Parameters and Reliability Estimation of Alpha Power Exponential Distribution under Type-II Progressive Hybrid Censoring with Applications of Engineering and Management Fields

Presented by

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Abstract

The mixture of Type-I and Type-II censoring schemes, called the hybrid censoring scheme (HCS) is quite common in life testing or reliability experiments. Recently Type-II progressive censoring scheme (Type-II PCS) becomes quite popular for analyzing highly reliable data. One drawback of the Type-II PCS is that the length of the experiment can be quite large. In this paper we introduce the estimating problems of the unknown parameters of alpha power exponential distribution (APED) using Type-II progressive hybrid censoring scheme (Type-II PCS) will be considered. The maximum likelihood estimation (MLE) and Bayesian estimations of the unknown parameters based on both squared error loss (SE) and LINEX loss functions are obtained. We propose to apply the Markov Chain Monte Carlo (MCMC) technique to carry out a Bayes estimation procedure. The approximate and credible confidence intervals for the unknown parameters are obtained. Also, we introduced the estimating problems of reliability and hazard rate function of the APED under Type-II PHCS and the corresponding approximate confidence intervals. Finally, all the theoretical results obtained are assessed and compared using two real datasets, coming from engineering and management fields.

Keywords: Maximum likelihood ; Alpha Power Exponential Distribution ; Type-II Progressive Hybrid Censoring ; Bayesian estimations

1. Introduction

Recently, Mahdavi and Kundu (2017) introduced a new method for generating distribution and applied the proposed method to generate a new extension of the exponential distribution which called APED. They studied the statistical properties of the APED and used the method of maximum likelihood to estimate the unknown parameters under the complete sample. For the APED the estimation procedures available in the literature are not capable to include censored data.

The random variable *X* is said to have a two parameter APED denoted by APED (α, λ) as $\alpha > 0$ and $\lambda > 0$, then its cumulative distribution function of *X*, for $x > 0$ is given by

$$F(x;\alpha,\lambda) = \begin{cases} \dfrac{\alpha^{(1-e^{-\lambda x})}-1}{\alpha-1} & if\ \alpha \neq 1 \\ 1-e^{-\lambda x} & if\ \alpha = 1 \end{cases} \qquad (1)$$

The probability density function corresponding to (1) is given by

$$f(x;\alpha,\lambda) = \begin{cases} \dfrac{\log\alpha}{\alpha-1}\lambda e^{-\lambda x}\alpha^{1-e^{-\lambda x}} & if\ \alpha \neq 1 \\ \lambda e^{-\lambda x} & if\ \alpha = 1 \end{cases} \qquad (2)$$

The survival function *S(x)*, and the hazard rate function *h(x)*, for the APED for *x >0*, are in the following forms

$$S(x;\alpha,\lambda) = \begin{cases} \dfrac{\alpha}{\alpha-1}(1-\alpha^{-e^{-\lambda x}}) & if\ \alpha \neq 1 \\ e^{-\lambda x} & if\ \alpha = 1 \end{cases} \qquad (3)$$

and

$$h(x;\alpha,\lambda)=\begin{cases}\dfrac{\lambda e^{-\lambda x}\alpha^{-e^{-\lambda x}}\log\alpha}{1-\alpha^{-e^{-\lambda x}}} & if\ \alpha\neq 1\\ \lambda & if\ \alpha=1\end{cases} \tag{4}$$

Mahdavi and Kundu (2017), studied the statistical properties of the APED including, moments, moment generating function, stress strength parameter, order statistics. They also studied the shape behavior of the density and the hazard rate functions. They showed that the APED is more flexible to model life time data than some well-known distributions like Weibull and gamma based on a real data example. Nassar et al. (2020) used method of moments, method of percentile, maximum product of spacing method, weighted least squares, methods of L-moments, method of Anderson-Darling, ordinary least square and method of cramer-von-Mises to estimate the parameters of the APED. Salah (2020) studied parameters estimation of APED under progressive Type-II censored data using the MLE, Fisher information matrix and derived the approximate best linear unbiased estimators for the unknown parameters. Salah et al. (2021) considered parameters estimation of APED under type-II hybrid censored sample using the MLE by using the Newton-Raphson method and expectation maximizations algorithm. Also, they evaluated the estimate reliability and hazard functions by applying the invariance property of MLEs. In addition, the fisher information matrix is computed by applying the missing information rule to finding the asymptotic confidence interval.

Even though supposition on progressive Type-II censored (PCS -Type II) trials has continued to be examined in the past researches quite extensively for a while, PCS -Type II owns one major problem; the time taken to perform the experiment can be very immense. As a result of this, Kundu and Joarder (2006) pioneered censoring scheme known as a Type-II PHCS, an amalgamation of PCS -Type II and hybrid censoring schemes. The Type-II PHCS allows for the life investigating tests to conclude at a time T specified earlier. Extensive study in regard to Type-II PHCS and its significance, allude to Kundu and Joarder (2006) and Childs et al. (2007). This progressively hybrid scheme in the last few years has also become more accepted in studying reliability and life-investigating tests.

Suppose n identical items are put on a test and the lifetime distributions of the n items are denoted by $X_{1:n},.....,X_{n:n}$. The integer $r<n$ is fixed at the beginning of the experiment, and $R_1,R_2,....,R_r$ are r prefixed integers satisfying $R_1+R_2+....+R_r+r=n$. The time point T is also fixed beforehand. At the time of the first failure X_1,R_1 of the remaining units are randomly removed. Similarly, at the time of the second failure X_2,R_2 of the remaining units are removed and so on. If $X_r>T$, then the experiment is terminated at the r^{th} failure with the withdrawals occurring after each failure according to the prespecified progressive censoring scheme ($R_1,R_2,....,R_r$). However, if $X_r<T$, then instead of terminating the experiment by removing the remaining R_r units after the r^{th} failure, we continue to observe failures (without any further withdrawals) up to time T. Therefore, $R_r=R_{r+1}=...=R_D=0$,where $X_D<T<X_{D+1}$ and X_D is the D^{th} failure time occur before time T, and D denote the number of failures that occur before the time point T, the observed data will be one of the following two forms

$$Case\ I: \quad X_1 < X_2 < \ldots < X_r \qquad\qquad if \quad X_r \geq T,$$

Or

$$Case\ II: \quad X_1 < X_2 < \ldots < X_r < X_{r+1} < \ldots < X_D \qquad if \quad X_r < T.$$

Childs (2007) proposed the likelihood function of the observed data (without constant term) as follow:

$$L \propto \begin{cases} \prod_{i=1}^{r} f(x_i)[1-F(x_i)]^{R_i} & Case\ I \\ \prod_{i=1}^{r} f(x_i)[1-F(x_i)]^{R_i} \prod_{i=r+1}^{D} f(x_i)[1-F(T)]^{R_D^*} & Case\ II \end{cases} \tag{5}$$

where R_D^* is the number of remaining units left at the time point T for case II.

Mokhtari et al. (2011) carried out a deduction on Weibull distribution beneath Type-II PHCS data. Also Lin et al. (2011) estimated parameters of generalized Rayleigh distribution in regard to Type-II PHCS. Salem and Abo-Kasem (2011) considered approximation of parameters of exponentiated Weibull distribution based on Type-II PHCS prototypes. Recently, Ma Yongming and ShiYimin (2013) studied the inference of Lomax distribution grounded onto Type-II PHCS. They evaluated estimates due to the parameters employing the ML technique and made a comparison with those obtained using the Bayesian approaches. For more literature and outcome in regard to Type-II PHCS; refer to Hemmati and Khorram (2013), Bhattacharya et al. (2014), Cho et al. (2015), Chan et al. (2015), Li and Huang (2011), Chan et al. (2015), Gorny and Cramer (2016), El-Sherpieny et al. (2020) and Abuel Fotouh et al. (2021) among others.

The aim of this paper is the estimation of the unknown parameters, reliability and hazard rate functions of APED under Type-II PHCS .In section 2, The MLEs and the information matrix will be discussed to obtain asymptotic confidence intervals for the parameters and estimate reliability and hazard rate functions. Further, Bayesian estimation using SE and LINEX loss functions will be discussed in section 3. Finally a numerical proposed methods using two real data sets is compared in Section 4.

2. The Maximum Likelihood Estimation

In this section, MLE and its information matrix for the unknown parameters of the APED (2) will be obtained using Type-II PHCS (5).

The likelihood function is given by

$$L=\begin{cases} Case\ I:\ c\left[\prod_{i=1}^{r}\frac{\lambda\ln\alpha}{\alpha-1}e^{-\lambda x_i}\ \alpha^{1-e^{-\lambda x_i}}\right]\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i} \\ Case\ II:c\left[\prod_{i=1}^{r}\frac{\lambda\ln\alpha}{\alpha-1}e^{-\lambda x_i}\ \alpha^{1-e^{-\lambda x_i}}\right]\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}\left[\prod_{i=r+1}^{D}\frac{\lambda\ln\alpha}{\alpha-1}e^{-\lambda x_i}\ \alpha^{1-e^{-\lambda x_i}}\right]\left(\frac{\alpha-\alpha^{1-e^{-\lambda T}}}{\alpha-1}\right)^{R_D^*} \end{cases} \tag{6}$$

By taking the natural logarithm of the likelihood function (6), we get

$$\ln L=\begin{cases} Case\ I:\ln c+r\ln\left(\frac{\lambda\ln\alpha}{\alpha-1}\right)-\sum_{i=1}^{r}\lambda x_i+r\ln\alpha-\ln\alpha\sum_{i=1}^{r}e^{-\lambda x_i}+R_i\ln\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right) \\ Case\ II:\ln c+r\ln\left(\frac{\lambda\ln\alpha}{\alpha-1}\right)-\sum_{i=1}^{r}\lambda x_i+r\ln\alpha-\ln\alpha\sum_{i=1}^{r}e^{-\lambda x_i}+R_i\ln\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right) \\ \qquad +D\ln\left(\frac{\lambda\ln\alpha}{\alpha-1}\right)-\sum_{i=r+1}^{D}\lambda x_i+D\ln\alpha-\ln\alpha\sum_{i=r+1}^{D}e^{-\lambda x_i}+R_D^*\ln\left(\frac{\alpha-\alpha^{1-e^{-\lambda T}}}{\alpha-1}\right) \end{cases} \tag{7}$$

