The Topp-Leone Ailamujia Distribution: Properties & Applications

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Abstract

The Topp-Leone Ailamujia (TL-A) distribution is constructed as a new two-parameter lifetime model. The proposed model's hazard rate function can be a bathtub or a reversed J-shaped. A variety of statistical qualities and reliability features of the TL-A distribution have been investigated, including the moments, the moment generating function, incomplete moments, mean deviation, and the curves of Bonferroni and Lorenz. The functions of mean residual life and mean inactive time are also taken into account. The maximum likelihood procedure is highlighted for estimating the model parameters. To check the behaviour of the estimates, Monte Carlo simulations are being performed. Last, the novel model's efficiency is tested with a set of real data.

Keywords: Burr–Hatke distribution; Hazard Rate Function; Moments; Residual Analysis; Maximum Likelihood Estimation; Monte Carlo Simulation.

1. Introduction

Lifetime distributions are widely applied in biology, engineering, and management. The hazard rate of classical models such as exponential, gamma, and Weibull distributions exhibit monotonically increasing, monotonically decreasing, or constant hazard rates, this is a significant issue because many longevity systems utilize bathtub forms for their hazard rates. Many recent efforts have been undertaken to develop more flexible statistical distributions in modelling life data. Lv and Chen, (2002) presented the Ailamujia distribution (AD) as a one-parameter lifetime model. This model is flexible and diverse in terms of modelling repair or delays time of a system. The function of cumulative distribution (cdf) for AD is expressed in the form:

$$H(y;\lambda)=1-(1+\lambda y)\exp(-\lambda y);\; y\geq 0,\, \lambda>0,$$

thus, the relevant probability density function (pdf) is deduced as follows:

$$h(y;\lambda)=\lambda^2\, y\, \exp(-\lambda y);\; y\geq 0,\, \lambda>0.$$

Ailamujia distribution has profited greatly from the efforts of several authors. Pan et al. (2009) investigated small-sample interval estimation and hypothesis testing. Bing, (2015) has been using Type II censoring and three independent priors based on incomplete data to implement the Bayesian estimation of the AD. Li (2016) utilized the three-loss functions to evaluate the minimax estimate of the parameter of AD under a non-informative prior. Jan et al. (2017) proposed and investigated the weighted analogue of the AD. Rather and Subramanian (2022) established a new size-biased AD with applications in engineering and medicine. Jamel et al. (2021) established and exhibited the power version of AD. established and exhibited the power version of AD. Recently, Rather et al. (2018) studied the exponentiated version of AD in detail, utilizing statistical inference and biomedical data applications.

The necessity to assemble new generators for univariate lifetime distributions by introducing one or more shape parameters to the baseline model has lately increased. This parameter induction is useful in detecting tail features as well as optimizing the goodness-of-fit of the proposed generating distribution. These structures were created by adding one or more parameters to the baseline model's cdf to create a fresh family of distributions that are more analytically and accommodative. As a result, various classes of continuous distributions have been evolved in the literature including Eugene et al. (2002) produced the beta-G, Zografos and Balakrishnan (2009) who proposed the gamma-G. Cordeiro and de Castro (2011) built the Kumaraswamy-G, Alexander et al. (2012) created the McDonald-G, the Weibull-G has been introduced by Bourguignon et al. (2014), and several other new families of distributions has been devolved.

Topp and Leone (1955) suggested the Topp-Leone (TL) distribution as an alternative to the Beta distribution. The TL model received far less attention before Nadarajaha and Kotoz (2003) investigated it. Al-Shomrani et al. (2016) recently defined the Topp-Leone-G family of distributions, using the TL distribution as a generator to create this class. The TL-G family's cdf is formulated as:

$$G(y;\delta,\zeta)=\left[H(y;\zeta)\right]^{\delta}\left[2-H(y;\zeta)\right]^{\delta}$$

$$=\left[1-\left\{\bar{H}(y;\zeta)\right\}^{2}\right]^{\delta},\; y\in R;\; \delta>0, \qquad (3)$$

where $\bar{H}(y;\zeta)=1-H(y;\zeta)$ is a survival function of the baseline model that is affected by a parameter vector ζ. The related pdf is obtained as:

$$g(y;\delta,\zeta)=2\delta h(y;\zeta)\bar{H}(y;\zeta)\left[H(y;\zeta)\right]^{\delta-1}\left[2-H(y;\zeta)\right]^{\delta-1}$$

$$=2\delta h(y;\zeta)\bar{H}(y;\zeta)\left[1-\left\{\bar{H}(y;\zeta)\right\}^{2}\right]^{\delta-1};\; \delta>0, (4)$$

where, $h(y;\zeta)$ and $\bar{H}(y;\zeta)=1-H(y;\zeta)$ respectively, are the pdf and survival function of the baseline model.

