta', \gamma')'$, $\alpha = (\alpha_1, \dots, \alpha_p)'$, $\beta = (\beta_1, \dots, \beta_r)'$ and $\gamma = (\gamma_1, \dots, \gamma_q)'$ are vectors of unknown regression parameters: $(p + r + q < n)$,

The likelihood function of EIDTPR model is derived by substituting (7) in (42) as follows:

$$L\left(\underline{\vartheta}; \underline{y}\right) = \prod_{i=1}^{n} \left([\varphi_{oi}][\varphi_{1i}] \left[\varphi_2 \frac{\lambda^{y_i}}{y_i!\left(\sum_{k=0}^{m} \frac{\lambda^k}{k!} - 1 - \frac{\lambda^m}{m!}\right)} \right] \right). \tag{43}$$

By substituting (39), (40), (41) in (43), the natural logarithm of (43) can be obtained as follows

$$\ell(\underline{\vartheta}) = \sum_{\{i:y_i=0\}} [\eta_{1i} - ln(1 + e^{\eta_{1i}} + e^{\eta_{2i}})] + \sum_{\{i:y_i=m\}} [\eta_{2i} - ln(1 + e^{\eta_{1i}} + e^{\eta_{2i}})]$$
$$+ \sum_{\{i:0<y_i<m\}} \left[-ln(1 + e^{\eta_{1i}} + e^{\eta_{2i}}) + y_i\eta_{3i} - ln(y_i!) - ln\left(\sum_{k=0}^{m} \frac{e^{k\eta_{3i}}}{k!} - 1 - \frac{e^{m\eta_{3i}}}{m!} \right) \right]. \tag{44}$$

The elements of the score vector is obtained by taking the partial derivatives of the log likelihood function (44) with respect to the unknown regression parameters, α, β and γ as follows:

The partial derivative of (44) with respect to α is given by:

$$U_\alpha(\alpha)' = \frac{\partial \ell(\underline{\vartheta})}{\partial \alpha} = \sum_{\{i:y_i=0\}} a_i \left(1 - \frac{e^{\eta_{1i}}}{1 + e^{\eta_{1i}} + e^{\eta_{2i}}} \right) - \sum_{\{i:0<y_i\le m\}} a_i \left(\frac{e^{\eta_{1i}}}{1 + e^{\eta_{1i}} + e^{\eta_{2i}}} \right). \tag{45}$$

The partial derivative of (44) with respect to β is given by:

$$U_\beta(\beta)' = \frac{\partial \ell(\underline{\vartheta})}{\partial \beta} = \sum_{\{i:y_i=m\}} b_i \left(1 - \frac{e^{\eta_{2i}}}{1 + e^{\eta_{1i}} + e^{\eta_{2i}}} \right) - \sum_{\{i:0\le y_i<m\}} b_i \left(\frac{e^{\eta_{2i}}}{1 + e^{\eta_{1i}} + e^{\eta_{2i}}} \right). \tag{46}$$

The partial derivative of (44) with respect to γ is given by:

$$U_\gamma(\gamma)' = \frac{\partial \ell(\underline{\vartheta})}{\partial \gamma} = \sum_{\{i:0<y_i<m\}} c_i \left(y_i - \frac{\sum_{k=0}^{m} \frac{k e^{k\eta_{3i}}}{k!} - \frac{m e^{m\eta_{3i}}}{m!}}{\sum_{k=0}^{m} \frac{e^{k\eta_{3i}}}{k!} - 1 - \frac{e^{m\eta_{3i}}}{m!}} \right). \tag{47}$$

The score vector for α, β and γ can be form as follows:

$$U(\underline{\vartheta}) = \left(U_\alpha(\alpha)', U_\beta(\beta)^T, U_\gamma(\gamma)'\right)'. \quad (48)$$

The maximum likelihood estimators of α, β and γ are obtained as the solution of the nonlinear system $U(\theta) = \underline{0}$. Such estimators do not have closed form and must be computed numerically.

