The Additive Flexible Weibull Extension-Lomax Distribution: Properties and Estimation with Applications to COVID-19 Data

H. N. Salem, G. R. AL-Dayian, A. A. EL-Helbawy and R. E. Abd EL-Kader

Statistics Department, Faculty of Commerce,
AL-Azhar University (Girls' Branch), Cairo, Egypt

Abstract

This paper introduces a four-parameter competing risks model called the additive flexible Weibull extension-Lomax distribution. It has a very flexible hazard rate function accommodates different shapes, the most important shapes of them are the bathtub and the modified bathtub shapes. Moreover, it has several new and well-known models as special cases. Some main properties of the additive flexible Weibull extension-Lomax distribution are derived. The model parameters, reliability and hazard rate functions are estimated via the maximum likelihood method based on Type II censored samples. Also, the asymptotic confidence intervals of the parameters, reliability function and the hazard rate function are obtained. A simulation study is carried out to evaluate the performance of the maximum likelihood estimates. The superiority of the proposed distribution over some known distributions is demonstrated through some applications on COVID-19 data in some countries.

Keywords: *Competing risks, additive model, flexible Weibull extension distribution, the additive flexible Weibull extension-Lomax distribution, modified bathtub hazard shape.*

1. Introduction

In reliability studies, lifetime testing, human mortality studies, engineering modeling, electronic sciences and biological surveys, there are different types of lifetime data. So, different shapes of lifetime distributions are required for fitting these types of lifetime data. Researchers have proposed several extensions and modifications to provide more flexibility than the existing distributions. Therefore, several methods for constructing, extending and generalizing lifetime distributions are presented [see Lai (2013)], such as: the transformations of variables and distribution functions, probability integral transforms, compound distributions, finite and infinite mixture distributions.

Another method for constructing new lifetime distributions is the competing risks approach, which is based on the concept of the competing risks. In many life-testing studies, often the failure of the tested item may be associated to more than one cause or mode of failure. These failure modes in some sense compete with each other in order to cause the failure of the tested item. Due to this reason, in the statistical literature this is well known as competing risks. Moreover, competing risks arise in series systems, in which the components are arranged in series. Each component has a certain distribution with certain parameters and these components are statistically independent of each other, therefore the lifetime of the series system is the minimum of its components lifetimes. Competing risks often occurred in reliability studies, demographic, medical and biological sciences and engineering applications. Furthermore, the competing risks model is also known as series model, additive model and multi-risk model.

Based on the concept of the competing risks, there are many lifetime distributions that have been introduced in literature such as: Xie and Lai (1995) introduced the *additive Weibull* (AW) distribution. Wang (2000) presented the additive Burr XII distribution. Bousquet and Bertholon (2006) proposed a competing risks distribution, called the B distribution. Almalki and Yuan (2013) derived a *new modified Weibull* (NMW) distribution by combining the Weibull distribution with the modified Weibull distribution presented by Lai *et al.* (2003) in a series system. Cordeiro *et al.* (2013) constructed the exponential-Weibull. He *et al.* (2016) obtained the additive modified Weibull distribution. Oluyede *et al.* (2016) introduced the log-logistic Weibull distribution. Singh (2016) obtained the additive Perks-Weibull distribution. Mdlongwa *et al.* (2017) derived the Burr XII modified Weibull distribution. Tarvirdizade and Ahmadpour (2019) introduced Weibull-Chen distribution. Shakhatreh *et al.* (2019) proposed the log-normal modified Weibull distribution. Osagie and Osemwenkhae (2020) constructed the *Lomax-Weibull* (L-W) distribution. Kamal and Ismail (2020) presented the flexible Weibull extension-Burr XII distribution by combining the *flexible Weibull extension distribution* (FWE) obtained by Bebbington *et al.* (2007a) and Burr XII distribution in a series system. Thach and Bris (2021) introduced the additive Chen-Weibull distribution. Khalil *et al.* (2021) presented the flexible additive Weibull distribution by combining three Weibull distributions. Makubate *et al.* (2021) proposed the Lindley-Burr XII distribution. Abba *et al.* (2022) introduced the flexible additive Chen-Gompertz distribution by combining Chen and a special case of Gompertz distributions in a series system. Recently, Xavier *et al.* (2022) proposed the additive power-transformed half-logistic model by combining two power-transformed half-logistic distributions in a series system. More recently, Thach (2022) presented the three-component additive Weibull distribution.

This paper aims to introduce a new competing risks model, called the *additive flexible Weibull extension-Lomax* (AFWE-L) distribution, by considering a series system with two components functioning independently in series. The lifetime of the first component, X_1, has the FWE distribution and the lifetime of the second component, X_2, has *Lomax* (L) distribution. Therefore, the lifetime of the system is $X = \min\{X_1, X_2\}$ has AFWE-L distribution with four parameters. Its *hazard rate function* (hrf) can be expressed as the sum of the hrfs of the FWE and L distributions, which shows different hazard shapes, the most important shapes are the bathtub and the modified bathtub (where the failure initially increases at the beginning for a short period; maybe due to manufacturing defects, then it is followed by a bathtub shape).

The FWE distribution is a very flexible extension of Weibull distribution that was introduced by Bebbington *et al.* (2007a) as a member of the class of distributions which was presented by Gurvich *et al.* (1997). The *reliability function* (rf) and the hrf of the FWE distribution are given, respectively, by:

$$R_1(x;\alpha,\beta) = e^{-e^{\alpha x-\frac{\beta}{x}}}, \qquad x>0;\ \alpha,\beta>0, \qquad (1)$$

and

$$h_1(x;\alpha,\beta) = \left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}, \qquad x>0;\ \alpha,\beta>0, \qquad (2)$$

where α and β are shape parameters.