Some of the TL-distributions have been investigated and analyzed, including the TL-exponential reported by Al-Shomrani et al. (2016), the TL-generalized inverted exponential derived by Al-Saiary and Bakoban (2020), and the Tl-Gompertz evaluated by Nzei et al. (2020). In this work, we propose and investigate the Topp Leone-Ailamujia (TL-A) distribution, as a new extension of the Ailamujia distribution. The rate of failure for the TL-A distribution demonstrates a bathtub and a unimodal pattern, which is the primary motivation for adopting this model.

The remainder of the paper is structured as follows. The Topp-Leone Ailamujia distribution is introduced in Section 2. Section 3 examine a variety of mathematical aspects of the TL-A distribution including: the r^{th} moment, moment generating function, the sth incomplete moment, conditional moments, mean deviation, mean residual life, mean inactivity times and the entropy. Section 4 goes on estimation and simulation; maximum likelihood method has been used to estimate the distribution parameters. The numerical simulations for maximum likelihood estimates are utilized to study the behavior of the estimate. Section 5 uses a real-life data set to demonstrate the TL-A model's utility. Finally, Section 6 discusses some findings.

2. The Topp Leone-Ailamujia Distribution

In this section, we will define the Topp Leone-Ailamujia (TL-A) distribution by taking $H(y;\zeta)$ to be the cdf of Ailamujia distribution with the parameter λ. The recommended model's cdf and pdf respectively, are generated using Eqs. (1&2) in Eqs. (3&4) as indicated below:

$$G(y;\delta,\lambda)=\left[1-(1+\lambda y)^2 e^{-2\lambda y}\right]^{\delta},\quad y>0;\ \delta,\lambda>0, \qquad (5)$$

and

$$g(y;\delta,\lambda)=2\delta\lambda^2 y\,(1+\lambda y)\,e^{-2\lambda y}\left[1-(1+\lambda y)^2 e^{-2\lambda y}\right]^{\delta-1}. \qquad (6)$$

Applying the generalized binomial theorem, for any real number ε that is a positive integer, we have:

$$(1+\omega)^{\varepsilon}=\sum_{k=0}^{\infty}\binom{\varepsilon}{k}\omega^k\ ,\quad \text{for } |\omega|>0. \qquad (7)$$

The cdf and pdf of TL-A(δ,λ) distribution can be reformulated using Eq.(7) as:

$$G(y;\delta,\lambda)=\sum_{i,k=0}^{\infty}\Lambda_{i,k}\,y^{k}\,e^{-i\lambda y}\ ; \qquad (8)$$

where, $\Lambda_{i,k}=(-1)^{i}\,\lambda^{k}\binom{\delta}{i}\binom{2i}{k}$.

and

$$g(y;\delta,\lambda)=\sum_{i,k=0}^{\infty}\Pi_{i,k}\,y^{k+1}\,e^{-2(i+1)\lambda y}\ ; \qquad (9)$$

where, $\Pi_{i,k}=2\delta\,\lambda^{k+2}(-1)^{i}\binom{\delta-1}{i}\binom{2i+1}{k}$.

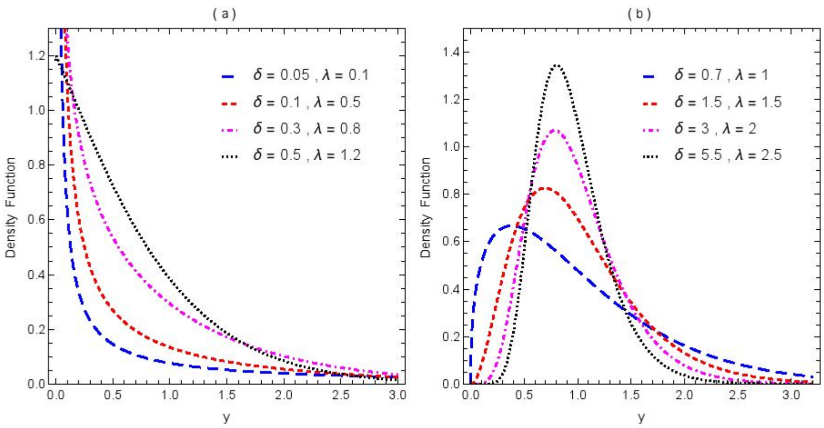

Figure (1) The TL-A density function plots

The representations of the TL-A density function for various model attribute values are shown in Figure (1) The suggested model's pdf is slanted to right, monotonically decreasing, and unimodal.