The variance-covariance matrix

The observed information matrix contains the negative Hessian matrix. The elements of the observed information matrix are obtained by taking the negative second derivatives of the log likelihood function with respect to the unknown parameters as follows:

$$[J(\underline{\vartheta})]_{i,j} = -\left[\frac{\partial^2}{\partial\vartheta_i \partial\vartheta_j}\ell(\underline{\vartheta})\right]\Bigg|_{\hat{\theta}_i, \hat{\theta}_j}, \qquad i,j = 1,2,3.$$

The observed information matrix can be written as follows:

$$J(\underline{\vartheta}) = \begin{pmatrix} J_{\alpha\alpha} & J_{\alpha\beta} & J_{\alpha\gamma} \\ J_{\beta\alpha} & J_{\beta\beta} & J_{\beta\gamma} \\ J_{\gamma\alpha} & J_{\gamma\beta} & J_{\gamma\gamma} \end{pmatrix}, \quad (49)$$

where

$$J_{\alpha\alpha} = \frac{\partial^2 \ell(\underline{\vartheta})}{\partial\alpha^2} = -\sum_{\{i:0\le y_i\le m\}} a_i{}^2 \frac{e^{\eta_{1i}} + e^{\eta_{2i}+\eta_{1i}}}{(1 + e^{\eta_{1i}} + e^{\eta_{2i}})^2},$$

$$J_{\beta\beta} = \frac{\partial^2 \ell(\underline{\vartheta})}{\partial\beta^2} = -\sum_{\{i:0\le y_i\le m\}} b_i{}^2 \frac{e^{\eta_{2i}} + e^{\eta_{1i}++\eta_{2i}}}{(1 + e^{\eta_{1i}} + e^{\eta_{2i}})^2},$$

$$J_{\gamma\gamma} = \frac{\partial^2 \ell(\underline{\vartheta})}{\partial\gamma^2} = -\sum_{\{i:0< y_i< m\}} c_i{}^2 \frac{\left(\sum_{k=0}^{m} \frac{e^{k\eta_{3i}}}{k!} - 1 - \frac{e^{m\eta_{3i}}}{m!}\right)\left(\sum_{k=0}^{m} \frac{k^2 e^{k\eta_{3i}}}{k!} - \frac{m^2 e^{m\eta_{3i}}}{m!}\right)}{\left(\sum_{k=0}^{m} \frac{e^{k\eta_{3i}}}{k!} - 1 - \frac{e^{m\eta_{3i}}}{m!}\right)^2}$$

$$+ \sum_{\{i:0< y_i< m\}} c_i{}^2 \frac{\left(\sum_{k=0}^{m} \frac{k e^{k\eta_{3i}}}{k!} - \frac{m e^{m\eta_{3i}}}{m!}\right)^2}{\left(\sum_{k=0}^{m} \frac{e^{k\eta_{3i}}}{k!} - 1 - \frac{e^{m\eta_{3i}}}{m!}\right)^2},$$

$$J_{\alpha\beta} = J_{\beta\alpha} = \frac{\partial^2 \ell(\underline{\vartheta})}{\partial\alpha\partial\beta} = \frac{\partial^2 \ell(\underline{\vartheta})}{\partial\beta\partial\alpha} = \sum_{\{i:0\le y_i\le m\}} a_i b_i \frac{e^{(\eta_{1i}+\eta_{2i})}}{(1 + e^{\eta_{1i}} + e^{\eta_{2i}})^2},$$

$$J_{\alpha\gamma} = J_{\gamma\alpha} = \frac{\partial^2 \ell(\underline{\vartheta})}{\partial\alpha\partial\gamma} = \frac{\partial^2 \ell(\underline{\vartheta})}{\partial\gamma\partial\alpha} = 0,$$

and

$$J_{\beta\gamma} = J_{\gamma\beta} = \frac{\partial^2 \ell(\underline{\vartheta})}{\partial\beta\partial\gamma} = \frac{\partial^2 \ell(\underline{\vartheta})}{\partial\gamma\partial\beta} = 0.$$