The flexibility of the FWE distribution is due to its hrf which has different shapes: increasing failure rate, increasing failure rate average and modified bathtub-shaped failure rate. Due to its flexibility, it has many applications in engineering, life testing experiments, applied statistics, reliability analysis and clinical studies [see Bebbington *et al.* (2007a), Bebbington *et al.* (2007b) and Choquet *et al.* (2013)].

The L distribution which is also named Pareto Type II distribution was pioneered by Lomax (1954) to model business failure data. It is known that it is a special case of Pearson Type VI distribution and it can be obtained as a compound of the exponential and gamma distributions. The L distribution belongs to the family of decreasing hrf. The L distribution has various applications in several fields such as income and wealth inequality, actuarial, medical and biological sciences, engineering, lifetime and reliability analysis. Its rf and hrf are expressed as:

$$R_2(x;\lambda,\theta) = \left(1+\frac{x}{\lambda}\right)^{-\theta}, \qquad x>0;\lambda,\theta>0, \quad (3)$$

and

$$h_2(x;\lambda,\theta) = \frac{\theta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1}, \qquad x>0;\lambda,\theta>0, \quad (4)$$

where λ is a scale parameter and θ is a shape parameter.

This paper is organized as follows: The construction of the proposed model and the graphical description of the pdf, hrf and the *reversed hazard rate function* (rhrf) of the proposed model are introduced in Section 2. In Section 3, some main properties of AFWE-L distribution are derived. The *maximum likelihood* (ML) estimators of the parameters, rf and hrf and the *asymptotic confidence intervals* (ACIs) of the parameters, rf and the hrf of AFWE-L distribution based on Type II censored samples are developed in Section 4. A simulation study is presented in Section 5 to evaluate the performance of the ML estimates. In Section 6, applications on COVID-19 data in some countries are performed to demonstrate the superiority of the proposed distribution over some known distributions.

2. The Model

In this section, the construction of the proposed model based on the hrfs and rfs of the FWE and L distributions is derived. Also, the graphical description of the pdf, hrf and rhrf of the proposed model is introduced.

The hrf of AFWE-L distribution with parameter vector $\underline{\psi} = (\alpha,\beta,\lambda,\theta)$ can be expressed as the sum of the hrfs of FWE and L distributions as follows:

$$\begin{aligned} h\left(x;\underline{\psi}\right) &= h_1(x;\alpha,\beta) + h_2(x;\lambda,\theta) \\ &= \left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}} + \frac{\theta}{\lambda}\left(1+\frac{x}{\lambda}\right)^{-1}, \qquad x>0;\ \underline{\psi}>\underline{0}, \quad (5) \end{aligned}$$

and the rf of AFWE-L distribution can be obtained as

$$R\left(x;\underline{\psi}\right) = \prod_{i=1}^{2} R_i(x) = e^{-e^{\alpha x-\frac{\beta}{x}}}\left(1+\frac{x}{\lambda}\right)^{-\theta}, \qquad x>0;\ \underline{\psi}>\underline{0}, \quad (6)$$

where

$R_1(x)$ and $R_2(x)$ are the rfs of FWE and L distributions, respectively. Consequently, the *probability density function* (pdf) of AFWE-L distribution is given by:

$$f\left(x;\underline{\psi}\right) = h\left(x;\underline{\psi}\right) R\left(x;\underline{\psi}\right)$$

Hence,

$$f\left(x;\underline{\psi}\right) = \left[\left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} + \frac{\theta}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-1}\right] e^{-e^{\alpha x - \frac{\beta}{x}}} \left(1 + \frac{x}{\lambda}\right)^{-\theta}, \quad x > 0; \underline{\psi} > \underline{0}. \quad (7)$$

The corresponding *cumulative distribution function* (cdf) of AFWE-L distribution is given by:

$$F\left(x;\underline{\psi}\right) = 1 - e^{-e^{\alpha x - \frac{\beta}{x}}} \left(1 + \frac{x}{\lambda}\right)^{-\theta}, \qquad x > 0;\ \underline{\psi} > \underline{0}. \quad (8)$$

Moreover, the rhrf and the *cumulative hazard rate function* (chrf) of AFWE-L distribution are given, respectively, as:

$$r\left(x;\underline{\psi}\right) = \frac{f\left(x;\underline{\psi}\right)}{F\left(x;\underline{\psi}\right)} = \frac{\left[\left(\alpha + \frac{\beta}{x^2}\right) e^{\alpha x - \frac{\beta}{x}} + \frac{\theta}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-1}\right] e^{-e^{\alpha x - \frac{\beta}{x}}} \left(1 + \frac{x}{\lambda}\right)^{-\theta}}{1 - e^{-e^{\alpha x - \frac{\beta}{x}}} \left(1 + \frac{x}{\lambda}\right)^{-\theta}}, \quad x > 0; \underline{\psi} > \underline{0}, \quad (9)$$

and

$$H\left(x;\underline{\psi}\right) = -\ln R\left(x;\underline{\psi}\right) = e^{\alpha x - \frac{\beta}{x}} + \theta \ln\left(1 + \frac{x}{\lambda}\right), \quad x > 0;\ \underline{\psi} > \underline{0}. (10)$$

Plots of pdf, hrf and rhrf of AFWE-L distribution are given, respectively in Figures 1-3.

Plots of the pdf, hrf and rhrf of AFWE-L distribution are provided to show the flexibility of pdf and hrf of AFWE-L distribution, which allow this distribution to fit different types of lifetime data. Figure 1 displays AFWE-L pdf for selected values of the parameters, where one can observe that the pdf of AFWE-L distribution can be decreasing, unimodal or decreasing-unimodal. Also, Figure 2 shows AFWE-L distribution hrf for some values of the parameters. The hrf of AFWE-L distribution represents major shapes such that increasing, decreasing, bathtub, bi-bathtub and modified bathtub shapes. Moreover, plots of the rhrf of AFWE-L distribution for different values of the parameters are given in Figure 3, which indicates that the rhrf of AFWE-L can be decreasing or have the reversed shape of the modified bathtub shape.