The hazard rate (hr) function is used to illustrate the population's current failure rate. It is also significant in reliability assessment and socioeconomics, as well as in developing a model when dealing with longevity data. The hr of the TL-A(δ,λ) distribution is:

$$r(y;\delta,\lambda)=\frac{g(y;\delta,\lambda)}{1-G(y;\delta,\lambda)}$$

$$=\frac{2\delta\lambda^2 y(1+\lambda y)e^{-2\lambda y}\left[1-(1+\lambda y)^2 e^{-2\lambda y}\right]^{\delta-1}}{1-\left[1-(1+\lambda y)^2 e^{-2\lambda y}\right]^{\delta}}. \qquad (10)$$

The patterns of the hazard rate function are shown in Figure (2) for a variety of distribution parameter values. The TL-A distribution's hr is shaped like a bathtub or upside-down bathtub shaped (unimodal).

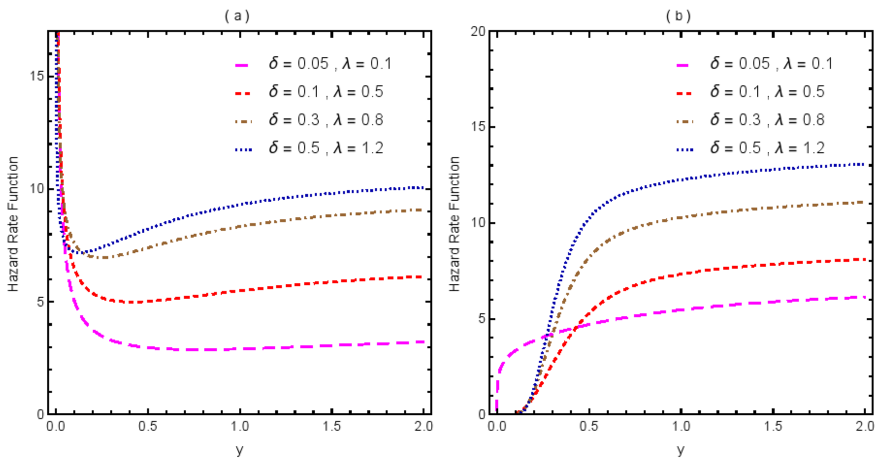

Figure (2) The plots of hazard rate function for the TL-A distribution

3. Statistical Properties

The basic statistical properties of the TL-A model, such as the r^{th} moment, moment generating function, the sth incomplete moment, conditional moments, mean deviation, mean residual life, mean inactivity times and the entropy, are deduced in this section.

3.1. The rth moment

Central tendency and dispersion measurements are the most common methods for describing the characteristics of a probability distribution. Mainly two measurements are expected value and variance. Skewness and kurtosis are two further features that can be highlighted. A moment is a mathematical quantity that includes all of these measures.

For the random variable Y with a TL-A distribution, the r^{th} moment about the origin is determined as follows:

$$\mu'_r(y) = E\left[Y^r\right] = \int_0^\infty y^r\, g(y)\, dy$$

By substituting Eq. (9) into the previous equation, we get:

$$\mu'_r(y) = \sum_{i,k=0}^\infty \Pi_{i,k} \int_0^\infty y^{r+k+1}\, e^{-2(i+1)\lambda y}\, dy$$

$$= \sum_{i,k=0}^\infty \Pi_{i,k} \int_0^\infty \frac{\left[2y(i+1)\lambda\right]^{r+k+2-1}}{\left[2(i+1)\lambda\right]^{r+k+1}}\, e^{-2(i+1)\lambda y}\, d\left[2y(i+1)\lambda\right]$$

$$= \sum_{i,k=0}^\infty \Pi_{i,k} \frac{\Gamma(r+k+2)}{\left[2(i+1)\lambda\right]^{r+k+2}}; \quad r = 1,2,3,\ldots$$

where, $\Gamma(\upsilon) = \int_0^\infty u^{\upsilon-1} \exp[-u]\, du$ is the gamma function. as a special case of the above equation, the mean of TL-A distribution is given by:

$$\mu = \mu'_1 = \sum_{i,k=0}^\infty \Pi_{i,k} \frac{\Gamma(k+3)}{\left[2(i+1)\lambda\right]^{k+3}}$$

The TL-A random variable's central moments are specified by:

$$\mu_n = E\left[(y-\mu)^n\right] = \sum_{r=0}^n \binom{n}{r} \mu'_r(y)\left(-\mu\right)^{n-r}$$

$$=\sum_{r=0}^{n}\kappa_{i,k,n}\frac{\Gamma(k+3)}{\left[2(i+1)\lambda\right]^{k+3}},$$

where, $\kappa_{i,k,n}=\sum_{i,k=0}^{\infty}\Pi_{i,k}\binom{n}{r}(-\mu)^{n-r}$.

The variance (Var) of the TL-A distribution is given from Eq. (13) for $n=2$. Utilizing Eq. (11), the skewness (Ske) and kurtosis (Kur) metrics can be computed by the following relationships:

$$Ske(y)=\frac{\mu_3'-3\mu_1'\mu_2'+2(\mu_1')^3}{Var^{\frac{3}{2}}(y)},$$

and

$$Kur(y)=\frac{\mu_4'-4\mu_1'\mu_3'+6\mu_2'(\mu_1')^2-3(\mu_1')^4}{Var^2(y)}.$$

Table (1) presents the values of these measures for some selected values of the distribution parameters. As the value of parameter δ increases while the value of parameter λ decreases. The mean and variance of the TL-A distribution rise, but skewness and kurtosis decrease.