The MLEs $\hat{\alpha}$ and $\hat{\lambda}$ can be obtained by equating the partial differentiation of equation (7) with respect to α and λ to zero. The partial differentiation of $\ln L$ are given by

$$\frac{\partial \ln L}{\partial \alpha} = \begin{cases} Case\ I : r\left(\frac{\frac{1}{\alpha}}{\ln\alpha} - \frac{1}{\alpha-1}\right) + \frac{r}{\alpha} - \frac{1}{\alpha}\sum_{i=1}^{r} e^{-\lambda x_i} + R_i \left(\frac{1-(1-e^{-\lambda x_i})\alpha^{-e^{-\lambda x_i}}}{\alpha - \alpha^{1-e^{-\lambda x_i}}} - \frac{1}{\alpha-1}\right) = 0, \\ Case\ II : (r+D)\left(\frac{\frac{1}{\alpha}}{\ln\alpha} - \frac{1}{\alpha-1}\right) + \frac{1}{\alpha}(r+D) - \frac{1}{\alpha}\left(\sum_{i=1}^{r} e^{-\lambda x_i} + \sum_{i=r+1}^{D} e^{-\lambda x_i}\right) \\ \qquad + R_i \left(\frac{1-(1-e^{-\lambda x_i})\alpha^{-e^{-\lambda x_i}}}{\alpha - \alpha^{1-e^{-\lambda x_i}}} - \frac{1}{\alpha-1}\right) + R_D^* \left(\frac{1-(1-e^{-\lambda T})\alpha^{-e^{-\lambda T}}}{\alpha - \alpha^{1-e^{-\lambda T}}} - \frac{1}{\alpha-1}\right) = 0 \end{cases} \tag{8}$$

and

$$\frac{\partial \ln L}{\partial \lambda} = \begin{cases} Case\ I : \frac{r}{\lambda} - \sum_{i=1}^{r} x_i + \ln\alpha \sum_{i=1}^{r} x_i e^{-\lambda x_i} - R_i \left(\frac{\alpha^{1-e^{-\lambda x_i}} \ln\alpha (x_i e^{-\lambda x_i})}{\alpha - \alpha^{1-e^{-\lambda x_i}}}\right) = 0, \\ Case\ II : \frac{(r+D)}{\lambda} - \sum_{i=1}^{r} x_i - \sum_{i=r+1}^{D} x_i + \ln\alpha \left(\sum_{i=1}^{r} x_i e^{-\lambda x_i} + \sum_{i=r+1}^{D} x_i e^{-\lambda x_i}\right) \\ \qquad - R_i \left(\frac{\alpha^{1-e^{-\lambda x_i}} \ln\alpha (x_i e^{-\lambda x_i})}{\alpha - \alpha^{1-e^{-\lambda x_i}}}\right) - R_D^* \left(\frac{\alpha^{1-e^{-\lambda T}} \ln\alpha (T e^{-\lambda T})}{\alpha - \alpha^{1-e^{-\lambda T}}}\right) = 0 \end{cases} \tag{9}$$

Since the equations (8) and (9) after equating them to zero are clearly transcendental equations in $\hat{\alpha}$ and $\hat{\lambda}$ that is, no closed form solutions are known they must be solved by iterative numerical techniques to provide solutions (estimates), $\hat{\alpha}$ and $\hat{\lambda}$, in the desired degree of accuracy.

By using the property of invariance (replacing α and λ by their MLEs $\hat{\alpha}$ and $\hat{\lambda}$) ,we can obtain the MLE of the survival and hazard function from equations (3) and (4) by

$$\hat{S}(x) = \frac{\hat{\alpha} - \hat{\alpha}^{1-e^{-\hat{\lambda}x}}}{\hat{\alpha}-1} \quad \text{and} \quad \hat{h}(x) = \frac{\hat{\lambda} e^{-\hat{\lambda}x} \hat{\alpha}^{-e^{-\hat{\lambda}x}} \ln\alpha}{1-\hat{\alpha}^{-e^{-\hat{\lambda}x}}} \tag{10}$$

To study the variation of the MLEs $\hat{\alpha}$ and $\hat{\lambda}$, the asymptotic variance of these estimators are obtained. The asymptotic variance covariance matrix of $\hat{\alpha}$ and $\hat{\lambda}$ is obtained by inverting the information matrix with elements that are negative expected values of the second order derivatives of natural logarithms of the likelihood function, for sufficiently large samples, a reasonable approximation to the asymptotic variance covariance matrix of the estimators can be obtained as

$$I^{-1}(\hat{\alpha},\hat{\lambda}) \cong \begin{bmatrix} -\dfrac{\partial^2 \ln L}{\partial \alpha^2} & -\dfrac{\partial^2 \ln L}{\partial \lambda \partial \alpha} \\ -\dfrac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & -\dfrac{\partial^2 \ln L}{\partial \lambda^2} \end{bmatrix}^{-1}_{(\hat{\alpha},\hat{\lambda})} \cong \begin{bmatrix} Var(\hat{\alpha}) & Cov(\hat{\alpha},\hat{\lambda}) \\ Cov(\hat{\alpha},\hat{\lambda}) & Var(\hat{\lambda}) \end{bmatrix} \qquad (11)$$

The elements of the previous sample information matrix can be obtained such that

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = \begin{cases} Case\ I: r\left[\dfrac{-(1+\ln\alpha)}{(\alpha\ln\alpha)^2} + \dfrac{1}{(\alpha-1)^2}\right] - \dfrac{r}{\alpha^2} + \dfrac{1}{\alpha^2}\displaystyle\sum_{i=1}^{r} e^{-\lambda x_i} \\ \qquad + R_i\left[\dfrac{\left(\alpha-\alpha^{1-e^{-\lambda x_i}}\right)\left(e^{-\lambda x_i}(1-e^{-\lambda x_i})\alpha^{-e^{-\lambda x_i}}-1\right) - \left(1-(1-e^{-\lambda x_i})\alpha^{-e^{-\lambda x_i}}\right)^2}{(\alpha-\alpha^{1-e^{-\lambda x_i}})^2} + \dfrac{1}{(\alpha-1)^2}\right] \\ Case\ II: (r+D)\left[\dfrac{-(1+\ln\alpha)}{(\alpha\ln\alpha)^2} + \dfrac{1}{(\alpha-1)^2}\right] - \dfrac{1}{\alpha^2}(r+D) + \dfrac{1}{\alpha^2}\left(\displaystyle\sum_{i=1}^{r} e^{-\lambda x_i} + \sum_{i=r+1}^{D} e^{-\lambda x_i}\right) \\ \qquad + R_i\left[\dfrac{\left(\alpha-\alpha^{1-e^{-\lambda x_i}}\right)\left(e^{-\lambda x_i}(1-e^{-\lambda x_i})\alpha^{-e^{-\lambda x_i}}-1\right) - \left(1-(1-e^{-\lambda x_i})\alpha^{-e^{-\lambda x_i}}\right)^2}{\left(\alpha-\alpha^{1-e^{-\lambda x_i}}\right)^2} + \dfrac{1}{(\alpha-1)^2}\right] \\ \qquad + R_D^*\left[\dfrac{\left(\alpha-\alpha^{1-e^{-\lambda T}}\right)\left(e^{-\lambda T}(1-e^{-\lambda T})\alpha^{-e^{-\lambda T}}-1\right) - \left(1-(1-e^{-\lambda T})\alpha^{-e^{-\lambda T}}\right)^2}{\left(\alpha-\alpha^{1-e^{-\lambda T}}\right)^2} + \dfrac{1}{(\alpha-1)^2}\right] \end{cases}$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = \begin{cases} Case\ I: \dfrac{-r}{\lambda^2} - \ln\alpha \sum_{i=1}^{r} x_i^2 e^{-\lambda x_i} + R_i \left[\dfrac{\psi}{(\alpha - \alpha^{1-e^{-\lambda x_i}})^2} \right] \\ Case\ II: \dfrac{-(r+D)}{\lambda^2} - \ln\alpha \left(\sum_{i=1}^{r} x_i^2 e^{-\lambda x_i} + \sum_{i=r+1}^{D} x_i^2 e^{-\lambda x_i} \right) + R_i \left[\dfrac{\psi}{(\alpha - \alpha^{1-e^{-\lambda x_i}})^2} \right] + R_D^* \left[\dfrac{\delta}{(\alpha - \alpha^{1-e^{-\lambda T}})^2} \right] \end{cases}$$

and

$$\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} = \begin{cases} Case\ I: \dfrac{1}{\alpha} \sum_{i=1}^{r} x_i e^{-\lambda x_i} - R_i \dfrac{\psi}{\left(\alpha - \alpha^{1-e^{-\lambda x_i}}\right)^2}, \\ Case\ II: \dfrac{1}{\alpha} \left(\sum_{i=1}^{r} x_i e^{-\lambda x_i} + \sum_{i=r+1}^{D} x_i e^{-\lambda x_i} \right) - R_i \dfrac{\psi}{\left(\alpha - \alpha^{1-e^{-\lambda x_i}}\right)^2} - R_D^* \dfrac{\delta}{\left(\alpha - \alpha^{1-e^{-\lambda T}}\right)^2} \end{cases} \quad (12)$$

where

$$\psi = \ln\alpha x_i e^{-\lambda x_i} \alpha^{1-e^{-\lambda x_i}} \left[(\alpha - \alpha^{1-e^{-\lambda x_i}})(\ln\alpha e^{-\lambda x_i} - 1)(x_i) + (\ln\alpha x_i e^{-\lambda x_i} \alpha^{1-e^{-\lambda x_i}}) \right]$$

and $\delta = \ln\alpha T e^{-\lambda T} \alpha^{1-e^{-\lambda T}} \left[(\alpha - \alpha^{1-e^{-\lambda T}})(\ln\alpha e^{-\lambda T} - 1)(T) + (\ln\alpha T e^{-\lambda T} \alpha^{1-e^{-\lambda T}}) \right]$

Diagonal elements of $I^{-1}(\hat{\alpha}, \hat{\lambda})$ provides the asymptotic variance of α and λ respectively. Then using large sample theory a two sided $100(1-\vartheta)\%$ approximate confidence interval (ACI) for α can be constructed as $\hat{\alpha} \pm Z_{1-\vartheta/2}\sqrt{Var(\hat{\alpha})}$ and similarly, for λ the two sided $100(1-\vartheta)\%$ approximate confidence interval can be obtained as $\hat{\lambda} \pm Z_{1-\vartheta/2}\sqrt{Var(\hat{\lambda})}$.

To construct the ACIs of $S(x)$ and $h(x)$, The variances of them is needed Therefore, the delta method is considered to obtain the approximate estimates of the variance of $\hat{S}(x)$ and $\hat{h}(x)$. Delta method is a general approach for computing ACIs for any function of the MLEs $\hat{\alpha}$ and $\hat{\lambda}$, (See

Greene (2012)). According to this method, the variance of $\hat{S}(x)$ and $\hat{h}(x)$, can be approximated, by

$$\hat{\sigma}^2_{\hat{S}(x)} = [\nabla \hat{S}(x)]^T I_0^{-1} [\nabla \hat{S}(x)] \quad \textbf{and} \quad \hat{\sigma}^2_{\hat{h}(t)} = [\nabla \hat{h}(t)]^T I_0^{-1} [\nabla \hat{h}(t)]$$

respectively, where the gradient vector of first partial derivatives of $S(x)$ and $h(x)$ with respect to α and λ obtained at $\hat{\alpha}$ and $\hat{\lambda}$ are given by

$$[\nabla \hat{S}(x)]^T = \left[\frac{\partial \nabla \hat{S}(x)}{\partial(\alpha)}, \frac{\partial \nabla \hat{S}(x)}{\partial(\lambda)} \right]_{(\hat{\alpha},\hat{\lambda})} \quad \text{and} \quad [\nabla \hat{H}(t)]^T = \left[\frac{\partial \nabla \hat{h}(x)}{\partial(\alpha)}, \frac{\partial \nabla \hat{h}(x)}{\partial(\lambda)} \right]_{(\hat{\alpha},\hat{\lambda})}$$

Hence, the $100(1-\vartheta)\%$ ACIs of $S(x)$ and $h(x)$, are given by

$$\hat{S}(x) \pm Z_{1-\vartheta/2}\sqrt{\hat{\sigma}^2_{\hat{S}(x)}} \quad \text{and} \quad \hat{h}(x) \pm Z_{1-\vartheta/2}\sqrt{\hat{\sigma}^2_{\hat{h}(x)}}$$

respectively.