The variance-covariance matrix for the ML parameter estimators is the inverse of the observed information matrix. The inverse of the observed information matrix can be obtained as in (24), where $\left|J(\underline{\vartheta})\right|$ is the determinant of $J(\underline{\vartheta})$, can be obtained as follows:

$$\left|J(\underline{\vartheta})\right| = \begin{vmatrix} J_{\alpha\alpha} & J_{\alpha\beta} & 0 \\ J_{\beta\alpha} & J_{\beta\beta} & 0 \\ 0 & 0 & J_{\gamma\gamma} \end{vmatrix} = J_{\gamma\gamma}\left[J_{\alpha\alpha}J_{\beta\beta} - J_{\alpha\beta}J_{\beta\alpha}\right], \quad (50)$$

and $adj\, J(\underline{\vartheta})$ is the adjoint of $J(\underline{\vartheta})$, can be obtained as follows:

$$\text{cof}\, J(\underline{\vartheta}) = \begin{pmatrix} J_{\beta\beta}J_{\gamma\gamma} & -(J_{\beta\alpha}J_{\gamma\gamma}) & 0 \\ -(J_{\alpha\beta}J_{\gamma\gamma}) & J_{\alpha\alpha}J_{\gamma\gamma} & 0 \\ 0 & 0 & J_{\alpha\alpha}J_{\beta\beta} - J_{\alpha\beta}J_{\beta\alpha} \end{pmatrix}, \quad (51)$$

The transpose of (44) can be obtained as follows:

$$adj\, J(\underline{\vartheta}) = \begin{pmatrix} J_{\beta\beta}J_{\gamma\gamma} & -(J_{\alpha\beta}J_{\gamma\gamma}) & 0 \\ -(J_{\beta\alpha}J_{\gamma\gamma}) & J_{\alpha\alpha}J_{\gamma\gamma} & 0 \\ 0 & 0 & J_{\alpha\alpha}J_{\beta\beta} - J_{\alpha\beta}J_{\beta\alpha} \end{pmatrix}, \quad (52)$$

By substituting (50) and (52) in (24), then the variance-covariance matrix can be written as follows:

$$J(\underline{\vartheta})^{-1} = \begin{pmatrix} J^{\alpha\alpha} & J^{\alpha\beta} & 0 \\ J^{\beta\alpha} & J^{\beta\beta} & 0 \\ 0 & 0 & J^{\gamma\gamma} \end{pmatrix}. \quad (53)$$

where

$$J^{\alpha\alpha} = \frac{J_{\beta\beta}}{J_{\alpha\alpha}J_{\beta\beta} - J_{\alpha\beta}J_{\beta\alpha}},$$

$$J^{\beta\beta} = \frac{-J_{\alpha\alpha}}{J_{\alpha\alpha}J_{\beta\beta} - J_{\alpha\beta}J_{\beta\alpha}},$$

$$J^{\gamma\gamma} = \frac{1}{J_{\gamma\gamma}},$$

and

$$J^{\alpha\beta} = J^{\beta\alpha} = \frac{-J_{\alpha\beta}}{J_{\alpha\alpha}J_{\beta\beta} - J_{\alpha\beta}J_{\beta\alpha}} = \frac{-J_{\beta\alpha}}{J_{\alpha\alpha}J_{\beta\beta} - J_{\alpha\beta}J_{\beta\alpha}}.$$

The diagonal elements, J^{u} of the variance-covariance matrix, $J(\underline{\vartheta})^{-1}$ in (53) are the estimated variance of the ML estimators, $(\hat{\alpha},\hat{\beta},\hat{\gamma})$ and the square root of the diagonal elements of the variance-covariance matrix, are the estimated standard errors of the ML estimators, $(\hat{\alpha},\hat{\beta},\hat{\gamma})$. Thus, $(1-\delta)100\%$ asymptotic confidence intervals (CIs) of $\hat{\alpha},\hat{\beta}$ and $\hat{\gamma}$ can be obtained as follows:

$$\hat{\vartheta} \pm z_{\frac{\delta}{2}}(J^{u})^{\frac{1}{2}},$$

where

$\hat{\vartheta} = (\hat{\alpha},\hat{\beta},\hat{\gamma})$, and z_{δ} represent the δ^{th} quantile of the $N(0,1)$ distribution.

Numerical study

In this section, a simulation study is conducted to evaluate the performance of the proposed methods. A real data set is analyzed to demonstrate how the methods can be applied in practice.