Table (1) The mean, variance, skewness and kurtosis for the TL-A distribution

$\delta\uparrow$	$\lambda\downarrow$	$\mu(y)\uparrow$	$Var(y)\uparrow$	$Ske(y)\downarrow$	$Kur(y)\downarrow$
0.5	5	0.170112	0.023650	1.601120	6.52357
1	3.5	0.357143	0.056122	1.260870	5.33058
1.5	2	0.753518	0.178726	1.141790	5.01119
3	1	1.965050	0.724575	1.030940	4.77877
6	0.5	4.858080	2.832560	0.991012	4.73435

3.2. The moment generating function

The moment-generating function should be considered as an alternative method for expressing a random variable's probability distribution. This alternate formulation is quite valuable since it provides somewhat superior analytic controllability than density or cumulative distribution functions. Likewise, Eq. (9) can be used to construct the moment generating function of Y as follow:

$$M_Y(t) = E\left[e^{tY}\right] = \int_0^\infty e^{ty} g(y)\, dy$$

$$= \sum_{i,k=0}^\infty \Pi_{i,k} \int_0^\infty y^{k+1} e^{-[2(i+1)\lambda - t y]}\, dy$$

$$= \sum_{i,k=0}^\infty \Pi_{i,k} \frac{\Gamma(k+2)}{\left[2(i+1)\lambda - t\right]^{k+2}} \;; \qquad t < 2(i+1)\lambda .$$

3.3. The sth incomplete moment

The Lorenz l(p) and Bonferroni b(p) curves can be obtained using the first incomplete moment. These economic inequality measures can assist other disciplines such as reliability, biology, and insurance. The sth lower incomplete moment for TL-A distribution is calculated as:

$$\varphi_s(t) = E\left[Y^s | Y < t\right] = \int_0^t y^s g(y) dy \;,$$

$$= \sum_{i,k=0}^\infty \Pi_{i,k} \frac{\gamma\left[s+k+2,\, 2\lambda(i+1)\, t\right]}{\left[2(i+1)\lambda\right]^{s+k+2}}$$

where $\gamma(m,t) = \int_0^t u^{m-1} \exp[-u]\, du$ denotes the lower incomplete gamma function.

Setting $s=1$ in Eq. (15), the first order of lower incomplete monent is:

$$\varphi_1(t) = \sum_{i,k=0}^\infty \Pi_{i,k} \frac{\gamma\left[k+3,\, 2\lambda(i+1)\, t\right]}{\left[2(i+1)\lambda\right]^{k+3}} .$$

Using Eqs. (12 & 16), the Lorenz and Bonferroni curves have been calcautleted from these relations: $l(p)=\frac{1}{\mu_1}\int_0^q y\, g(y)\, dy$ and $b(p)=\frac{1}{p\,\mu_1}\int_0^q y\, g(y)\, dy$, where $p=G(y)$ and $q=G^{-1}(p)=\inf\{y : G(y)\geq p\}$.

3.4. The conditional moments

The conditional moments of the first order is most commonly used to calculate the mean deviation about the mean (or the median) and the mean residual life function. For the TL-A distribution, the conditional moments are as follows:

$$E\left[Y^s \,\middle|\, Y > t\right]=\frac{o_s(t)}{\bar{G}(t)},$$

and,

$$o_s(t)=\int_t^\infty y^s\, g(y)dy$$

$$=\sum_{i,k=0}^{\infty}\Pi_{i,k}\,\frac{\Gamma\left[s+k+2,\; 2\lambda(i+1)\,t\right]}{\left[2(i+1)\lambda\right]^{s+k+2}}, \tag{17}$$

where $\Gamma(m,t)=\int_t^\infty u^{m-1}\exp[-u]\,du$ gives the upper incomplete gamma function. From the above equation with $s=1$, the first upper incompete moment is given by:

$$o_1(t)=\sum_{i,k=0}^{\infty}\Pi_{i,k}\,\frac{\Gamma\left[k+3,\; 2\lambda(i+1)\,t\right]}{\left[2(i+1)\lambda\right]^{k+3}}, \qquad ($$

3.5. The mean deviation

The overall amount of variations from the mean and median refers to the amount of scattering in a population to some level. The mean deviation about the mean is given by:

$$D(\mu)=E\left[\left|y-\mu\right|\right]=\int_{0}^{\infty}\left|y-\mu\right|g(y)dy$$

$$=\int_{0}^{\mu}[\mu-y]g(y)dy+\int_{\mu}^{\infty}[y-\mu]g(y)dy$$

$$D(\mu)=2\mu G(\mu)-2\mu+2\int_{\mu}^{\infty}y\ g(y)dy=2\mu G(\mu)-2\mu+2o_1(\mu)$$