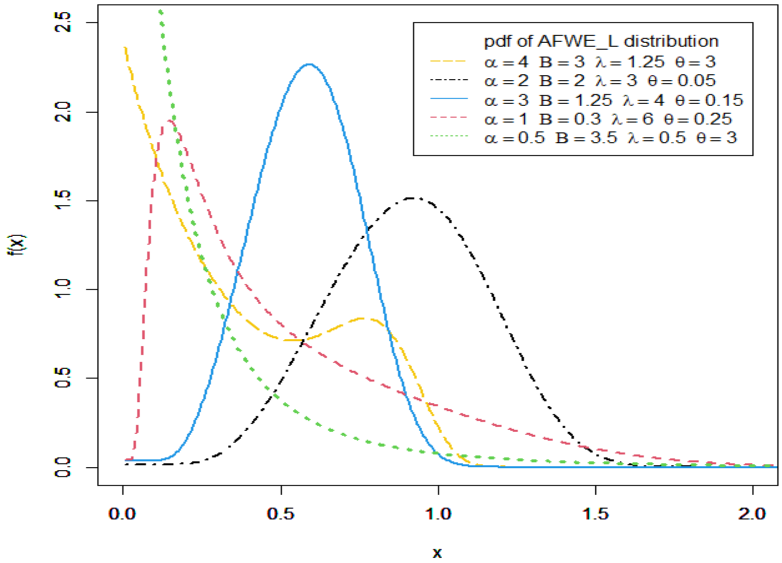

Figure 1: Plots of AFWE-L pdf

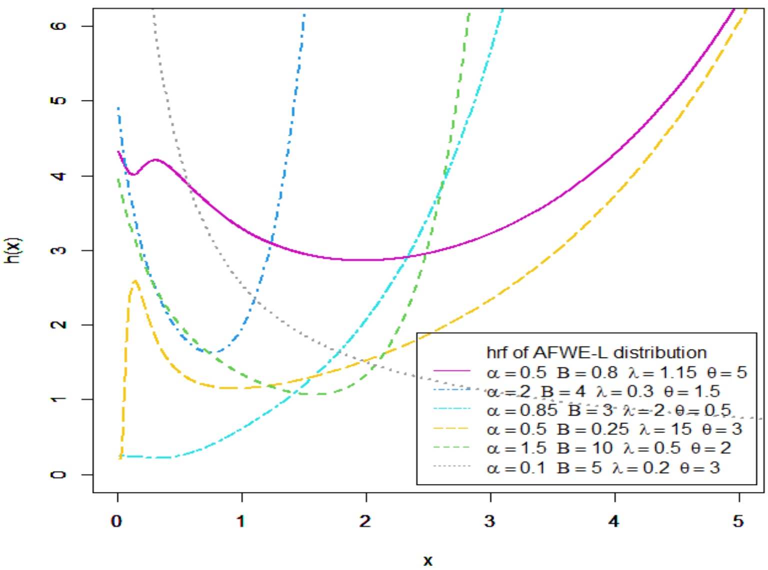

Figure 2: Plots of AFWE-L hrf

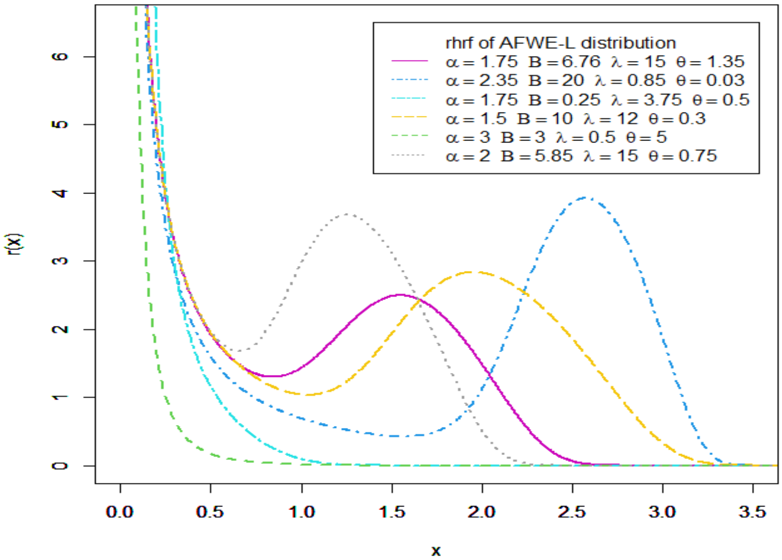

Figure 3: Plots of AFWE-L rhrf

An interpretation of AFWE-L distribution is as following: the lifetime of a series system with two components functioning independently, the lifetime of the first component has the FWE distribution, the lifetime of the second component has L distribution and the lifetime of the series system is the minimum of the lifetimes of the two components. Also, this new additive model can be interpreted as the lifetime of an item or an individual which is subject to two independent failure modes or causes, acting simultaneously on it and one of these failure modes can cause the failure of this item or this individual. The lifetime of one of these failure modes has the FWE distribution and the second has L distribution.

3. Statistical Properties

In this section some main properties of AFWE-L distribution are studied including: the quantile function, mode, central and non-central moments, moment generating function, r^{th} incomplete moment and inequality curves, *mean residual life* (MRL) and *mean inactivity time* (MIT), Rényi entropy and Tsallis entropy (q-entropy), the order statistics and some new and well-known sub-models of the proposed distribution.