3. Bayesian Estimation

In this section, Bayesian method is used to obtain the estimators for the unknown parameters of APED using SE and LINEX loss functions. We consider the gamma prior distribution for the parameters α and λ as

$$\begin{aligned} \pi_1(\alpha) &\propto \alpha^{a_1-1} e^{-b_1\alpha} \quad , \qquad \alpha, a_1, b_1 > 0 \\ \pi_2(\lambda) &\propto \lambda^{a_2-1} e^{-b_2\lambda} \quad , \qquad \lambda, a_2, b_2 > 0 \end{aligned} \tag{13}$$

Combining (13) with equation (6) and using Bayes theorem, the joint posterior distribution can be obtained as follow:

$$\pi(\alpha,\lambda\backslash \underline{x})=\begin{cases} Case\ I:\dfrac{1}{\omega_1}\lambda^{r+a_2-1}\alpha^{r+a_1-1}e^{b_1\alpha}e^{b_2\lambda}\prod\limits_{i=1}^{r}\dfrac{\ln\alpha}{\alpha-1}\prod\limits_{i=1}^{r}e^{-\lambda x_i}\prod\limits_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\dfrac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i} \\ Case\ \ II:\dfrac{1}{\omega_2}\lambda^{r+D+a_2-1}\alpha^{r+D+a_1-1}e^{b_1\alpha}e^{b_2\lambda}\prod\limits_{i=1}^{r}\dfrac{\ln\alpha}{\alpha-1}\prod\limits_{i=1}^{r}e^{-\lambda x_i}\prod\limits_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\dfrac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i} \\ \qquad\times\prod\limits_{i=r+1}^{D}\dfrac{\ln\alpha}{\alpha-1}\prod\limits_{i=r+1}^{D}e^{-\lambda x_i}\prod\limits_{i=r+1}^{D}\alpha^{-e^{-\lambda x_i}}\left(\dfrac{\alpha-\alpha^{1-e^{-\lambda T}}}{\alpha-1}\right)^{R^*_D} \end{cases}$$

where

$$\omega_1=\int_0^\infty\int_0^\infty\lambda^{r+a_2-1}\alpha^{r+a_1-1}e^{b_1\alpha}e^{b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}d\alpha d\lambda \text{ and}$$

$$\omega_2=\int_0^\infty\int_0^\infty\lambda^{r+D+a_2-1}\alpha^{r+D+a_1-1}e^{b_1\alpha}e^{b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}\prod_{i=r+1}^{D}\frac{\ln\alpha}{\alpha-1}$$

$$\times\prod_{i=r+1}^{D}e^{-\lambda x_i}\prod_{i=r+1}^{D}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda T}}}{\alpha-1}\right)^{R^*_D}d\alpha d\lambda$$

Marginal posteriors of a parameter is obtained by integrating the joint posterior distribution with respect to the other parameter and hence the marginal posterior of α can be written, after simplification, as

$$\pi(\alpha\backslash x)=\begin{cases} Case\ I:\dfrac{\omega_3}{\omega_1} \\ Case\ \ II:\dfrac{\omega_4}{\omega_2} \end{cases},$$

where

$$\omega_3=\alpha^{r+a_1-1}e^{b_1\alpha}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\int_0^\infty\lambda^{r+a_2-1}e^{b_2\lambda}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}d\lambda \text{ and}$$

$$\omega_4 = \alpha^{r+D+a_1-1} e^{b_1\alpha} \prod_{i=1}^{r} \frac{\ln\alpha}{\alpha-1} \prod_{i=r+1}^{D} \frac{\ln\alpha}{\alpha-1} \int_0^{\infty} \lambda^{r+D+a_2-1} e^{b_2\lambda} \prod_{i=1}^{r} e^{-\lambda x_i} \prod_{i=1}^{r} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda x_i}}}{\alpha-1} \right)^{R_i}$$
$$\times \prod_{i=r+1}^{D} e^{-\lambda x_i} \prod_{i=r+1}^{D} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda T}}}{\alpha-1} \right)^{R^*_D} d\lambda$$

Similarly integrating the joint posterior with respect to α, the marginal posterior of λ can be obtained as

$$\pi(\lambda \setminus x) = \begin{cases} Case\ I: \dfrac{\omega_5}{\omega_1} \\ Case\ II: \dfrac{\omega_6}{\omega_2} \end{cases},$$

where

$$\omega_5 = \lambda^{r+a_2-1} e^{b_2\lambda} \prod_{i=1}^{r} e^{-\lambda x_i} \int_0^{\infty} \alpha^{r+a_1-1} e^{b_1\alpha} \prod_{i=1}^{r} \frac{\ln\alpha}{\alpha-1} \prod_{i=1}^{r} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda x_i}}}{\alpha-1} \right)^{R_i} d\alpha \quad \text{and}$$

$$\omega_6 = \lambda^{r+D+a_2-1} e^{b_2\lambda} \prod_{i=1}^{r} e^{-\lambda x_i} \prod_{i=r+1}^{D} e^{-\lambda x_i} \int_0^{\infty} \alpha^{r+D+a_1-1} e^{b_1\alpha} \prod_{i=1}^{r} \frac{\ln\alpha}{\alpha-1} \prod_{i=r+1}^{D} \frac{\ln\alpha}{\alpha-1} \prod_{i=1}^{r} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda x_i}}}{\alpha-1} \right)^{R_i}$$
$$\times \prod_{i=r+1}^{D} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda T}}}{\alpha-1} \right)^{R^*_D} d\alpha$$

Under SE loss function the Bayes estimators for parameters α and λ of APED are

$$\hat{\alpha}_{SE} = \begin{cases} Caes\ I: \dfrac{\omega_7}{\omega_1} \\ Case\ II: \dfrac{\omega_9}{\omega_2} \end{cases} \quad \text{and} \quad \hat{\lambda}_{SE} = \begin{cases} Case\ I: \dfrac{\omega_8}{\omega_1} \\ Case\ II: \dfrac{\omega_{10}}{\omega_2} \end{cases}. \qquad (14)$$

where

$$\omega_7=\int_0^\infty\int_0^\infty \lambda^{r+a_2-1}\alpha^{r+a_1}e^{b_1\alpha}e^{b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}d\alpha\, d\lambda\ ,$$

$$\omega_8=\int_0^\infty\int_0^\infty \lambda^{r+a_2}\alpha^{r+a_1-1}e^{b_1\alpha}e^{b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}d\alpha\, d\lambda\ ,$$

$$\omega_9=\int_0^\infty\int_0^\infty \lambda^{r+D+a_2-1}\alpha^{r+D+a_1}e^{b_1\alpha}e^{b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}\prod_{i=r+1}^{D}\frac{\ln\alpha}{\alpha-1}\prod_{i=r+1}^{D}e^{-\lambda x_i}$$
$$\times\prod_{i=r+1}^{D}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda T}}}{\alpha-1}\right)^{R^*_D}d\alpha\, d\lambda$$

and

$$\omega_{10}=\int_0^\infty\int_0^\infty \lambda^{r+D+a_2}\alpha^{r+D+a_1-1}e^{b_1\alpha}e^{b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}\prod_{i=r+1}^{D}\frac{\ln\alpha}{\alpha-1}\prod_{i=r+1}^{D}e^{-\lambda x_i}$$
$$\times\prod_{i=r+1}^{D}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda T}}}{\alpha-1}\right)^{R^*_D}d\alpha\, d\lambda$$

and the form of reliability function and hazard function (for $\alpha\neq 1$) are given as the following equations,

$$\hat{S}(x)_{SE}=\begin{cases}Case\ I:\int_\alpha\int_\lambda\frac{\alpha^{r+a_1}}{\alpha-1}(1-\alpha^{-e^{-\lambda x}})\frac{1}{\omega_1}\lambda^{r+a_2-1}e^{b_1\alpha+b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}d\alpha\, d\lambda\\ Case\ II:\int_\alpha\int_\lambda\frac{\alpha^{r+D+a_1}}{\alpha-1}(1-\alpha^{-e^{-\lambda x}})\frac{1}{\omega_2}\lambda^{r+D+a_2-1}e^{b_1\alpha+b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}\\ \quad\times\prod_{i=r+1}^{D}\frac{\ln\alpha}{\alpha-1}\prod_{i=r+1}^{D}e^{-\lambda x_i}\prod_{i=r+1}^{D}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda T}}}{\alpha-1}\right)^{R^*_D}d\alpha\, d\lambda\end{cases}\qquad(15)$$

and

$$\hat{h}(x)_{SE}=\begin{cases}Case\ I:\int_{\alpha}\int_{\lambda}\frac{\alpha^{r+a_1-e^{-\lambda x}-1}\log\alpha}{1-\alpha^{-e^{-\lambda x}}}\frac{1}{\omega_1}\lambda^{r+a_2}e^{-\lambda x+b_1\alpha+b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}d\alpha\, d\lambda\\ Case\ II:\int_{\alpha}\int_{\lambda}\frac{\alpha^{r+D+a_1-e^{-\lambda x}-1}\log\alpha}{1-\alpha^{-e^{-\lambda x}}}\frac{1}{\omega_2}\lambda^{r+D+a_2}e^{-\lambda x+b_1\alpha+b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}\\ \qquad\times\prod_{i=r+1}^{D}\frac{\ln\alpha}{\alpha-1}\prod_{i=r+1}^{D}e^{-\lambda x_i}\prod_{i=r+1}^{D}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda T}}}{\alpha-1}\right)^{R^*_D}d\alpha\, d\lambda\end{cases}\quad(16)$$

respectively.

Following Zellner (1986), the Bayes estimators under LINEX loss function are

$$\hat{\alpha}_{LINEX}=-\frac{1}{c}\ln(E\,(\mathrm{e}^{-c\,\alpha}))\ \text{and}\ \hat{\lambda}_{LINEX}=-\frac{1}{c}\ln(E\,(\mathrm{e}^{-c\,\lambda}))$$

respectively, where $E(\cdot)$ denotes the posterior expectation. These estimators for parameters of APED α and λ can be expressed as

$$\hat{\alpha}_{LINEX}=\begin{cases}Case\ I:-\frac{1}{c}\ln(\frac{\omega_{11}}{\omega_1})\\ Case\ II:-\frac{1}{c}\ln(\frac{\omega_{13}}{\omega_2})\end{cases}\quad\text{and}\quad\hat{\lambda}_{LINEX}=\begin{cases}Case\ I:-\frac{1}{c}\ln(\frac{\omega_{12}}{\omega_1})\\ Case\ II:-\frac{1}{c}\ln(\frac{\omega_{14}}{\omega_2})\end{cases}.\qquad(17)$$

where

$$\omega_{11}=\int_0^{\infty}\int_0^{\infty}\lambda^{r+a_2-1}\alpha^{r+a_1-1}e^{b_1\alpha-c\alpha+b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}d\alpha\, d\lambda\ ,$$

$$\omega_{12}=\int_0^{\infty}\int_0^{\infty}\lambda^{r+a_2-1}\alpha^{r+a_1-1}e^{b_1\alpha+b_2\lambda-c\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i}\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}d\alpha\, d\lambda\ ,$$