Substituting Eqs. (8 &16) into Eq. (19), yields:

$$D(\mu)=2\sum_{i,k=0}^{\infty}\Lambda_{i,k}\ \mu^{k+1}e^{-i\lambda\mu}-2\mu+2\sum_{i,k=0}^{\infty}\Pi_{i,k}\ \frac{\Gamma\left[k+3,\ 2\lambda(i+1)\ \mu\right]}{\left[2(i+1)\lambda\right]^{k+3}}\ .$$

Now, the mean deviation about the median is defined by:

$$D(m)=E\left[\left|y-m\right|\right]=\int_{0}^{\infty}\left|y-m\right|g(y)dy$$

$$=-\mu+2\int_{m}^{\infty}y\ g(y)dy=-\mu+2o_1(m)$$

$$=-\mu+2\sum_{i,k=0}^{\infty}\Pi_{i,k}\ \frac{\Gamma\left[k+3,\ 2\lambda(i+1)\ m\right]}{\left[2(i+1)\lambda\right]^{k+3}}\ .$$

3.6. The mean residual life

Major applications of the mean residual life (MRL) exist in the fields of biology, insurance, service quality control and social science. A product or device's MRL is the expected length of time it will last after achieving age *t*. The MRL of TL-A distribution is derived as:

$$MR(t)=\frac{1}{\bar{G}(t)}\int_{t}^{\infty}y\ g(y)dy-t=\frac{o_1(t)}{\bar{G}(t)}-t\ \ ;\ \ t>0$$

Inserting Eq. (18) into Eq. (22), yields:

$$MR(t)=\frac{1}{1-G(t)}\sum_{i,k=0}^{\infty}\Pi_{i,k}\frac{\Gamma\left[k+3,\,2\lambda(i+1)\,t\right]}{\left[2(i+1)\lambda\right]^{k+3}}-t\ .$$

3.7. The mean inactivity times

A well-known reliability measure with applications in forensics and reliability theory is the mean inactivity time (MIT) function. In terms of the TL-A distribution, the MIT of a random variable Y is calculated as follows:

$$MT(t)=t-\frac{1}{G(t)}\int_{0}^{t}y\,g(y)dy=t-\frac{\varphi_1(t)}{G(t)}\ ;\ t>0$$

Using Eq. (16) into Eq. (24), we have:

$$MT(t)=t-\frac{1}{G(t)}\sum_{i,k=0}^{\infty}\Pi_{i,k}\frac{\gamma\left[k+3,\,2\lambda(i+1)\,t\right]}{\left[2(i+1)\lambda\right]^{k+3}}\ .$$

Table (2) presents the numeric values of the MRL and MIT for the proposed model at point $t=1$ and a fixed value of the parameter $\lambda=2$, with various parameter selections (α,β). As the parameters α and β increase, the MRL decreases while the MIT increases. Figure (3) depicts the activity of the MRL and the MIT for various values of distribution parameters. The MRL (left side) is decreasing, whereas the MIT (right side) is rising.

Table (2) The MRL and MIT function for the TL-A distribution

$\delta\uparrow$	$\lambda\downarrow$	$t=2$	$MR\uparrow$	$MT\downarrow$
0.5	7		0.076349	1.878490
1	5		0.109504	1.750000
2.5	3.5		0.161834	1.473550
4	1.5		0.439874	0.737483
5.5	1		0.833005	0.411506
6.5	0.5		2.9850800	0.217965

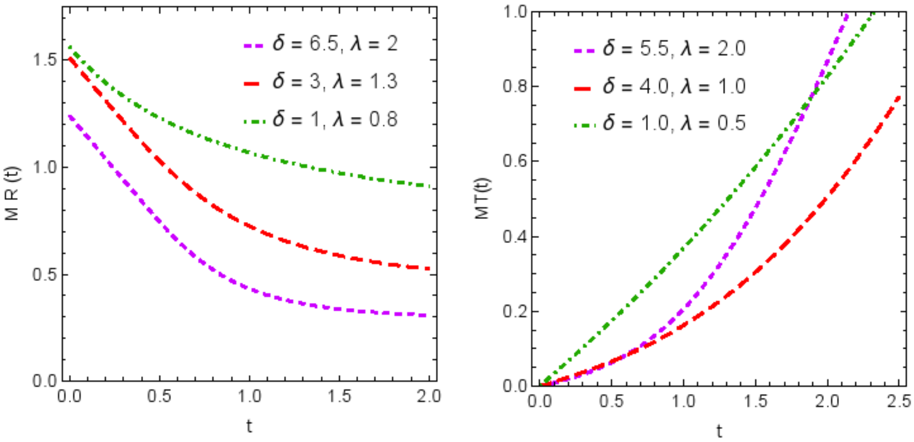

Figure (3) The Plots of MRL and MIT functions for the TL-A distribution

3.8. *Entropy*

A measure of the variation in uncertainty is the entropy. The more uncertainty in the data is indicated by a high value for entropy. An explanation of the Rényi entropy (Rényi & others, 1961) for TL-A distribution is derived as:

$$R_\gamma = \frac{1}{1-\gamma}\log\int_{-\infty}^{\infty} g^{\gamma}(y)\,dy \quad ; \ \gamma > 0, \gamma \neq 1$$

$$= \frac{1}{1-\gamma}\log\left[\left(2\delta\lambda^2\right)^{\gamma}\int_0^{\infty} y^{\gamma}\left(1+\lambda y\right)^{\gamma} e^{-2\gamma\lambda y}\left[1-\left(1+\lambda y\right)^2 e^{-2\gamma\lambda y}\right]^{\gamma(\delta-1)} dy\right]$$

Applying the generalized binomial theorem which is given by Eq. (7), yields:

$$R_\gamma = \frac{1}{1-\gamma}\log\left[\sum_{i,k=0}^{\infty}\Pi_{i,k}^{\gamma}\int_0^{\infty} y^{k+\gamma} e^{-2\gamma\lambda(i+1)y}\,dy\right]$$

$$= \frac{1}{1-\gamma} \log\left[\sum_{i,k=0}^{\infty} \Pi_{i,k}^{\gamma} \frac{\Gamma(k+\gamma+1)}{[2\gamma\lambda(i+1)]^{k+\gamma+1}} \right],$$

where, $\Pi_{i,k}^{\gamma} = (2\delta)^{\gamma} \lambda^{k+2\gamma} (-1)^{i} \binom{(\delta-1)\gamma}{i} \binom{2i+\gamma}{k}.$

4. Estimation and Simulation

Out all the several techniques for estimating the parameters that have been reported in the literature, the maximum likelihood approach (MLEs) is the technique that is most frequently used. The MLEs have attributional properties and can be used to create confidence intervals and test statistics. The unknown parameters of the TL-A distribution are estimated in this section using the MLEs technique. Also, we will use Monte Carlo simulation to demonstrate the behavior of estimates.

4.1. Maximum likelihood estimation

Suppose $y_1, y_2, \ldots, y_n$ be a random sample of size n drawn from the TL-A distribution with pdf (6). Thus, the log-likelihood function of the suggested model indicates:

$$l = n \,\mathrm{Log}[2\delta\lambda^2] + \sum_{i=1}^{n} \mathrm{Log}[y_i] + \sum_{i=1}^{n} \mathrm{Log}[(1+\lambda y_i)] - 2\lambda \sum_{i=1}^{n} y_i$$
$$+ (\delta-1) \sum_{i=1}^{n} \mathrm{Log}\left[1 - \left((1+\lambda y_i) e^{-\lambda y_i}\right)^2\right].$$

By differentiating Eq. (26) with respect to parameters δ and λ, respectively, one can construct the likelihood equations for the purposed model as:

$$\frac{\partial l}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^{n} \mathrm{Log}\left[1 - e^{-2\lambda y_i} (1+\lambda y_i)^2\right],$$

and

$$\frac{\partial l}{\partial \lambda}=(\delta-1)\sum_{i=1}^{n}\frac{2y_i(\lambda y_i+1)^2 e^{-2\lambda y_i}-(2\lambda y_i^2+2y_i)e^{-2\lambda y_i}}{1-(\lambda y_i+1)^2 e^{-2\lambda y_i}}$$
$$+\sum_{i=1}^{n}\frac{y_i}{\lambda y_i+1}-2\sum_{i=1}^{n}y_i+\frac{2n}{\lambda}.$$

By equating the system of nonlinear equations (27-28) to zero and solving them concurrently, the MLEs of parameters δ and λ can be acquired. Because these equations are nonlinear, we should solve them using analytical techniques such as Newton-Raphson.

4.2. Mont Carlo simulation

The effectiveness of the MLEs of the parameters of TL-A distribution is investigated using Monte-Carlo simulation. MLE accuracy is discussed using bias term and mean square error (MSE). Using Eq (5), different samples of size 25, 50, 75, and 100 are generated, this simulation study is being evaluated using 1000 replicates, we consider the following cases:

Case I: The true values of the distribution parameters δ and λ are: $\delta=0.5$ and $\lambda=2$.

Case II: The values of δ and λ are: $\delta=3$ and $\lambda=0.75$.