3.1 The quantile function and the mode

The quantile function of AFWE-L distribution can be obtained by inverting

$$R\left(x;\underline{\psi}\right) = 1 - q, \qquad 0 < q < 1.$$

So, the quantile function can be obtained by solving the following nonlinear equation

$$e^{\alpha x_q - \frac{\beta}{x_q}} + \theta \ln\left(1 + \frac{x_q}{\lambda}\right) + \ln(1 - q) = 0, \qquad 0 < q < 1. \quad (11)$$

As special cases of the quantile function are the median of AFWE-L distribution, denoted by x_m, the first quartile, denoted by $x_{0.25}$, and the third quartile, denoted by $x_{0.75}$, which can be obtained, respectively, by setting $q = 0.5, q = 0.25$ and $q = 0.75$ into (11).

The mode of AFWE-L distribution is the value of x_0 which maximize $f\left(x; \underline{\psi}\right)$. So, the mode of AFWE-L distribution can be obtained by solving the following nonlinear equation numerically,

$$\left[\left(\alpha + \frac{\beta}{x_0^2}\right)^2 \left(1 - e^{\alpha x_0 - \frac{\beta}{x_0}}\right) - \frac{2\beta}{x_0^3} + \frac{2\theta}{\lambda}\left(\alpha + \frac{\beta}{x_0^2}\right)\left(1 + \frac{x_0}{\lambda}\right)^{-1}\right] \times e^{\alpha x_0 - \frac{\beta}{x_0}} - (\theta + 1)\frac{\theta}{\lambda^2}\left(1 + \frac{x_0}{\lambda}\right)^{-2} = 0. \quad (12)$$

The mathematical derivative of the mode of AFWE-L distribution is obtained in Appendix I.

Some numerical values of the first quartile, the median and the third quartile as special cases of the quantile and the mode for different parameter values $\underline{\psi} = (\alpha, \beta, \lambda, \theta)$are listed in Table1.

Table 1
Some quartiles and the mode of AFWE-L distribution
for different parameter values

α	β	λ	θ	$x_{0.25}$	x_m	$x_{0.75}$	**Mode**
4	3	1.25	3	0.1258	0.3247	0.6490	0.7669
1.5	0.5	4	3	0.2132	0.3623	0.5724	0.2276
1.5	3	4	0.5	0.9546	1.2335	1.4845	1.2930
4	3	4	0.5	0.6917	0.8048	0.8979	0.8425
2	2	3	0.05	0.6655	0.9082	1.0820	0.9227
3	1.25	4	0.15	0.4637	0.5828	0.6991	0.5909
1	0.3	6	0.25	0.2030	0.3866	0.7217	0.1457
0.85	3	2	0.5	0.9465	1.4387	1.9083	1.4846

3.2 Central and non-central moments

The r^{th} non-central moment of a random variable X has AFWE-L distribution is given by:

$$\mu'_r = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+k}\, i^{j+k}\, \alpha^j\, \beta^k}{i!\, j!\, k!} \big[r\lambda^{r+j-k} \times \mathbf{B}\big(r+j-k, \theta-(r+j-k)\big)\big], \quad r = 1,2,\ldots, \tag{13}$$

where

$\mathbf{B}(.,.)$ is the beta function and $0 < r+j-k < \theta$. [for more details see Appendix II].

By substituting $r = 1$ into (13), the mean of AFWE-L distribution can be obtained as follows:

$$\mu = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+k}\, i^{j+k}\, \alpha^j\, \beta^k\, \lambda^{j-k+1}}{i!\, j!\, k!} \times \mathbf{B}\big(1+j-k, \theta-(1+j-k)\big). \tag{14}$$

where

$$0 < 1+j-k < \theta.$$

The r^{th} central moment of a random variable X has AFWE-L distribution is:

$$\mu_r = \sum_{l=0}^{r} \binom{r}{l} (-1)^l \mu^l \mu'_{r-l}, \qquad r = 1,2,\ldots. \tag{15}$$

Substituting $r = 2$ in (15), the variance of AFWE-L distribution is:

$$V(X) = \mu_2 = \mu'_2 - \mu^2. \tag{16}$$

The *coefficient of variation* (CV), the *coefficient of skewness* (CS) and the *coefficient of kurtosis* (CK) are given, respectively, by:

$$CV = \frac{\sqrt{\mu_2}}{\mu}, \quad CS = \frac{\mu_3}{\mu_2^{3/2}} \quad \text{and} \quad Ck = \frac{\mu_4}{\mu_2^2} \tag{17}$$

where

μ and μ_2 are evaluated using (14) and (16), respectively. μ_3 and μ_4 can be calculated, respectively, by setting $r = 3$ and $r = 4$ into (15).

Numerical results of the first four non-central moments, variance, CV, CS and CK of AFWE-L distribution for some parameter values are presented in Tables 2 and 3.

Table 2
Moments of AFWE-L distribution for different values of α and β,
$\lambda = 1.25$ and $\theta = 0.5$

α	β	μ	μ_2'	μ_3'	μ_4'	μ_2	CV	CS	Ck
4		0.7083	0.5633	0.4689	0.4020	0.0617	0.3506	-1.1354	3.6768
3		0.7985	0.7290	0.7030	0.7025	0.0915	0.3788	-0.9092	3.1637
2	3	0.9421	1.0466	1.2477	1.5587	0.1590	0.4232	-0.5983	2.6480
1		1.2390	1.9410	3.3916	6.3780	0.4060	0.5143	-0.0748	2.2939
0.85		1.3191	2.2456	4.3133	8.9877	0.5055	0.5390	0.0489	2.3074
	3	1.0570	1.3520	1.8821	2.7724	0.2348	0.4585	-0.3808	2.4212
	2	0.8787	0.9375	1.1041	1.3937	0.1653	0.4627	-0.1534	2.4776
1.5	1.25	0.7074	0.6188	0.6123	0.6620	0.1184	0.4865	0.1709	2.5913
	0.5	0.4648	0.2932	0.2251	0.1979	0.0772	0.5978	0.7958	3.2633
	0.3	0.3718	0.2055	0.1454	0.1205	0.0673	0.6975	1.0867	3.8558