$$\omega_{13} = \int_0^\infty \int_0^\infty \lambda^{r+D+a_2-1} \alpha^{r+D+a_1-1} e^{b_2\lambda + b_1\alpha - c\alpha} \prod_{i=1}^{r} \frac{\ln\alpha}{\alpha-1} \prod_{i=1}^{r} e^{-\lambda x_i} \prod_{i=1}^{r} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda x_i}}}{\alpha-1} \right)^{R_i}$$
$$\times \prod_{i=r+1}^{D} \frac{\ln\alpha}{\alpha-1} \prod_{i=r+1}^{D} e^{-\lambda x_i} \prod_{i=r+1}^{D} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda T}}}{\alpha-1} \right)^{R^*_D} d\alpha\, d\lambda ,$$

and

$$\omega_{14} = \int_0^\infty \int_0^\infty \lambda^{r+D+a_2-1} \alpha^{r+D+a_1-1} e^{b_1\alpha + b_2\lambda - c\lambda} \prod_{i=1}^{r} \frac{\ln\alpha}{\alpha-1} \prod_{i=1}^{r} e^{-\lambda x_i} \prod_{i=1}^{r} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda x_i}}}{\alpha-1} \right)^{R_i}$$
$$\times \prod_{i=r+1}^{D} \frac{\ln\alpha}{\alpha-1} \prod_{i=r+1}^{D} e^{-\lambda x_i} \prod_{i=r+1}^{D} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda T}}}{\alpha-1} \right)^{R^*_D} d\alpha\, d\lambda$$

and form of reliability function and hazard function (for $\alpha \neq 1$) are given as the following

$$\hat{S}(x)_{LINEX} = \begin{cases} Case\ I: -\frac{1}{c}\ln \int_\alpha \int_\lambda \frac{\alpha^{r+a_1}}{\alpha-1}(1-\alpha^{-e^{-\lambda x}}) \frac{1}{\omega_1} \lambda^{r+a_2-1} e^{-c\alpha - c\lambda b_1\alpha + b_2\lambda} \prod_{i=1}^{r} \frac{\ln\alpha}{\alpha-1} \prod_{i=1}^{r} e^{-\lambda x_i} \\ \quad \times \prod_{i=1}^{r} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda x_i}}}{\alpha-1} \right)^{R_i} d\alpha\, d\lambda \\ Case\ II: -\frac{1}{c}\ln \int_\alpha \int_\lambda \frac{\alpha^{r+D+a_1}}{\alpha-1}(1-\alpha^{-e^{-\lambda x}}) \frac{1}{\omega_2} \lambda^{r+D+a_2-1} e^{-c\alpha - c\lambda b_1\alpha + b_2\lambda} \prod_{i=1}^{r} \frac{\ln\alpha}{\alpha-1} \prod_{i=1}^{r} e^{-\lambda x_i} \\ \quad \times \prod_{i=1}^{r} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda x_i}}}{\alpha-1} \right)^{R_i} \prod_{i=r+1}^{D} \frac{\ln\alpha}{\alpha-1} \prod_{i=r+1}^{D} e^{-\lambda x_i} \prod_{i=r+1}^{D} \alpha^{-e^{-\lambda x_i}} \left(\frac{\alpha - \alpha^{1-e^{-\lambda T}}}{\alpha-1} \right)^{R^*_D} d\alpha\, d\lambda \end{cases} \quad (18)$$

and

$$\hat{h}(x)_{LINEX}=\begin{cases} Case\ I:-\frac{1}{c}\ln\int\limits_{\alpha}\int\limits_{\lambda}\frac{\alpha^{r+a_1-e^{-\lambda x}-1}\log\alpha}{1-\alpha^{-e^{-\lambda x}}}\frac{1}{\omega_1}\lambda^{r+a_2}e^{-c\alpha-c\lambda-\lambda x+b_1\alpha+b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i} \\ \qquad\times\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}d\alpha\, d\lambda \\ Case\ II:\int\limits_{\alpha}\int\limits_{\lambda}\frac{\alpha^{r+D+a_1-e^{-\lambda x}-1}\log\alpha}{1-\alpha^{-e^{-\lambda x}}}\frac{1}{\omega_2}\lambda^{r+D+a_2}e^{-c\alpha-c\lambda-\lambda x+b_1\alpha+b_2\lambda}\prod_{i=1}^{r}\frac{\ln\alpha}{\alpha-1}\prod_{i=1}^{r}e^{-\lambda x_i} \\ \qquad\times\prod_{i=1}^{r}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda x_i}}}{\alpha-1}\right)^{R_i}\prod_{i=r+1}^{D}\frac{\ln\alpha}{\alpha-1}\prod_{i=r+1}^{D}e^{-\lambda x_i}\prod_{i=r+1}^{D}\alpha^{-e^{-\lambda x_i}}\left(\frac{\alpha-\alpha^{1-e^{-\lambda T}}}{\alpha-1}\right)^{R^*_D}d\alpha\, d\lambda \end{cases} \tag{19}$$

respectively.

Equations (14), (15), (16), (17), (18) and (19) in general cannot be obtained in a closed form, so the approximate methods is employed. MCMC using MH algorithm has been used to carry out the Bayes estimates and also to construct the associate HPD credible intervals by using an R code program

4. Real Data Applications

This section deals with analyzing two real datasets, coming from engineering and management fields, to show the adaptability of the methodologies proposed in practical situations.

4.1 Electronic devices data

This application is devoted to illustrating the applicability and flexibility of the proposed distribution by analyzing a real-life data set from the engineering field. The data set, reported by Wang (2000), consists of 18 observations of failure times of electronic devices. These failure times have been ordered and listed in Table 1.

Table 1. Failure times of electronic devices.

5, 11, 21, 31, 46, 75, 98, 122, 145, 165, 196, 224, 245, 293, 321, 330, 350, 420

One question arises whether the data fit APED or not. For this purpose, based on an electronic devices dataset, the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D) and Cramér-von Mises CvM) goodness-of-fit test statistics with associated p-values are used. In addition, using selection measures reported in Table 2, the APED is compared to five competing lifetime models, namely; Lomax distribution (LD), gamma distribution (GD), Weibull distribution (WD), exponentiated-Weibull distribution (EWD) and generalized-exponential distribution (GED). The corresponding PDFs of the competing models (for $x > 0$) are written in Table 3.

Table 2. Some of useful selection measures.

Measure	Abbreviate	Formula
Negative log-likelihood criterion $\hat{\mathcal{L}}$	**NLC**	$-\log\left(L(\cdot)\big|_{\theta=\hat{\theta}}\right)$
Akaike's information criterion	**AIC**	$2(\tau+\hat{\mathcal{L}})$
Consistent Akaike's information criterion	**CAIC**	$2\hat{\mathcal{L}}+(2n\tau/(n-\tau-1))$
Bayesian information criterion	**BIC**	$2\hat{\mathcal{L}}+\tau\log(n)$
Hannan-Quinn information criterion	**HQIC**	$2(\tau\log(\log(n))+\hat{\mathcal{L}})$

Clearly, n and τ, as in Table 2, represent the sample size and the number of model parameters, respectively. Obviously, the best distribution corresponds to the lowest value of these criteria and highest p-value.

Table 3. Some competing models of the APE distribution.

Model	PDF	Author(s)
LD	$f(x)=\alpha\lambda(1+\lambda x)^{-(\alpha+1)}$	Lomax (1954)
GD	$f(x)=(\lambda^{\alpha}/\Gamma(\alpha))x^{\alpha-1}\exp(-\lambda x)$	Johnson et al. (1994)
WD	$f(x)=\alpha\lambda x^{\alpha-1}\exp(-\lambda x^{\alpha})$	Weibull (1951)
EWD	$f(x)=\alpha\lambda x^{\alpha-1}e^{-x^{\alpha}}(1-e^{-x^{\alpha}})^{\lambda-1}$	Mudholkar & Srivastava (1993)
GED	$f(x)=\alpha\lambda\exp(-\lambda x)\left(1-\exp(-\lambda x)\right)^{\alpha-1}$	Gupta and Kundu (2001)

To estimate the parameters of the considered distributions and also to evaluate the goodness-of-fit selection measures, the 'Adequacy Model' package proposed by Marinho et al. (2019) is implemented via R statistical programming software. However, the calculated MLEs with their standard errors (SEs) of the model parameters and corresponding selection measures are computed and listed in Table 4. It shows that the APE lifetime model is the best distribution among all fitted competitive models under electronic devices dataset, since it has the smallest goodness of statistic values and highest p-value.

Table 4. Summary fit based on electronic devices dataset.

Model	**MLE(SE)**		**NLC**	**AIC**	**CAIC**	**BIC**	**HQIC**	**K-S (p-value)**	**A-D**	**CvM**
	α	λ								
APED	**3.0804 (3.8588)**	**0.0074 (0.0022)**	**110.32**	**224.64**	**225.44**	**226.43**	**224.89**	**0.1035 (0.979)**	**0.3366**	**0.0487**
LD	**3.1887 (0.9886)**	**0.0023 (0.0006)**	**111.95**	**227.90**	**228.70**	**229.68**	**228.14**	**0.1534 (0.735)**	**0.5408**	**0.0850**
GD	**1.1156 (0.3214)**	**0.0065 (0.0023)**	**110.60**	**225.21**	**226.01**	**226.99**	**225.45**	**0.1208 (0.927)**	**0.3974**	**0.0597**
WD	**1.1458 (0.0702)**	**0.0026 (0.0008)**	**110.45**	**224.89**	**225.69**	**226.67**	**225.14**	**0.1132 (0.955)**	**0.3649**	**0.0540**
EWD	**18.734 (5.9207)**	**0.2603 (0.0222)**	**113.33**	**230.65**	**231.45**	**232.43**	**230.90**	**0.1713 (0.607)**	**0.8329**	**0.1381**
GED	**1.0915 (0.3312)**	**0.0061 (0.0017)**	**110.63**	**225.25**	**226.05**	**227.03**	**225.50**	**0.1214 (0.925)**	**0.4005**	**0.0602**

Note: The best model is corresponding to bold values.

Moreover, for goodness-of-fit of distributions using graphical presentation method, we draw quantile-quantile (Q-Q) plots of the competitive models using depicts the points $\{F^{-1}((i-0.5)/n;\hat{\theta}), x_{(i)}\}$, $i=1,2,...,n$, where $\hat{\theta}$ is the MLE of θ, which are shown in Figure 1. From Tables 8 and 9, it can be seen that the APE model is the best model comparing with other fitted models in the literature for fitting lifetime real data, since it has the smallest goodness of statistic values and highest p-values. One may also show that the APED is a good competitor and may be used as an alternative to the considered distributions. Furthermore, the Q-Q plots support our findings. For more fitting illustration, in Figure 2, we have also provided two plots computed at the estimated model parameters of APED, LD, GD, WD, EWD and GED; Plot (a) represents the histogram of the electronic devices data and the fitted PDFs, Plot (b) represents the fitted and empirical survival functions. It is observed that, from Figure 2, the graphical presentations support our numerical findings.

To prove the existence and uniqueness of the MLEs $\hat{\alpha}$ and $\hat{\lambda}$ of α and λ, respectively, the contour plot of the log-likelihood function with respect to the two-parameter APED using the complete electronic devices dataset as is plotted and displayed in Figure 3. The maximum of the log-likelihood function is denoted by point x in the innermost contour. The coordinates of x-point provide the MLEs of α and λ which are becomes $\hat{\alpha} \cong 3.0805$ and $\hat{\lambda} \cong 0.0074$. Further, it shows that the MLEs are exist and are also unique.