The average MSEs and biases of the simulated estimates $\hat{\eta}_i=(\hat{\delta},\hat{\lambda})$ are calculated using the following relationships:

$$MSE(\eta)=\frac{1}{M}\sum_{i=1}^{M}(\hat{\eta}_i-\eta)^2 \quad \text{and} \quad Bias(\eta)=\frac{1}{M}\sum_{i=1}^{M}(\hat{\eta}_i-\eta).$$

Tables (3-4) show the average values of MSE and the bias term for the simulated parameters. These tables clearly show that as sample size increases, MSEs and biases decrease. Furthermore, for large samples, the estimated value of parameter is close to the parametric values.

Table (3) The simulation results for the Case I

n	$\hat{\delta}$	$MSE(\delta)$	$Bias(\delta)$	$\hat{\lambda}$	$MSE(\lambda)$	$Bias(\lambda)$
30	0.54808	0.021901	0.048077	2.13964	0.229899	0.139639
40	0.53555	0.014170	0.035545	2.10099	0.15308	0.100987
50	0.52748	0.010595	0.027475	2.07729	0.121251	0.077292
75	0.51558	0.006237	0.01558	2.04823	0.076914	0.048226
100	0.51240	0.004369	0.012403	2.03701	0.058677	0.037014
150	0.50847	0.002812	0.008465	2.02662	0.036508	0.026617
175	0.50715	0.002399	0.007150	2.02133	0.031071	0.021327
200	0.50624	0.002113	0.006238	2.01885	0.027346	0.018850

Table (4) The simulation results for the Case I

n	$\hat{\delta}$	$MSE(\delta)$	$Bias(\delta)$	$\hat{\lambda}$	$MSE(\lambda)$	$Bias(\lambda)$
30	3.53651	2.35888	0.536515	0.78146	0.014358	0.031456
40	3.38133	1.46840	0.381335	0.77210	0.010040	0.022101
50	3.32773	1.08446	0.327735	0.76914	0.008399	0.019142
75	3.17431	0.573069	0.174314	0.76086	0.005339	0.010857
100	3.13061	0.363478	0.130611	0.75872	0.003588	0.008715
150	3.11464	0.274776	0.114642	0.75764	0.002680	0.007642
175	3.09989	0.214687	0.099887	0.75728	0.002237	0.007277
200	3.08706	0.179063	0.087061	0.75625	0.001841	0.006249

5. Applications

In this section, we investigate a real data set to show how the TL-A distribution can be used. Nassar and Nada (Nassar & Nada, 2011) examined a data set containing the actual tax information, from January 2006 to November 2010, the data represents Egypt's monthly actual tax income (in 1000 million Egyptian pounds). The data are as follows: 5.9, 20.4, 14.9, 16.2, 17.2, 7.8, 6.1, 9.2, 10.2, 9.6, 13.3, 8.5, 21.6, 18.5, 5.1, 6.7, 17, 8.6, 9.7, 39.2, 35.7, 15.7, 9.7, 10, 4.1, 36, 8.5, 8, 9.2, 26.2, 21.9, 16.7, 21.3, 35.4, 14.3, 8.5, 10.6, 19.1, 20.5, 7.1, 7.7, 18.1, 16.5, 11.9, 7, 8.6, 12.5, 10.3, 11.2, 6.1, 8.4, 11, 11.6, 11.9, 5.2, 6.8, 8.9, 7.1, 10.8.

To demonstrate the TL-A distribution's flexibility; the goodness of fit criterion for the proposed model is compared to the fit of the following lifetime distributions: gamma (GM), generalized Rayleigh (GR) by Kundu and Raqab (2005), Weibull (W) by Weibull (1951), power Ailamujia (PA) and Ailamujia (A).

The MLE technique is used to estimate the distribution parameters for all fitted models [Table (5)]. The following information criterion (IC) metrics are employed for each model: Akaike's IC (AIC), Bayesian IC (BIC) and Hannan-Quinn IC (HQIC). Also, the Anderson-Darling (AD), Cramér-Von Mises (CV), and the Kolmogorov Smirnov (KS) statistics and their p-values. In particular, the best model for fitting the data is the one with smaller value for these metrics and a higher p-value for the K-S statistic. The estimated density and estimated survival are also depicted.

Table (5) The MLE of the estimated parameters for the tax data

Model	Parameters Estimate	
$\text{TL-A}(\delta,\lambda)$	$\hat{\delta}=1.96497$	$\hat{\lambda}=0.125684$
	(0.45099)	(0.0145518)
$GM(\alpha,\beta)$	$\hat{\alpha}=3.67824$	$\hat{\beta}=3.667$
	(0.64885)	(0.693112)
$GR(\sigma,\gamma)$	$\hat{\sigma}=0.0644648$	$\hat{\gamma}=1.03096$
	(0.00569)	(0.184452)
$W(\eta,\nu)$	$\hat{\eta}=0.0653337$	$\hat{\nu}=1.84037$
	(0.00491428)	(0.171157)
$PA(\tau,\lambda)$	$\hat{\tau}=1.30231$	$\hat{\lambda}=0.0636523$
	(0.118822)	(0.0222917)
$A(\lambda)$	-	$\hat{\lambda}=0.148278$
	-	(0.0136501)

Tables (6-7) compares the fit of the TL-A distribution to other models. Figure (4) depicts the estimated density and estimated survival plots for the proposed model. According to these tables and figure, the proposed model outperforms all other distributions in terms of fit. As a result, the TL-A distribution could be regarded as the best data fitting model.