8.1 Simulation Study

A simulation study is conducted to evaluate the performance of the proposed $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$ distribution and the EIDTPR model. A simulation study is performed for a set of initial parameter values, sample sizes and right endpoints. For each combination of the parameter values, sample sizes and right endpoints, the $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$ is fitted and the variance, bise^2, mean square error (MSE) are calculated using the following formulae:

$$\text{bise}^2 = (\text{estimate of the parameter} - \text{true value of the parameter})^2. \quad (54)$$

$$\text{MSE} = \text{variance (estimate)} + \text{bise}^2(\text{estimate}). \quad (55)$$

The following steps are used to compute the ML estimates for $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$.

1. For given values of the parameters $\varphi_0 = 0.4$, $\varphi_1 = 0.3$ and $\lambda = 3$ counts are generated from $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$ using (7) for different sample sizes [n =50,100,300 and 500] and different values of the right endpoint [m=8, 13 and 20].
2. Obtain the ML estimates by solving (17), (18) and (19), respectively.
3. Compute the bise^2, MSE for each estimate using (54), (55) respectively.
4. Repeat the above steps for all sample sizes 500 times.
5. The results of the simulation study are illustrated in Table 1.

   It is observed from Table 1 that the bise^2, and MSE decreased as n increased.

Table 1: The Estimates bise^2, MSE and C.I. for the Parameters for the EIDTP$(\varphi_0, \varphi_1, \lambda)$

		$m = 8$			$m = 13$			$m = 20$		
Sample Size	Parameter	Estimates	MSE	bise^2	Estimates	MSE	bise^2	Estimates	MSE	bise^2
	φ_0	0.4031	0.0048	9.745E-06	0.3986	0.005	1.965E-06	0.4018	0.0045	3.092E-06
n=50	φ_1	0.3009	0.0038	7.748E-07	0.2988	0.0041	1.432E-06	0.2986	0.004	1.842E-06
	λ	3.2563	0.0726	0.0657	3.2579	0.0765	0.0665	3.2556	0.071	0.0653
	φ_0	0.4007	0.0022	5.205E-07	0.4052	0.0025	2.666E-05	0.3995	0.0025	2.701E-07
n=100	φ_1	0.3005	0.0021	2.135E-07	0.2992	0.002	6.71E-07	0.2992	0.0021	5.735E-07
	λ	3.2561	0.0690	0.0656	3.2554	0.0695	0.0652	3.2572	0.0703	0.0662
	φ_0	0.4028	0.0008	8.054E-06	0.4016	0.0008	2.413E-06	0.3994	0.0008	3.080E-07
n=300	φ_1	0.2989	0.0007	1.133E-06	0.2995	0.0007	2.485E-07	0.3001	0.0007	1.179E-08
	λ	3.2480	0.0645	0.0615	3.2522	0.0664	0.0636	3.2483	0.0648	0.0616
	φ_0	0.3980	0.0005	4.066E-06	0.4003	0.0005	9.126E-08	0.3991	0.0005	8.261E-07
n=500	φ_1	0.3016	0.0004	2.468E-06	0.3003	0.0004	1.065E-07	0.2997	0.0004	1.087E-07
	λ	3.2455	0.0629	0.0603	3.2478	0.0643	0.0614	3.2485	0.0641	0.0618

The EIDTPR model is used with the function relations (36), (37) and (38).Independent samples of size n are generated from $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$ using initial values $\varphi_0 = 0.4$, $\varphi_1 = 0.3$ and $\lambda = 3$. The distribution parameters are related to a set of explanatory variables with unknown coefficients through link function.

The following steps are used to compute the ML estimates for the EIDTPR model for different sample sizes [n=50,100,300 and 500] and different values of the right endpoint [m=8, 13 and 20].

1. Three covariates are generated from $U(1, 2)$.
2. Let the intercept $\alpha_o = \beta_o = \gamma_o = 0$
3. The initial values of the regression coefficient are set to be $\alpha_1 = \beta_1 = \gamma_1 = 0.4$
4. Obtain the ML estimates by solving (45), (46) and (47), respectively.
5. Compute the bias, MSE for each estimate using (54), (55) respectively.
6. Repeat the above steps for all sample sizes 500 times.
7. The results of the simulation study are illustrated in Table 2.