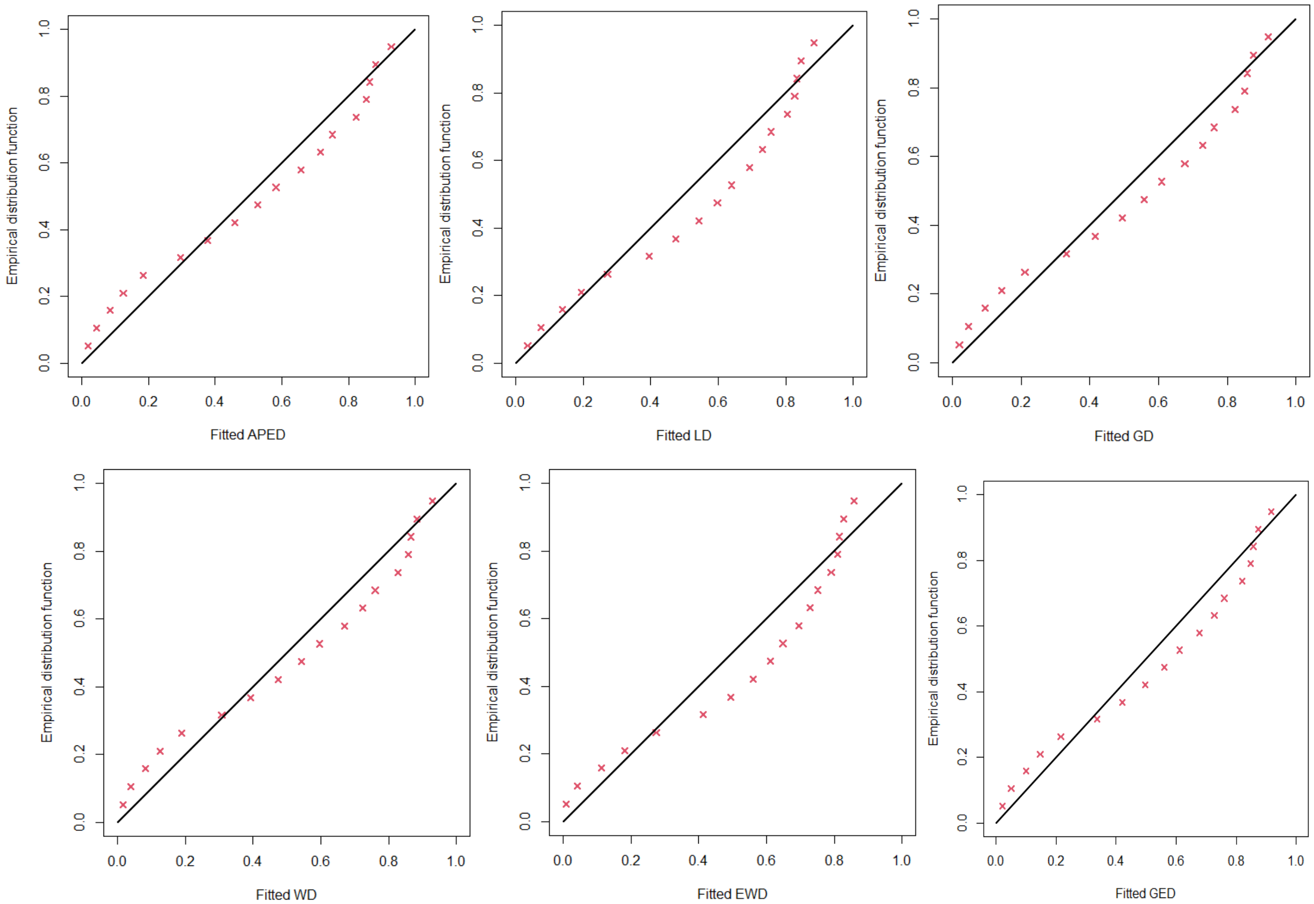
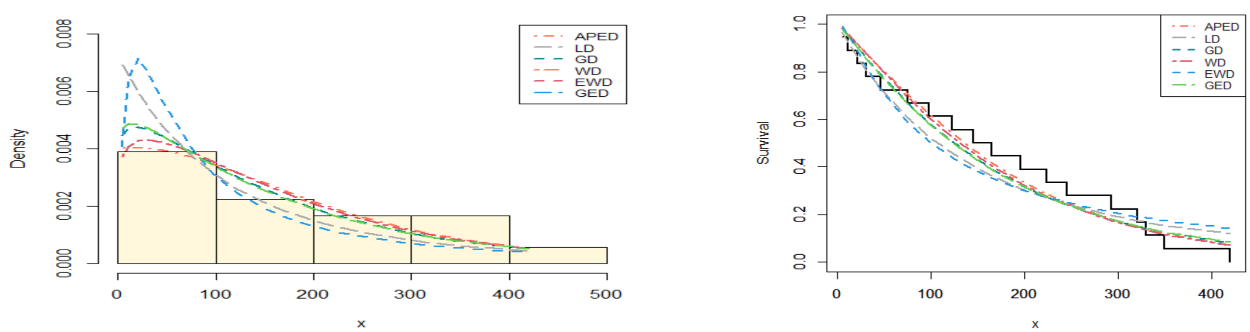


Figure 1. The Q-Q plots of the APED and various models for electronic devices dataset.



(a) (b)

Figure 2. Estimated the density and survival functions of the APED and various models for electronic devices dataset.

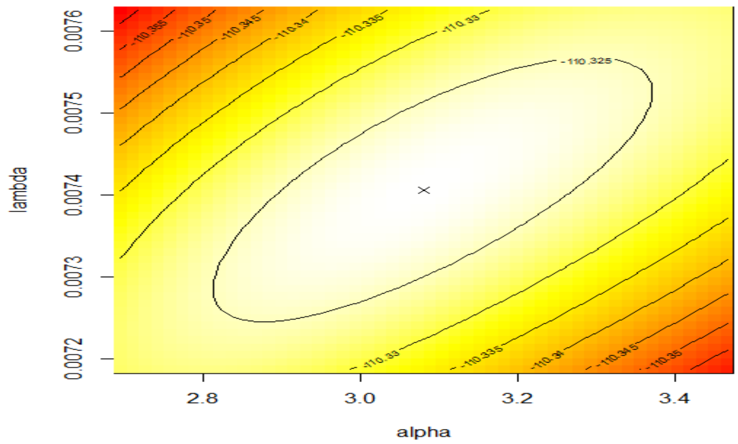


Figure 3. Contour plot of log-likelihood function for different values of α and λ for an electronic devices dataset.

Now, from the electronic devices dataset as in Table 1, various artificial data by Type-II PHCS, when $r=8$ and $R_i=1,\ i=1,2,...,r$, for different choices of threshold time T, are created and provided in Table 5. Using data sets of Table 5, the MLEs and the Bayes MCMC estimates with their SEs of the unknown parameters α and λ, as well as, the reliability characteristics $S(t)$ and $h(t)$ at given mission time $t=50$, are computed and listed in Table 6. The initial values for the unknown parameters for running the MCMC sampler algorithm were taken to be their MLEs. Moreover, two-sided 95% ACI/HPD credible intervals with their lengths are calculated and listed in Table 7.

Since there is no any prior information about the model parameters, the Bayes estimates are developed using SE and LINEX (for $\upsilon(=-5,-0.05,+5)$) loss functions under improper gamma priors, i.e., $a_i=b_i=0,\ i=1,2$. However, for computational convenience, all given hyper-parameters are set to be 0.0001. Using the MCMC algorithm, we generate 30,000 MCMC samples and then first 5000 iterations (burn-in period) have been discarded from the generated sequence. Moreover, some important characteristics such as: mean, median, mode, standard deviation (SD) and skewness (Sk.) for MCMC posterior distributions of α, λ, $S(t)$ and $h(t)$ after bun-in; are computed and provided in Table 7.

Table 5. Two Type-II PHCS samples generated from electronic devices data.

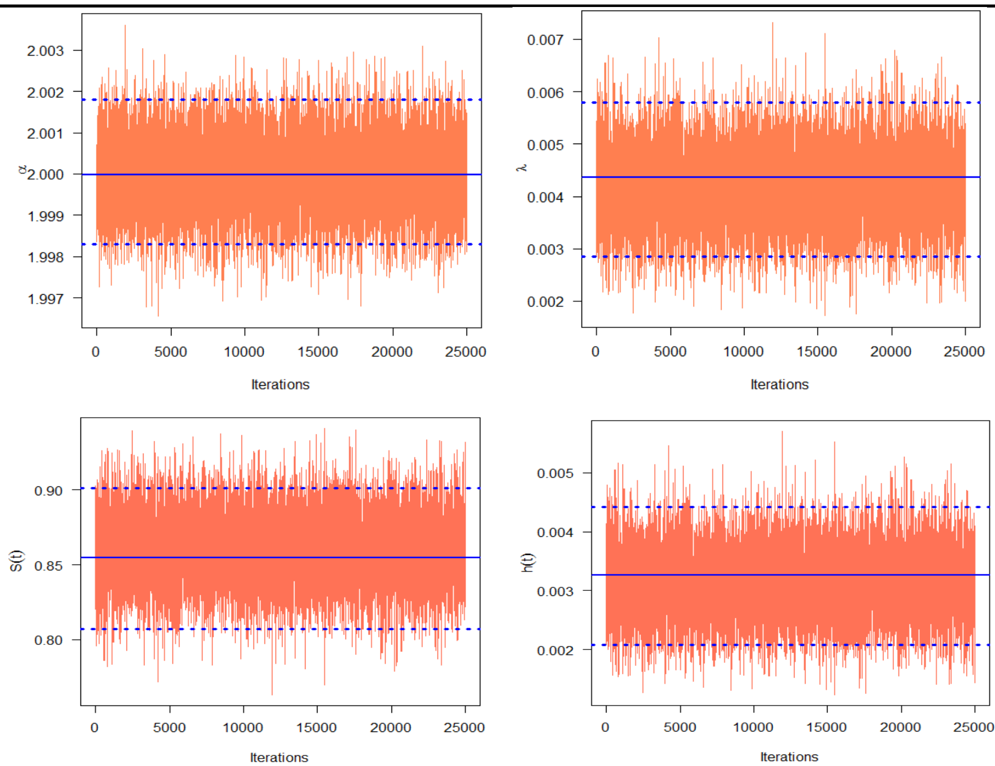
Sample	$T(D)$	R^*	Type-II PHCS samples
1	100(7)	3	5, 21, 46, 98, 145, 196, 245, 321
2	180(10)	1	5, 21, 46, 98, 145, 196, 245, 321, 330, 350

Table 6. The classical and Bayes estimates (with their SEs) of α, λ, $S(t)$ and $h(t)$ for an electronic devices dataset.