Table 3
Moments of AFWE-L distribution for different values of λ and θ,
$\alpha = 0.5$ and $\beta = 1.5$

λ	θ	μ	μ_2'	μ_3'	μ_4'	μ_2	CV	CS	Ck
6		1.4461	2.7880	6.4991	17.2885	0.6968	0.5773	0.7773	3.2068
4		1.4024	2.6619	6.1420	16.2198	0.6951	0.5945	0.7926	3.2544
3	0.5	1.3638	2.5536	5.8419	15.3358	0.6936	0.6107	0.8088	3.2975
2		1.2981	2.3756	5.3603	13.9423	0.6907	0.6402	0.8423	3.3758
1.25		1.2056	2.1379	4.7389	12.1878	0.6846	0.6863	0.9023	3.5035
	0.75	0.8046	1.1961	2.4042	5.8130	0.5488	0.9207	1.3744	4.8618
	0.5	0.9869	1.6277	3.4818	8.7727	0.6538	0.8194	1.1068	3.9641
0.5	0.25	1.2293	2.2417	5.0862	13.3246	0.7306	0.6953	0.8556	3.3528
	0.15	1.3481	2.5563	5.9328	15.7767	0.7389	0.6376	0.7782	3.2070
	0.05	1.4823	2.9205	6.9292	18.6980	0.7234	0.5738	0.7409	3.1189

It can be noticed from Tables 2 and 3 that:

- For fixed $\beta = 3, \lambda = 1.25$ and $\theta = 0.5$
  As α decreases, the first four non-central moments, variance and CV of AFWE-L distribution increase, and the CS increases and shifts from the left (negatively) skewed shape to the right (positively) skewed shape. Moreover, the CK decreases, and the distribution changes from the leptokurtic shape to the platykurtic shape.

- For fixed $\alpha = 0.5, \lambda = 1.25$ and $\theta = 0.5$
  As β decreases, the first four non-central moments and variance of AFWE-L distribution decrease, the CV and CS increase and shifts from the left (negatively) skewed shape to the right (positively) skewed shape. The CK increases, and the distribution changes from the platykurtic shape to the leptokurtic shape.
- For fixed $\alpha = 0.5, \beta = 1.5$ and $\theta = 0.5$
  As λ decreases, the first four non-central moments and variance of AFWE-L distribution decrease, the CV and CS increase to be more skewed to the right. Also, the CK increases and tends to be more leptokurtosis.
- For fixed $\alpha = 0.5, \beta = 1.5$ and $\lambda = 0.5$
  As θ decreases, the first four non-central moments and variance of AFWE-L distribution increases, and the CV, CS and CK decrease.

3.3 The moment generating function

The moment generating function, denoted by $M_X(t)$, of a random variable X has AFWE-L distribution can be obtained as given below:

$$M_X(t) = E(e^{tx}) = \int_0^\infty e^{tx} f\left(x; \underline{\psi}\right) dx = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$= \sum_{r=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^r}{r!} \frac{(-1)^{i+k}\, i^{j+k}\, \alpha^j\, \beta^k}{i!\, j!\, k!} \big[r\, \lambda^{r+j-k} \times \mathbf{B}\big(r+j-k, \theta-(r+j-k)\big)\big], \quad (18)$$

where

$$0 < r + j - k < \theta.$$

3.4 Incomplete moments and inequality curves

The r^{th} incomplete moment of a random variable X has AFWE-L distribution is given by:

$$\mu_r(t) = \int_0^t x^r f\left(x; \underline{\psi}\right) dx$$

$$= -t^r R\left(t; \underline{\psi}\right) + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+k}\, i^{j+k}\, \alpha^j\, \beta^k}{i!\, j!\, k!} \times \left[r\lambda^{r+j-k}\, \mathbf{IB}_{\left(t/_{\lambda}\right)}\big(r+j-k, \theta-(r+j-k)\big)\right], (19)$$

where

$$R\left(t;\underline{\psi}\right)=e^{-e^{\alpha t-\frac{\beta}{t}}}\left(1+\frac{t}{\lambda}\right)^{-\theta},$$

$\mathbf{IB}_{\left(t/_{\lambda}\right)}\big(r+j-k,\theta-(r+j-k)\big)$ is a lower incomplete beta function and

$$0<r+j-k<\theta.$$

Lorenz and Bonferroni curves are well known inequality curves that have been extensively used in different fields such as economics, demography, insurance, reliability analysis and life testing. These curves are important applications of the first incomplete moment. Lorenz and Bonferroni curves are denoted, respectively, by $L_F(p)$ and $B_F(p)$ which are defined by:

$$L(p)=\frac{1}{\mu}\int_0^q xf(x)dx=\frac{\mu(q)}{\mu}, \tag{20}$$

and

$$B(p)=\frac{1}{p\mu}\int_0^q xf(x)dx=\frac{L(p)}{p}, \tag{21}$$

where μ is obtained from (14), $\mu(q)$ is the first incomplete moment which can be obtained by substituting $r=1$ and $t=q$ into (19) and $q=F^{-1}(p)$ for $0<p<1$.