It is observed from Table 2 that the bias, and MSE decreased when n increased.

Table 2: The simulation results of the MLE, MSE, $\mathbf{bise^2}$, and SE for the Parameters for the EIDTPR model

		$m = 8$				$m = 13$				$m = 20$			
Sample Size	Parameter	Estimates	MSE	SE	$bise^2$	Estimates	MSE	SE	$bise^2$	Estimates	MSE	SE	$bise^2$
	α_1	0.2045	0.1475	0.2303	0.3070	0.1877	0.1517	0.2308	0.3135	0.1917	0.1548	0.2307	0.3184
n=50	β_1	-0.0036	0.2834	0.2404	0.4748	0.0042	0.2698	0.2387	0.4610	-0.0017	0.2831	0.2396	0.4748
	γ_1	0.6706	0.0991	0.1116	0.2939	0.6671	0.0929	0.1020	0.2867	0.6574	0.0878	0.1023	0.2776
	α_0	0.1818	0.0969	0.1587	0.2678	0.1964	0.0919	0.1597	0.2577	0.1906	0.0938	0.1596	0.2614
n=100	β_1	-0.0257	0.2387	0.1693	0.4582	0.0075	0.2109	0.1689	0.4269	0.0041	0.2150	0.1688	0.4318
	γ_1	0.6878	0.0952	0.0783	0.2984	0.6813	0.0909	0.0732	0.2923	0.6890	0.0944	0.0723	0.2985
	α_1	0.1861	0.0626	0.0902	0.2334	0.1820	0.0629	0.0902	0.2341	0.1872	0.0615	0.0902	0.2311
n=300	β_1	-0.0064	0.1840	0.0969	0.4179	-0.0049	0.1831	0.0967	0.4168	-0.0079	0.1857	0.0969	0.4198
	γ_1	0.6977	0.0931	0.0456	0.3016	0.6941	0.0902	0.0418	0.2974	0.6913	0.0884	0.0419	0.2943
	α_1	0.1796	0.0581	0.0698	0.2307	0.1836	0.0569	0.0699	0.2281	0.1887	0.0541	0.0699	0.2219
n=500	β_1	-0.0089	0.1785	0.0748	0.4158	-0.0050	0.1759	0.0749	0.4126	0.0003	0.1707	0.0749	0.4063
	γ_1	0.6972	0.0908	0.0351	0.2993	0.6907	0.0867	0.0323	0.2926	0.6913	0.0870	0.0324	0.2932

8.2 Application

An Application using real data set is introduced to demonstrate the importance and flexibility of the proposed methods. The performance of the distribution is assessed using goodness of fit test and different information criteria. The chi-squared (χ^2) test is applied for testing the goodness of fit of EIDTP($\varphi_0, \varphi_1, \lambda$) to the data set. -log-likelihood, Akaike information criteria (AIC) and Bayesian information criteria

(BIC) are used for comparing the methods, Smaller values of– log-likelihood, AIC and BIC indicate better models. For performing significance tests of hypothesis about parameters in the $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$, the Wald's statistics which has an approximate standard normal distribution is used.

The application is carried out using a sample of 16120 individuals in working ages (16-60 years) from 7526 family of the household income, expenditure and consumption survey (HIECS) carried out in Egypt at 2012- 2013 is conducted. The data set is obtained from the Central Agency for Public Mobilization and Statistics. Egypt, Arab Rep. 2012-2013. The sample contains the number of weekly worked days (NWWD) by 16120 individuals in the last week before the survey. The data set contains 8899 zeros and contains 4170 six; i.e. the data set contains non-negligible number of zeros (left –endpoint) and six (right-endpoint). Summary statistics of the NWWD by 16120 individuals of 7526 family in Egypt in 2012 - 2013 is presented in Table 3. The bar chart and the normal Q-Q plot of the number of the data is presented in Figures 1 and 2. It is noticed that the data contains inflation at two points 0 and 6.