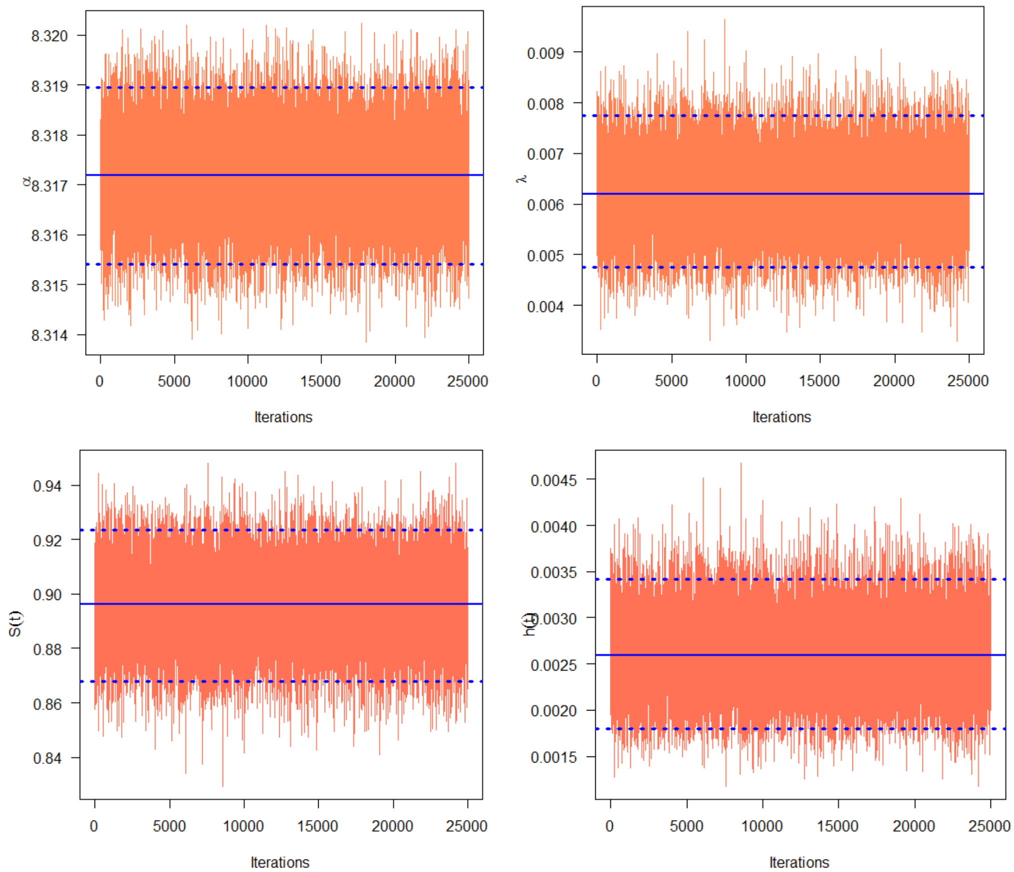
Sample	Parameter	MLE	MCMC			
			SE	LINEX		
$c \rightarrow$				-5	-0.05	+5
1	α	2.0007	1.9999	1.9998	1.9999	1.9999
		$(0.43\times10^{+1})$	(5.71×10^{-6})	(7.42×10^{-8})	(8.70×10^{-8})	(9.99×10^{-6})
	λ	0.0036	0.0044	0.0044	0.0043	0.0044
		(2.80×10^{-3})	(4.77×10^{-6})	(4.81×10^{-6})	(4.80×10^{-6})	(4.79×10^{-6})
	$S(t)$	0.8781	0.8548	0.8563	0.8548	0.8534
		(5.86×10^{-2})	(1.52×10^{-4})	(1.44×10^{-4})	(1.53×10^{-4})	(1.62×10^{-4})
	$h(t)$	0.0027	0.0033	0.0033	0.0033	0.0033
		(1.20×10^{-3})	(3.80×10^{-6})	(3.83×10^{-6})	(3.82×10^{-6})	(3.82×10^{-6})
2	α	8.3172	8.3171	8.3171	8.3172	8.3172
		$(0.71\times10^{+1})$	(5.72×10^{-6})	(4.72×10^{-8})	(6.00×10^{-8})	(7.31×10^{-8})
	λ	0.0064	0.0062	0.0062	0.0062	0.0062
		(1.80×10^{-3})	(4.81×10^{-6})	(1.24×10^{-6})	(1.25×10^{-6})	(1.26×10^{-6})
	$S(t)$	0.8926	0.8961	0.8966	0.8961	0.8956
		(4.02×10^{-2})	(8.99×10^{-5})	(2.56×10^{-5})	(2.25×10^{-5})	(1.93×10^{-5})
	$h(t)$	0.0027	0.0026	0.0026	0.0026	0.0026
		(1.00×10^{-3})	(2.61×10^{-6})	(6.08×10^{-7})	(6.11×10^{-7})	(6.13×10^{-7})

Table 7. Two-sided 95% asymptotic/credible intervals (first-line) with their lengths (second-line) of α, λ, $S(t)$ and $h(t)$ for an electronic devices dataset.

Sample	Parameter	ACI	HPD
1	α	(0.0000,10.589)	(1.9983,2.0018)
		10.589	0.0035
	λ	(0.0000,0.0091)	(0.0029,0.0058)
		0.0091	0.0029
	$S(t)$	(0.7632,0.9931)	(0.8073,0.9012)
		0.2298	0.0939
	$h(t)$	(0.0004,0.0050)	(0.0021,0.0044)
		0.0046	0.0023
2	α	(0.0000,22.156)	(8.3154,8.3190)
		22.156	0.0036
	λ	(0.0029,0.0099)	(0.0048,0.0078)
		0.0070	0.0030
	$S(t)$	(0.8138,0.9714)	(0.8678,0.9237)
		0.1576	0.0558
	$h(t)$	(0.0008,0.0046)	(0.0018,0.0034)
		0.0038	0.0016

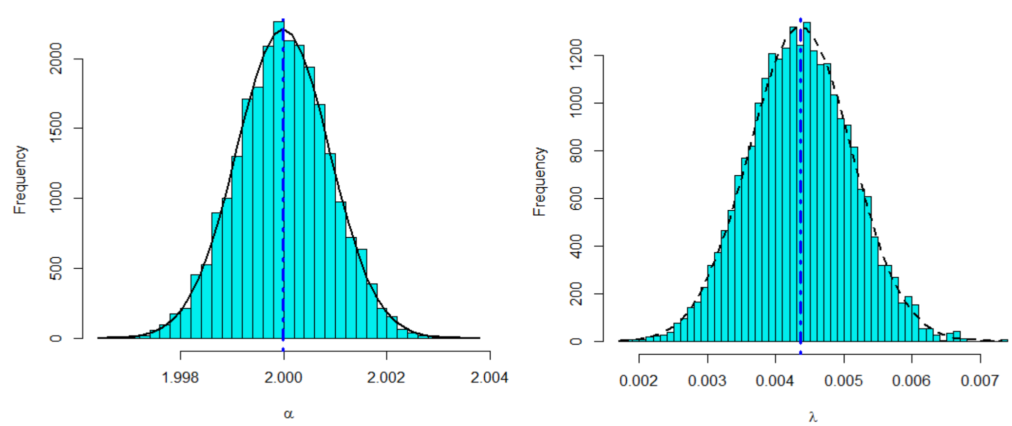


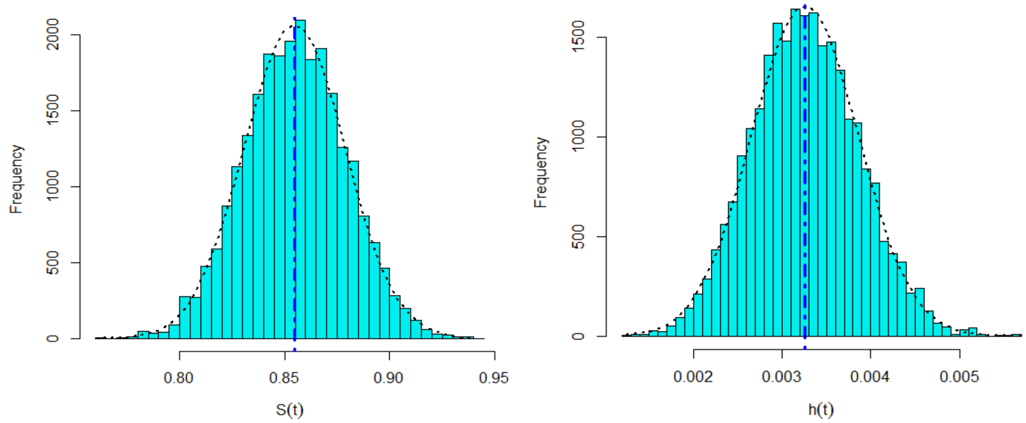
(a) Sample 1



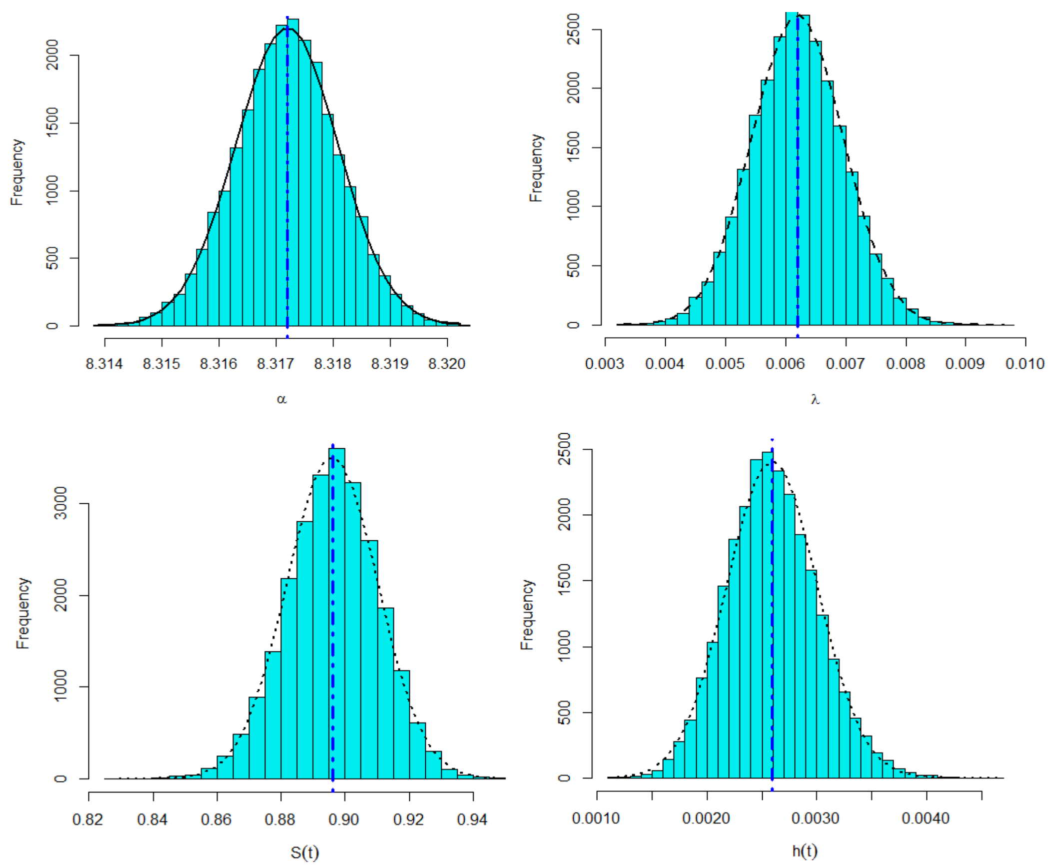
(b) Sample 2

Figure 4. MCMC trace plots of α, λ, $S(t)$ and $h(t)$ for an electronic devices dataset.