For AFWE-L distribution, the Lorenz and Bonferroni curves can be obtained, respectively, by:

$$L(p)=\frac{-qR\left(q;\underline{\psi}\right)+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\,i^{j+k}\,\alpha^j\,\beta^k\lambda^{j-k+1}}{i!\,j!\,k!}\mathbf{IB}_{\left(q/_{\lambda}\right)}\big(1+j-k,\theta-(1+j-k)\big)}{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\,i^{j+k}\,\alpha^j\,\beta^k\,\lambda^{j-k+1}}{i!\,j!\,k!}\mathbf{B}\big(1+j-k,\theta-(1+j-k)\big)}, \tag{22}$$

and

$$B(p)=\frac{-qR\left(q;\underline{\psi}\right)+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\,i^{j+k}\,\alpha^j\,\beta^k\lambda^{j-k+1}}{i!\,j!\,k!}\mathbf{IB}_{\left(q/_{\lambda}\right)}\big(1+j-k,\theta-(1+j-k)\big)}{p\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+k}\,i^{j+k}\,\alpha^j\,\beta^k\,\lambda^{j-k+1}}{i!\,j!\,k!}\mathbf{B}\big(1+j-k,\theta-(1+j-k)\big)}, \tag{23}$$

where

$$0<1+j-k<\theta.$$

3.5 The mean residual life and the mean inactivity time

The MRL function or the life expectation at age t, denoted by $m(t)$, which represents the expected additional life length for a system or a unit which is alive at age t, it is given by:

$$m(t) = E(X - t|X > t) = \frac{1}{R\left(t; \underline{\psi}\right)} \int_t^{\infty} R\left(x; \underline{\psi}\right) dx$$

$$= \frac{\left(1 + \frac{t}{\lambda}\right)^{\theta}}{e^{-e^{\alpha t - \frac{\beta}{t}}}} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+k} i^{j+k} \alpha^j \beta^k \lambda^{j-k+1}}{i!\, j!\, k!}$$

$$\times \mathbf{IB}^{\left(t/\lambda\right)}\left(1 + j - k, \theta - (1 + j - k)\right), \tag{24}$$

where

$\mathbf{IB}^{\left(t/\lambda\right)}\left(1 + j - k, \theta - (1 + j - k)\right)$ is an upper incomplete beta and

$$0 < 1 + j - k < \theta.$$

The MIT or the mean waiting time, also called the mean reversed residual life function, denoted by $M(t)$, which represents the waiting time elapsed since the failure of a system or a unit on the condition that this failure had occurred in $(0, t)$, is given by:

$$M(t) = E[(t - X)|X \le t] = \frac{1}{F\left(t; \underline{\psi}\right)} \int_0^{t} F\left(x; \underline{\psi}\right) dx$$

$$= \frac{1}{1 - e^{-e^{\alpha t - \frac{\beta}{t}}} \left(1 + \frac{t}{\lambda}\right)^{-\theta}} \left[t - \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^{i+k}\, i^{j+k}\, \alpha^j\, \beta^k\, \lambda^{j-k+1}}{i!\, j!\, k!} \times \mathbf{IB}_{\left(t/\lambda\right)}\left(1 + j - k, \theta - (1 + j - k)\right) \right], \tag{25}$$

where

$\mathbf{IB}_{\left(t/\lambda\right)}\left(1 + j - k, \theta - (1 + j - k)\right)$ is a lower incomplete beta function and

$$0 < 1 + j - k < \theta.$$

3.6 Entropy measures

In this subsection, entropy measures of AFWE-L distribution are derived. Entropy is a measure of uncertainty, randomness or variation of a random variable. One of the most important entropy measures is Rényi entropy which was proposed by Rényi (1961) as an extension of Shannon entropy and is defined by:

$$I_\delta(x) = \frac{1}{1-\delta}\ln\int_{-\infty}^{\infty} f^{\delta}(x)dx, \qquad \delta \neq 1, \delta > 0. \quad (26)$$

For a random variable X with AFWE-L distribution the Rényi entropy is given by:

$$I_\delta\left(x;\underline{\psi}\right) = \frac{1}{1-\delta}\ln\Bigg[\sum_{m=0}^{\delta}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{m}\binom{\delta}{m}\binom{m}{l}\frac{(-1)^{i+k}(m+i)^{j+k}\delta^{i}}{i!\,j!\,k!} \times \alpha^{j+l}\beta^{k+(m-l)}\lambda^{j-k-m+l-\delta+1}\theta^{(\delta-m)}\,\mathbf{B}\big(j-k-2(m-l) + 1, \delta(\theta+1)-m-(j-k-2(m-l)+1)\big)\Bigg], \qquad \delta \neq 1, \delta > 0, \quad (27)$$

where

$$0 < j - \mathrm{k} - 2(m-l) + 1 < \delta(\theta+1) - m.$$

As $\delta \to 1$ Rényi entropy tends to Shannon entropy.

Another entropy measure is Tsallis entropy (also called q-entropy) introduced by Tsallis (1988) is defined by:

$$I_q(x) = \frac{1}{1-q}\ln\left\{1 - \int_{-\infty}^{\infty} f^{q}(x)dx\right\}, \qquad q \neq 1, q > 0. \quad (28)$$

For a random variable X has AFWE-L distribution the Tsallis entropy is given by:

$$\mathrm{I}_q\left(x;\underline{\psi}\right) = \frac{1}{1-q}\ln\Bigg\{1 - \Bigg[\sum_{m=0}^{q}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{m}\binom{q}{m}\binom{m}{l}\frac{(-1)^{i+k}(m+i)^{j+k}q^{i}}{i!\,j!\,k!}\alpha^{j+l}\beta^{k+(m-l)} \times \lambda^{j-k-m+l-q+1}\theta^{(q-m)}\,\mathbf{B}\big(j-k-2(m-l)+1, q(\theta+1)-m-(j-k-2(m-l)+1)\big)\Bigg]\Bigg\}, \qquad q \neq 1, q > 0, \quad (29)$$

where $0 < j - k - 2(m - l) + 1 < q(\theta + 1) - m$.