Table 3: Sample Summary Statistics of the Number of Weekly Worked Days (NWWD) by 16120 Individuals of 7526 Family in Egypt in 2012 - 2013.

Mean	Min.	1st qu.	3rd qu.	Max.	SD
2.39553	0.0000	0.00	6.00	6.0000	2.7370

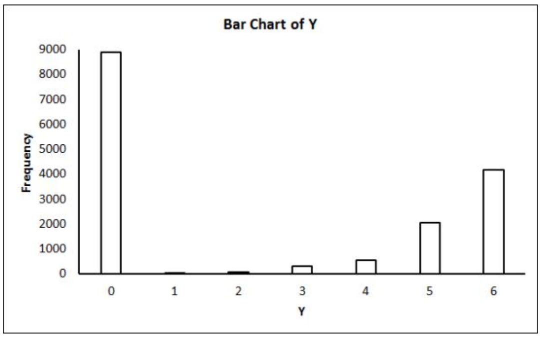

Figure 2: The Number of Weekly Worked Days by 16120 Individuals of 7526 Family in Egypt in 2012 – 2013

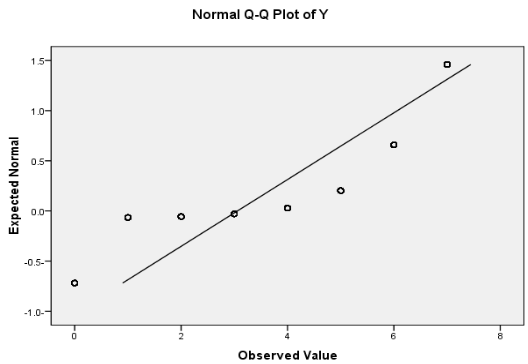

Figure 3: The normal Q-Q plot of the Number of Weekly Worked Days by 16120 Individuals of 7526 Family in Egypt in 2012 – 2013

The observed and fitted frequency distributions based on the MLE and ME of the $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$ are presented in Table 4.

Table 4: Observed and Fitted Frequency Distributions of the Number of Days Worked by 16120 Individuals of 7526 Family in Egypt in 2012 - 2013.

Observed Frequency		Number of days worked by an individual						
		0	1	2	3	4	5	6
		8899	54	89	307	562	2039	4170
MLE	$\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$	8899	14	81	301	634	2021	4170
ME	$\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$	8925	14	73	234	629	2000	4244

Point estimates with the corresponding standard errors and confidence intervals for the parameters of $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$ using the data set of the number of days worked by 16120 individuals of 7526 family in Egypt in 2012 – 2013, are summarized in Table 5.

Table 5: The Parameters Estimates and the Corresponding Standard Errors for the EIDTP($\varphi_0, \varphi_1, \lambda$) Using the Data Set of the Number of Weekly Worked Days By 16120 Individuals of 7526 Family in Egypt in 2012 – 2013

		MLE			ME		
		Point	Std. Error	Interval	Point	Std. Error	Interval
	$\hat{\lambda}$	11.4493	0.2607	(10.938, 11.9602)	11.4569	0.2610	(10.945, 11.9685)
	$\hat{\varphi}_o$	0.5520	0.0032	(0.5458, 0.5582)	0.5537	0.0030	(0.5479, 0.5594)
$\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$	$\hat{\varphi}_1$	0.2587	0.0032	(0.2525, 0.2649)	0.2633	0.001	(0.2617, 0.2649)
	$\widehat{E(Y)}$	2.3955			2.3955		
	$\widehat{V(Y)}$	7.4523			7.4911		

The estimated variance of the random variable $Y \sim \text{EIDTP}(\varphi_0, \varphi_1, \lambda)$ reflect the variation of the data which has more frequencies for some observations [namely zero counts and six counts]. Unlike the Poisson distribution where the mean and variance are equal, the $\text{EIDTP}(\varphi_0, \varphi_1, \lambda)$ can model data where mean and variance have different values.

The results of χ^2, - log-likelihood, AIC and BIC are summarized in Table 6.

Table 6: Validation of the EIDTP($\varphi_0, \varphi_1, \lambda$) to the Data Set of the Number of Weekly Worked Days by 16120 Individuals of 7526 Family in Egypt in 2012 – 2013.