(a) Sample 1



(b) Sample 2

Figure 5. Histogram and kernel density estimates of α, λ, $S(t)$ and $h(t)$ for an electronic devices dataset.

From Tables 6-7, due to the lack of prior information about α and λ, it can be seen that the estimated results of point and interval estimates are not much different, as expected. Also, the Bayes MCMC estimates of α, λ, $S(t)$ and $h(t)$ using LINEX loss function have performed superior than those obtained based on SE function and both better than the MLEs in terms of their standard errors and confidence interval lengths. To assess the convergence of 25,000 MCMC outputs, using PHCS-TII datasets reported in Table 5, trace plots of the conditional posterior distributions of α, λ, $S(t)$ and $h(t)$ are plotted in Figure 4. In each trace plot, the sample mean (solid (—) horizontal line) and 95% HPD credible intervals (dotes (⋯) horizontal line). Further, it indicates that the MH algorithm sampler converges very well. In addition, the approximate conditional PDF of α, λ, $S(t)$ and $h(t)$ with their histograms based on 25,000 chain values using the Gaussian kernel are presented in Figure 5. Similarly, in each histogram plot, the corresponding sample mean of each unknown parameter is displayed with vertical dash-dotted line (:). It is evident from the estimates that the generated posteriors of all unknown parameters of APE model are nearly symmetrical.

Table 7. The MCMC statistics of α, λ, $S(t)$ and $h(t)$ under electronic devices dataset.

Sample	Parameter	Mean	Median	Mode	SD	Sk.
1	α	1.99998	1.99999	1.72596	9.02×10^{-4}	-0.03563
	λ	0.00436	0.00434	0.00586	7.55×10^{-4}	0.06553
	$S(t)$	0.85481	0.85506	0.80742	2.41×10^{-2}	-0.01731
	$h(t)$	0.00326	0.00325	0.00448	6.01×10^{-4}	0.12223
2	α	8.31719	8.31720	8.31533	9.04×10^{-4}	-0.00725
	λ	0.00620	0.00619	0.00568	7.61×10^{-4}	0.05651
	$S(t)$	0.89610	0.89638	0.89787	1.42×10^{-2}	-0.12040
	$h(t)$	0.00260	0.00258	0.00231	4.13×10^{-4}	0.22360

4.2 Bank service waiting-time

To illustrate the usefulness and applicability of the proposed methodologies to real-life phenomena, we consider a dataset that representing the waiting times (in minutes) before service of 100 Bank customers, see Table 8. This data was first examined by Ghitany et al. (2008) and also recently analyzed by Irshad et al. (2021).

Table 8. Data of waiting times of 100 bank customers.

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7
2.9	3.1	3.2	3.3	3.5	3.6	4.0	4.1	4.2	4.2
4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8	4.9	4.9
5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3
6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0
8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11	11	11.1	11.2	11.2	11.5
11.9	12.4	12.5	12.9	13	13.1	13.3	13.6	13.7	13.9
14.1	15.4	15.4	17.3	17.3	18.1	18.2	18.4	18.9	19
19.9	20.6	21.3	21.4	21.9	23	27	31.6	33.1	38.5

To verify if these data are modeled by the APE distribution, the K-S distance with associated p-value is considered. First, we calculate the MLEs (with their SEs) of the unknown parameter α and λ which are 21.149(14.143) and 0.1831(0.0197), respectively. Thus, the K-S distance is 0.0528 with p-value 0.943. This result indicates that the APED a suitable model to fit head-neck cancer data. Further, using the full waiting-times of bank, the contour plot of the log-likelihood function with respect to α and λ is displayed in Figure 6 in order to show the existence and uniqueness of the MLEs $\hat{\alpha}$ and $\hat{\lambda}$. The coordinates of x-point provide the MLEs of α and λ which are close to 21.149 and 0.183, respectively. Further, one can conclude that the MLEs $\hat{\alpha}$ and $\hat{\lambda}$ of α and λ, respectively, are exists and are also unique.

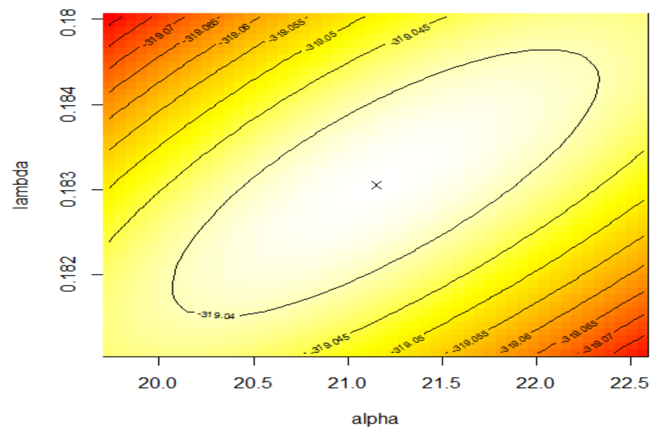


Figure 6. Contour plot of log-likelihood function for different values of α and λ under waiting-times of bank.

Using dataset in Table 1, we draw two artificial data by Type-II PHCS, when $r=20$ and $R_i=3,\ i=1,2,...,r$, for different choices of threshold time T, are generated and presented in Table 9. For both generated samples, the MLEs and the Bayes MCMC estimates with their SEs of the unknown APE parameters α and λ, as well as, the reliability characteristics $S(t)$ and $h(t)$ at given mission time $t=4$, are computed and listed in Table 10. Moreover, two-sided 95% ACI/HPD credible intervals with their lengths are calculated and listed in Table 11.

Table 9. Two Type-II PHCS samples generated from the waiting-times of bank.

Sample	$T\,(D)$	R^*	PHCS-TII samples
1	4(16)	23	0.8, 1.8, 2.6, 3.2, 4.0, 4.3, 4.6, 4.9, 5.5, 6.2, 6.7, 7.1, 7.7, 8.6, 8.9, 9.7, 11, 11.2, 12.5,13.3
2	5(30)	13	0.8, 1.8, 2.6, 3.2, 4.0, 4.3, 4.6, 4.9, 5.5, 6.2, 6.7, 7.1, 7.7, 8.6, 8.9, 9.7, 11, 11.2, 12.5, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2

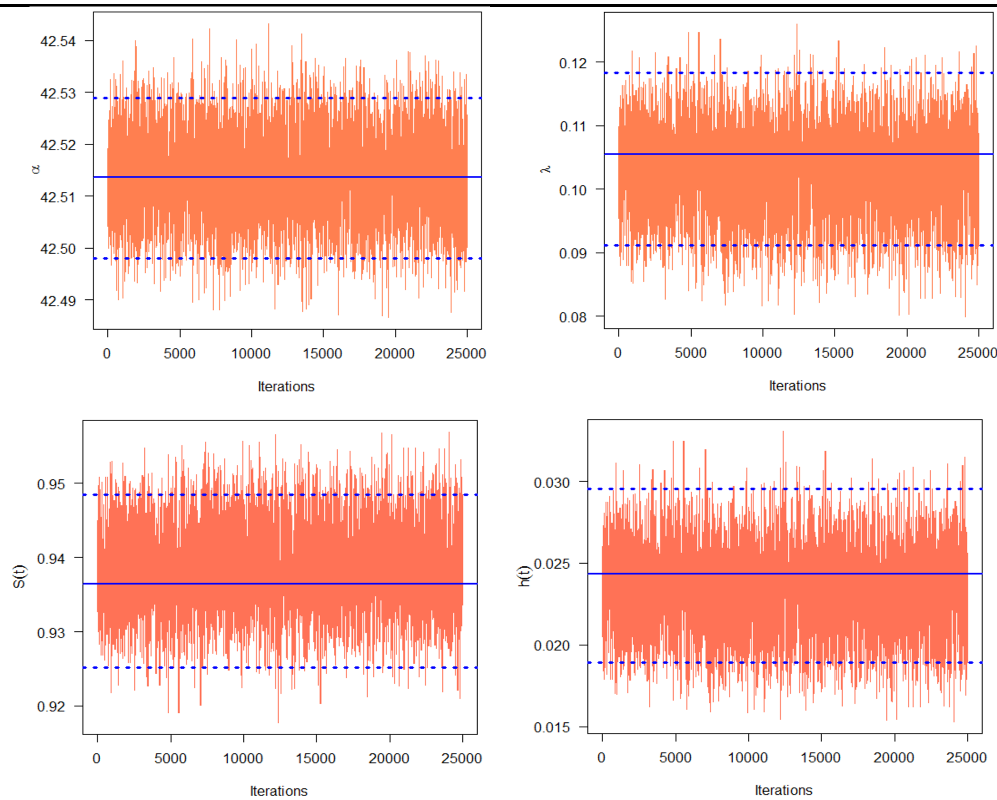
Table 10. The classical and Bayes estimates (with their SEs) of α, λ, $S(t)$ and $h(t)$ based on the waiting-times of bank.

Sample	Parameter	MLE	MCMC			
			SE	LINEX		
$c \rightarrow$				-5	-0.05	+5
1	α	42.514	42.514	42.513	42.513	42.513
		($0.60\times10^{+1}$)	(5.00×10^{-5})	(1.43×10^{-6})	(2.41×10^{-6})	(3.40×10^{-6})
	λ	0.0958	0.1055	0.1056	0.1055	0.1053
		(1.27×10^{-2})	(4.40×10^{-5})	(6.19×10^{-5})	(6.11×10^{-5})	(6.04×10^{-5})
	$S(t)$	0.9446	0.9364	0.9365	0.9364	0.9363
		(1.03×10^{-2})	(3.79×10^{-5})	(5.14×10^{-5})	(5.19×10^{-5})	(5.25×10^{-5})
	$h(t)$	0.0206	0.0243	0.0244	0.0243	0.0243
		(4.49×10^{-3})	(1.73×10^{-5})	(2.37×10^{-5})	(2.36×10^{-5})	(2.35×10^{-5})
2	α	440.04	440.04	440.39	440.39	440.39
		($0.48\times10^{+1}$)	(5.69×10^{-5})	(1.89×10^{-6})	(2.15×10^{-6})	(3.25×10^{-6})
	λ	0.1725	0.1784	0.1786	0.1784	0.1783
		(1.47×10^{-2})	(4.92×10^{-5})	(3.85×10^{-5})	(3.76×10^{-5})	(3.66×10^{-5})
	$S(t)$	0.9550	0.9513	0.9513	0.9513	0.9512
		(8.50×10^{-3})	(2.98×10^{-5})	(2.28×10^{-5})	(2.32×10^{-5})	(2.35×10^{-5})
	$h(t)$	0.0261	0.0285	0.0286	0.0285	0.0285
		(5.61×10^{-3})	(1.98×10^{-5})	(1.56×10^{-5})	(1.54×10^{-5})	(1.53×10^{-5})

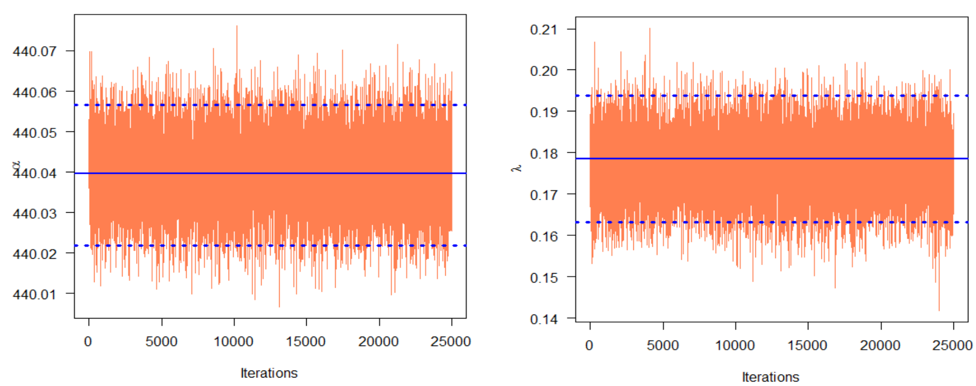
Table 11. Two-sided 95% asymptotic/credible intervals (first-line) with their lengths (second-line) of α, λ, $S(t)$ and $h(t)$ based on the waiting-times of bank.