3.7 The order statistics

Let $X_1, X_2, \dots, X_n$ be a random sample from AFWE-L distribution, X_i are i.i.d. random variables. Let $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ be the corresponding order statistics, then the pdf of the k^{th} order statistic is given by:

$$f_{k:n}\left(x; \underline{\psi}\right) = \sum_{j=0}^{k-1} C_{k,n,j} h\left(x; \underline{\psi}\right) \left[R\left(x; \underline{\psi}\right)\right]^{j+n-k+1}, \quad x_{(k)} > 0, \quad (30)$$

where

$$C_{k,n,j} = \frac{n!\,(-1)^j}{j!\,(k-j-1)!\,(n-k)!}.$$

Substituting (5) and (6) into (30), then the pdf of the k^{th} order statistics of AFWE-L distribution is:

$$f_{k:n}\left(x; \underline{\psi}\right) = \sum_{j=0}^{k-1} C_{k,n,j} \left[\left(\alpha + \frac{\beta}{x_{(k)}^2}\right) e^{\alpha x_{(k)} - \frac{\beta}{x_{(k)}}} + \frac{\theta}{\lambda}\left(1 + \frac{x_{(k)}}{\lambda}\right)^{-1}\right] \times \exp\left\{-(j + n - k + 1)\left[e^{\alpha x_{(k)} - \frac{\beta}{x_{(k)}}} + \theta \ln\left(1 + \frac{x_{(k)}}{\lambda}\right)\right]\right\}, \quad x_{(k)} > 0. \quad (31)$$

Special cases

a. The pdf of the smallest order statistics can be obtained when $k = 1$ as:

$$f_{1:n}\left(x; \underline{\psi}\right) = n\left[\left(\alpha + \frac{\beta}{x_{(1)}^2}\right) e^{\alpha x_{(1)} - \frac{\beta}{x_{(1)}}} + \frac{\theta}{\lambda}\left(1 + \frac{x_{(1)}}{\lambda}\right)^{-1}\right] \times \exp\left\{-n\left[e^{\alpha x_{(1)} - \frac{\beta}{x_{(1)}}} + \theta \ln\left(1 + \frac{x_{(1)}}{\lambda}\right)\right]\right\}, \quad x_{(1)} > 0. \quad (32)$$

b. The pdf of the largest order statistics can be obtained if $k = n$ as:

$$f_{n:n}\left(x; \underline{\psi}\right) = \sum_{j=0}^{k-1} C_{n,j} \left[\left(\alpha + \frac{\beta}{x_{(n)}^2}\right) e^{\alpha x_{(n)} - \frac{\beta}{x_{(n)}}} + \frac{\theta}{\lambda}\left(1 + \frac{x_{(n)}}{\lambda}\right)^{-1}\right] \times \exp\left\{-(j + 1)\left[e^{\alpha x_{(n)} - \frac{\beta}{x_{(n)}}} + \theta \ln\left(1 + \frac{x_{(n)}}{\lambda}\right)\right]\right\}, \quad x_{(n)} > 0, \quad (33)$$

where

$$C_{n,j} = \frac{n!\,(-1)^j}{i!\ (n-j-1)!}.$$

3.8 Some sub-models

There are several distributions that can be obtained as sub-models of AFWE-L distribution and are summarized in Table 4.

Table 4
Sub-models of AFWE-L distribution

Parameter	The resulting distribution	cdf
$\lambda \to \infty$	FWE distribution	$F(x;\alpha,\beta) = 1 - e^{-e^{\alpha x - \frac{\beta}{x}}}, x > 0;\ \alpha,\beta > 0.$
$\beta \to \infty$	Lomax distribution	$F(x;\lambda,\theta) = 1 - \left(1 + \frac{x}{\lambda}\right)^{-\theta},$ $x > 0;\ \lambda,\theta > 0.$
$\lambda = 1$	AFWE- compound exponential (or the AFWE- inverted Kumaraswamy $(\theta, 1)$ or the AFWE-beta Type II with $(1,\theta)$).	$F(x;\alpha,\beta,\theta) = 1 - e^{-e^{\alpha x - \frac{\beta}{x}}}(1+x)^{-\theta},$ $x > 0;\ \alpha,\beta,\theta > 0.$
$\theta = 1$	AFWE- log logistic (λ)	$F(x;\alpha,\beta,\lambda) = 1 - e^{-e^{\alpha x - \frac{\beta}{x}}}\left(1 + \frac{x}{\lambda}\right)^{-1},$ $x > 0;\ \alpha,\beta,\lambda > 0.$
$\lambda = 1$ and $\theta = 1$	AFWE-standard Lomax	$F(x;\alpha,\beta) = 1 - e^{-e^{\alpha x - \frac{\beta}{x}}}(1+x)^{-1},$ $x > 0;\ \alpha,\beta > 0.$

4. Maximum Likelihood Estimation

In this subsection, the ML estimators of the parameters, rf and hrf based on Type II censored samples are derived. Also, ACIs of the parameters, rf and the hrf are obtained.