	Expected frequencies	
	MLE	ME
$\chi 2$	128.371	154.6395
df	3	3
p-value	< .00001	< .00001
-Log-Likelihood	8296.73	8297.75
AIC	16599.46	16601.51
BIC	16606.08	16608.13

An application using a sample of 9874 individuals of the data set of the HIECS carried out in Egypt at 2012- 2013 is conducted. The data are obtained from the Central Agency for Public Mobilization and statistics.

The present application introduces a study of the effect of some explanatory variables, which are the age, average number of working hours per day (ANWHD) and average daily income per capita (ADIC), on the response variable Y, which is the NWWD. Outliers are detected according to ADIC, so the sample size became 9288 individuals.

Table 7 presents summary statistics of the NWWD by 9288 individuals and three quantitative random variables which are the age, the ANWHD and the ADIC that are assumed to affect the NWWD.

Table 7: Sample Summary Statistics of the NWWD, the age, the ANWHD and the ADIC for 9288 individuals in Egypt in 2012 - 2013.

Variable	Min.	1st qu.	Mean	Median	3rd qu.	Max.
NWWD	0.000	0.000	2.286	0.000	5.000	6.000
age	16.00	22.00	3303	30.00	44.00	60.00
ANWHD	0.000	0.000	3.474	3.000	8.000	24.000
ADIC	3.353	18.904	27.453	25.233	34.249	62.466

The bar chart of the NWWD by 9288 individuals, the histograms of the age, the ANWHD and ADIC for 9288 individuals in Egypt in 2012 – 2013 are presented in Figures 3, 4, 5 and 6 respectively.

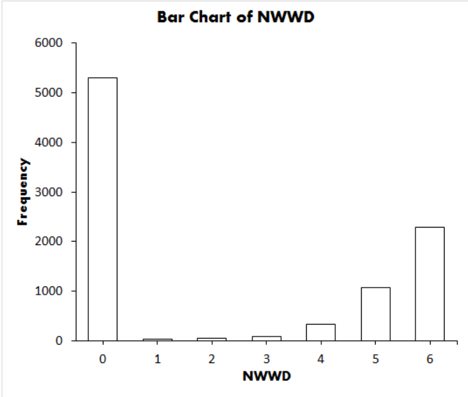

Figure 3: The Number of Days Worked by 9288 Individuals in Egypt in 2012 – 2013

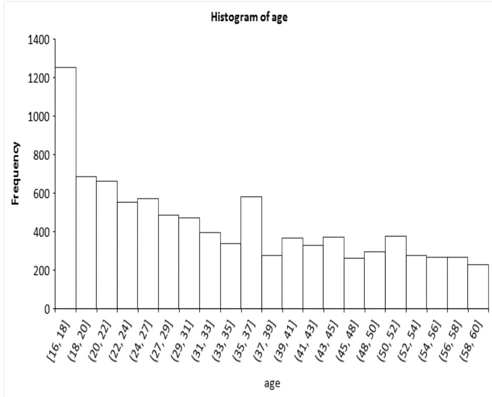

Figure 4: The Age of 9288 Individuals in Egypt in 2012 – 2013

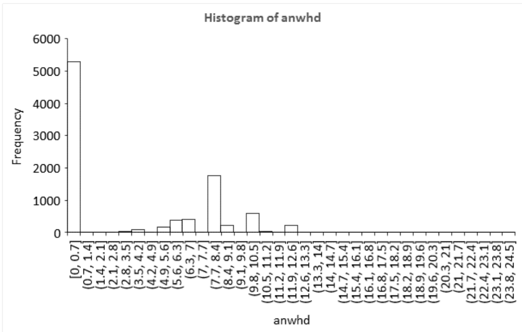

Figure 5: The Average Number of Working Hours Per Day (ANWHD) for 9288 Individuals in Egypt in 2012 – 2013