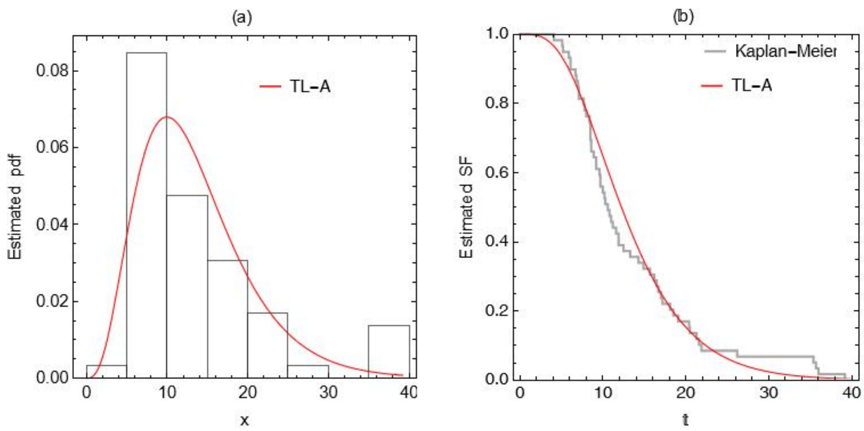

Figure (4): (a) The estimated pdf and (b) The estimated sf for tax data

Table (6) The AIC, BIC, HQC of each model for tax data

Model	AIC	BIC	HQC
$\text{TL-A}(\delta,\lambda)$	389.369	393.524	390.991
$\text{GM}(\alpha,\beta)$	390.164	394.319	391.786
$\text{GR}(\sigma,\gamma)$	399.393	403.548	401.015
$\text{W}(\eta,\nu)$	398.581	402.736	400.203
$\text{PA}(\tau,\lambda)$	393.109	397.264	394.731
$\text{A}(\lambda)$	398.193	400.270	399.004

Table (7) The values of AD, CV, K-S and their p-value of each model for tax data

Model	AD	CV	K-S	P-value
$TL-A(\delta,\lambda)$	1.2357	0.206047	0.131677	0.257948
$GM(\alpha,\beta)$	1.24717	0.204723	0.133632	0.242722
$GR(\sigma,\gamma)$	2.31655	0.400026	0.176353	0.0509594
$W(\eta,\nu)$	1.86497	0.282737	0.143165	0.177978
$PA(\tau,\lambda)$	1.4109	0.218055	0.134641	0.235132
$A(\lambda)$	2.33705	0.346263	0.167479	0.073043

6. Conclusions

In this article, we suggest and explore the Topp Leone-Ailamujia (TL-A) distribution, as a new extension of the Ailamujia distribution. The proposed model is more adaptable than the Ailamujia distribution because it includes bathtub and reversed J-shaped failure rates. The TL-A distribution's mathematical quantities, such as the rth moment, moment generating function, sth incomplete moment, conditional moments, mean deviation, mean residual life, mean inactivity times, and entropy, are clearly explained. The maximum likelihood method is used to estimate the TL-A parameters, Also, the simulation study is done to investigate the behavior of the estimate. The TL-A distribution fits a set of real data better than some well-known competing models.

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The Topp-Leone Ailamujia Distribution: Properties & Applications

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الملخص

في هذه الورقة البحثية تم اقتراح ودراسة توزيع Topp Leone-Ailamujia كامتداد جديد لتوزيع Ailamujia وقدم تم اشتقاقه باستخدام عائلة توب لين. يتميز التوزيع الجديد بأنه أكثر مرونة من التوزيع الأصلي، تم دراسة العديد من الخصائص الرياضية لنموذج المقترح كذلك تم دراسة بعض خصائص الموثوقية. قمنا باستخدام طريقة الأماكن الأعظم لتقدير معالم التوزيع الجديد. كما تم استخدام أسلوب محاكاة Mont Carlo لدراسة سلوك المعالم المقدرة، أخيراً تم استخدام عينة من البيانات الحقيقة وذلك لمقارنة التوزيع الجديد مع بعض توزيعات الحياة الكلاسيكية وقد أثبت التوزيع الجديد المرونة على نمذجة بيانات الحياة أفضل من التوزيعات المستخدمة في المقارنة.

الكلمات المفتاحية: توزيع Burr–Hatke؛ دالة معدل الخطر ؛ العزوم ؛ تحليل البواقى؛ تقدير الامكان الاعظم؛ محاكاة Mont Carlo.