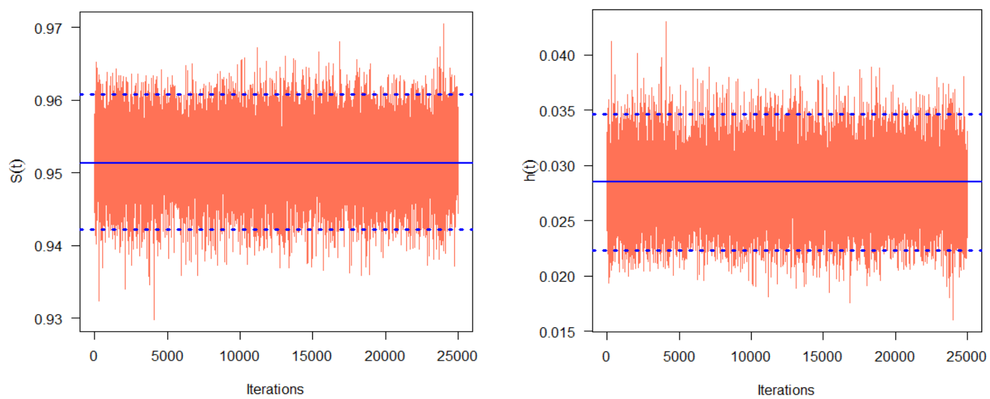
Sample	Parameter	ACI	HPD
1	α	(30.588,54.439)	(42.498,42.529)
		23.851	0.0309
	λ	(0.0708,0.1207)	(0.0912,0.1184)
		0.0499	0.0272
	$S(t)$	(0.9244,0.9649)	(0.9252,0.9485)
		0.0404	0.0233
	$h(t)$	(0.0118,0.0294)	(0.0189,0.0295)
		0.0176	0.0106
2	α	(430.55,449.54)	(440.02,440.06)
		18.986	0.0400
	λ	(0.1436,0.2013)	(0.1633,0.1939)
		0.0577	0.0306
	$S(t)$	(0.9833,0.9716)	(0.9422,0.9608)
		0.0333	0.0186
	$h(t)$	(0.0151,0.0371)	(0.0223,0.0346)

	0.0220	0.0123



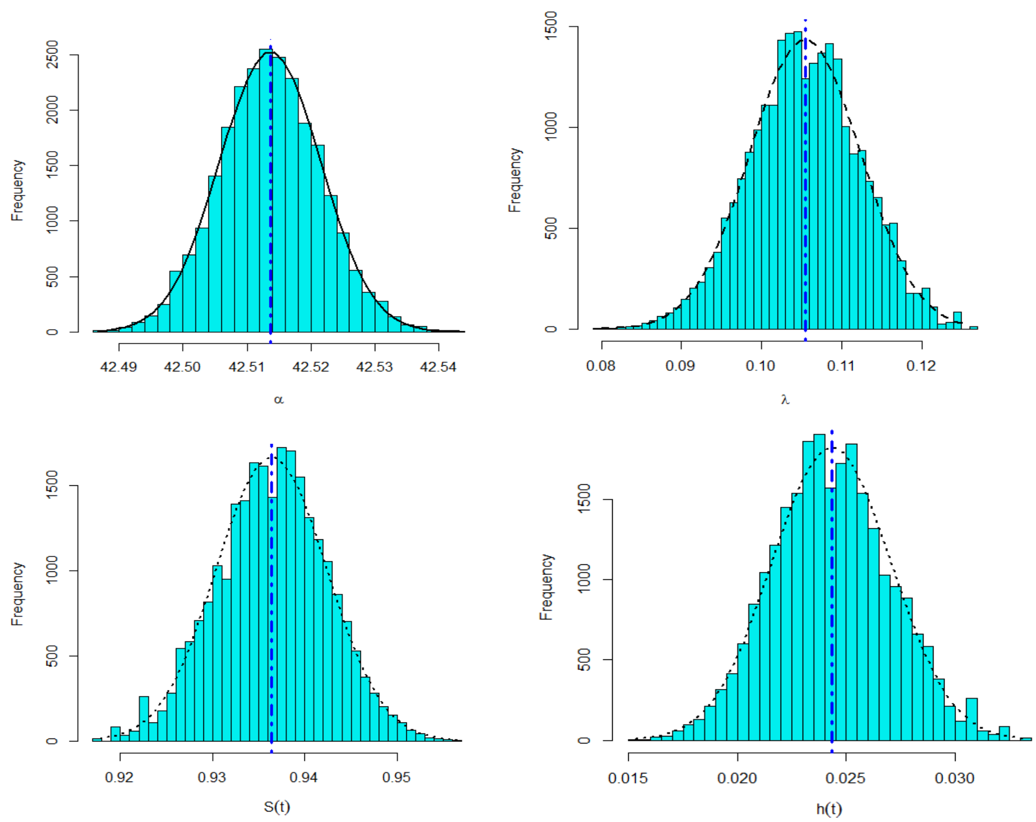
(a) Sample 1



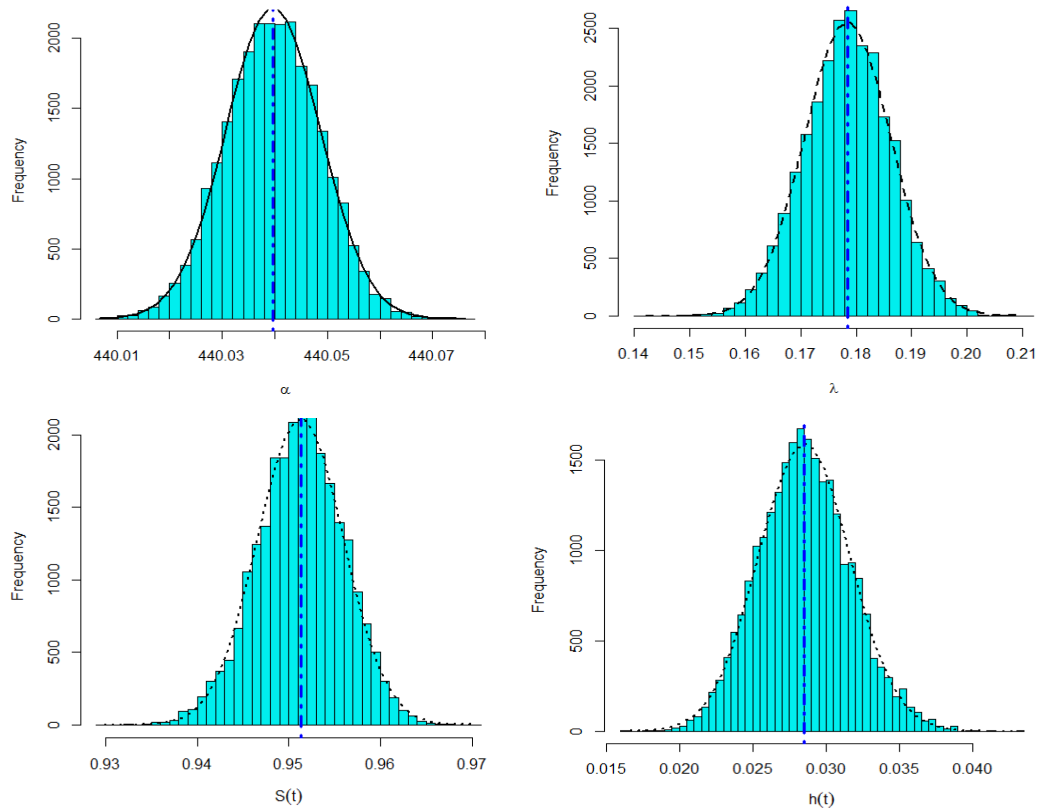


(b) Sample 2

Figure 7. MCMC trace plots of α, λ, $S(t)$ and $h(t)$ based on the waiting-times of bank.



(a) Sample 1



(b) Sample 2

Figure 8. Histogram and kernel density estimates of α, λ, $S(t)$ and $h(t)$ based on the waiting-times of bank.

Using SE and LINEX (for $\upsilon(=-5,-0.05,+5)$) loss functions, the Bayes estimates are obtained under non-informative priors, i.e., $a_i = b_i = 0.0001,\ i = 1,2$. Using the MCMC algorithm, we generate 30,000 MCMC samples and then first 5000 iterations (burn-in period) have been discarded from the generated sequence. Moreover, some important characteristics of MCMC outputs for α, λ, $S(t)$ and $h(t)$ after bun-in are computed and provided in Table 12.

All evaluations are implemented by R statistical programming language software by two useful statistical packages recommended by Elshahhat and Nassar (2021), namely; 'CODA' package used for carried out the computations of MCMC procudre proposed by Plummer et al. (2006),

’maxLik’ package which using N-R method of maximization in the computations, proposed by Henningsen and Toomet (2011).

Table 12. The MCMC statistics of α, λ, $S(t)$ and $h(t)$ based on the waiting-times of bank.

Sample	Parameter	Mean	Median	Mode	SD	Sk.
1	α	42.5136	42.5135	42.5119	7.90×10^{-3}	0.04154
	λ	0.10547	0.10539	0.12464	6.96×10^{-3}	-0.03940
	$S(t)$	0.93641	0.93659	0.91909	5.99×10^{-3}	-0.07665
	$h(t)$	0.02433	0.02422	0.03243	2.74×10^{-3}	0.13241
2	α	440.040	440.039	440.038	8.99×10^{-3}	-0.02686
	λ	0.17844	0.17851	0.19644	7.78×10^{-3}	-0.04362
	$S(t)$	0.95129	0.95139	0.93974	4.72×10^{-3}	-0.13743
	$h(t)$	0.02853	0.02846	0.00313	3.13×10^{-3}	0.16116

It can be seen that, from Tables 10-11, the estimated results of α, λ, $S(t)$ and $h(t)$ are quite close to each other, as expected. Also, in terms of minimum standard errors, the Bayes MCMC estimates have performed better than the MLEs. Also, it is observed that the length of the HPD credible interval is less than the corresponding length of the ACI. Similarly, for both samples 1 and 2 generated from the waiting-times of bank, trace plots of the conditional distributions of α, λ, $S(t)$ and $h(t)$ are plotted in Figure 7. It showed that the MCMC procedure converges very well. In addition, the approximate conditional PDF of α, λ, $S(t)$ and $h(t)$ with their histograms are also plotted in Figure 8. It is evident from the estimates that the generated posteriors of all unknown parameters of APE model are nearly symmetrical. Finally, our results based on the waiting-times of bank support our conclusion using electronic devices dataset.

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