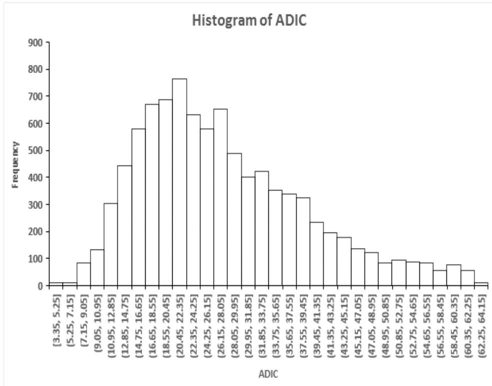

Figure 6: Average Daily Income per Capita (ADIC) for 9288 Individuals in Egypt in 2012 – 2013

It is noticed from Figure 3 that the data contains inflation at two points 0 and 6, such that the frequency of zero is 5297 and the frequency of 6 is 2295. The endpoint inflated Poisson regression model using the functional relations (36), (37) and (38) can be used, assumed that intercepts are set to be zero. The parameter estimates and the corresponding standard errors for the EIPR model are summarized in Table 8.

Table 8: The parameter estimates of the EIDTPR model

Parameter	Estimate	Standard Error	t-value	P-value
α	0.03863	0.00077	49.99	2e – 16
β	0.24183	0.00497	48.7	2e – 16
γ	0.09025	000127	70.83	2e – 16

It is noticed from Table 8 that age has a positive effect on the proportion of zero in the number of weekly worked days by the individuals. It means that as the age increase, the number of days worked by the individuals are set to be zero day. The average number of working hours per day (ANWHD) has a positive effect on the proportion of *m* in the number of weekly worked days by the individuals. It means that as the average number of working hours per day increase, the number of days worked by the individuals are set to be six days. The average daily income per capita (ADIC) has a positive effect that the number of days worked by the individuals rang between one and five days. It means that as the average daily income per capita increase, the number of days worked by the individuals rang between one and five days.

8. Conclusions

EIDTP$(\varphi_0, \varphi_1, \lambda)$ distribution is suggested for modeling data consisting of inflated counts of zeros and inflated counts of m, assuming all zeros and m are from one structural source rather than two sources (both structural and sampling). The distributional properties and two parameters estimation methods, maximum likelihood and method of moments are considered. The method of maximum likelihood estimators is shown to have better estimates on the real data set. The EIDTPR model is suggested to investigate the dependence of the response variable of count data containing both extra zeros (left-endpoints) and extra right-endpoint, on a set of explanatory variables. A simulation study is conducted to evaluate the performance of the proposed methods. A real data set is analyzed to demonstrate how the methods can be applied in practice. The numerical study is carried out using R program, version 4.1.2.

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الملخص العربي

ENDPOINT-INFLATED DOUBLE TRUNCATED POISSON MODEL

نموذج انحدار بواسون الموسع مزدوج البتر

تم اقتراح توزيع بواسون الموسع مزدوج البتر (Endpoint-Inflated Double Distribution Truncated Poisson) وذلك لتمثيل متغير عشوائي يأخذ قيم لبيانات (Count Data) والتي تحتوي على عدد كبير من الأصفار(Left Endpoint) بالإضافة الي وجود قيمة أخرى ذات تكرار كبير أيضًا (Right Endpoint) مقارنةً بالقيم الأخرى للبيانات، والتي لا يمكن تفسيرها بواسطة الفروض الأساسية لتوزيع بواسون. يعد توزيع بواسون الموسع مزدوج البتر خليطا من ثلاث مكونات، قيمة الاحتمال عند النقطة (صفر)، وقيمة الاحتمال عند النقطة (m) ، وباقي القيم التي يأخذها المتغير العشوائي يمثلها توزيع بواسون مزدوج البتر Double Truncated Poisson Distribution). تمت مناقشة بعض خصائص التوزيع. تم استخدام كل من طريقة الإمكان الأكبر وطريقة العزوم للحصول على المقدرات وفترات الثقة لمعالم التوزيع. تم اقتراح نموذج انحدار بواسون الموسع مزدوج البتر (Endpoint-Inflated Double Truncated Poisson Model). تم إجراء دراسة محاكاة (Simulation Study) لتقييم أداء الطرق المقترحة. تم تحليل مجموعة بيانات حقيقية لتوضيح كيف يمكن تطبيق الأساليب عمليًا.