4.1 Point estimation

Suppose that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(r)}$ is a censored sample of size r from AFWE-L distribution with parameter vector $\underline{\psi} = (\alpha,\beta,\lambda,\theta)$, then the likelihood function is given by:

$$L\left(\underline{\psi};\underline{x}\right) = \frac{n!}{(n-r)!}\left[\prod_{i=1}^{r} f\left(x_{(i)};\underline{\psi}\right)\right]\left[R\left(x_{(r)};\underline{\psi}\right)\right]^{n-r}$$

$$= \frac{n!}{(n-r)!}\left\{\prod_{i=1}^{r}\left[\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}+\frac{\theta}{\lambda}\left(1+\frac{x_{(i)}}{\lambda}\right)^{-1}\right]\right\}$$
$$\times\left[\prod_{i=1}^{r}\left(1+\frac{x_{(i)}}{\lambda}\right)^{-\theta}\right]\left(1+\frac{x_{(r)}}{\lambda}\right)^{-\theta(n-r)}$$
$$\times\exp\left(-(n-r)e^{\alpha x_{(r)}-\frac{\beta}{x_{(r)}}}-\sum_{i=1}^{r}e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}\right). \quad (34)$$

The natural logarithm of the likelihood function is

$$\ell = \ln L\left(\underline{\psi};\underline{x}\right)$$
$$= \ln\frac{n!}{(n-r)!}+\sum_{i=1}^{r}\ln\left[\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}+\frac{\theta}{\lambda}\left(1+\frac{x_{(i)}}{\lambda}\right)^{-1}\right]$$
$$-\theta\sum_{i=1}^{r}\ln\left(1+\frac{x_{(i)}}{\lambda}\right)-\sum_{i=1}^{r}e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}-\theta(n-r)\ln\left(1+\frac{x_{(r)}}{\lambda}\right)$$
$$-(n-r)e^{\alpha x_{(r)}-\frac{\beta}{x_{(r)}}}. \quad (35)$$

By differentiating the log likelihood function in (35) with respect to the parameters α, β, λ and θ as follows:

$$\frac{\partial\ell}{\partial\alpha}=\sum_{i=1}^{r}\frac{h_{\alpha}\left(x_{(i)};\underline{\psi}\right)}{h\left(x_{(i)};\underline{\psi}\right)}-\sum_{i=1}^{r}x_{(i)}e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}$$
$$-(n-r)x_{(r)}e^{\alpha x_{(r)}-\frac{\beta}{x_{(r)}}}, \quad (36)$$

$$\frac{\partial\ell}{\partial\beta}=\sum_{i=1}^{r}\frac{h_{\beta}\left(x_{(i)};\underline{\psi}\right)}{h\left(x_{(i)};\underline{\psi}\right)}+\sum_{i=1}^{r}\frac{1}{x_{(i)}}e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}$$
$$+\frac{(n-r)}{x_{(r)}}e^{\alpha x_{(r)}-\frac{\beta}{x_{(r)}}}, \quad (37)$$

$$\frac{\partial\ell}{\partial\lambda}=\sum_{i=1}^{r}\frac{h_{\lambda}\left(x_{(i)};\underline{\psi}\right)}{h\left(x_{(i)};\underline{\psi}\right)}+\frac{\theta}{\lambda^2}\sum_{i=1}^{r}\frac{x_{(i)}}{\left(1+\frac{x_{(i)}}{\lambda}\right)}$$
$$+\frac{\theta(n-r)}{\lambda^2}\frac{x_{(r)}}{\left(1+\frac{x_{(r)}}{\lambda}\right)}, \quad (38)$$

and

$$\frac{\partial \ell}{\partial \theta}=\sum_{i=1}^{r}\frac{h_\theta\left(x_{(i)};\underline{\psi}\right)}{h\left(x_{(i)};\underline{\psi}\right)}-\sum_{i=1}^{r}\ln\left(1+\frac{x_{(i)}}{\lambda}\right)$$
$$-(n-r)\ln\left(1+\frac{x_{(r)}}{\lambda}\right), \qquad (39)$$

where $h\left(x_{(i)};\underline{\psi}\right)$ is defined in (5),

$$h_\alpha\left(x_{(i)};\underline{\psi}\right)=\frac{\partial h\left(x_{(i)};\underline{\psi}\right)}{\partial \alpha}=x_{(i)}\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}+e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}},$$

$$h_\beta\left(x_{(i)};\underline{\psi}\right)=\frac{\partial h\left(x_{(i)};\underline{\psi}\right)}{\partial \beta}$$
$$=\frac{1}{x_{(i)}^2}e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}-\frac{1}{x_{(i)}}\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}},$$

$$h_\lambda\left(x_{(i)};\underline{\psi}\right)=\frac{\partial h\left(x_{(i)};\underline{\psi}\right)}{\partial \lambda}=-\frac{\theta}{\lambda^2\left(1+\frac{x_{(i)}}{\lambda}\right)^2},$$

and

$$h_\theta\left(x_{(i)};\underline{\psi}\right)=\frac{\partial h\left(x_{(i)};\underline{\psi}\right)}{\partial \theta}=\frac{1}{\lambda\left(1+\frac{x_{(i)}}{\lambda}\right)}.$$

The ML estimates of the parameters $\underline{\psi}=(\alpha,\beta,\lambda,\theta)$ can be obtained by equating Equations $(36)-(39)$ to zero and solving numerically.

The ML estimators of $R\left(x;\underline{\psi}\right)$ and $h\left(x;\underline{\psi}\right)$ can be obtained, using the invariance property of the ML estimators, by replacing the parameters $\underline{\psi}=(\alpha,\beta,\lambda,\theta)$ in (5) and (6) by their ML estimators, then the ML estimators of $R\left(x;\underline{\psi}\right)$ and $h\left(x;\underline{\psi}\right)$ can be given, respectively, as follows:

$$\hat{R}\left(x;\underline{\hat{\psi}}\right)=e^{-e^{\hat{\alpha}x-\frac{\hat{\beta}}{x}}}\left(1+\frac{x}{\hat{\lambda}}\right)^{-\hat{\theta}}, \qquad x>0, \quad (40)$$

and

$$\hat{h}\left(x;\underline{\hat{\psi}}\right)=\left(\hat{\alpha}+\frac{\hat{\beta}}{x^2}\right)e^{\hat{\alpha}x-\frac{\hat{\beta}}{x}}+\frac{\hat{\theta}}{\hat{\lambda}}\left(1+\frac{x}{\hat{\lambda}}\right)^{-1}, \qquad x>0. \quad (41)$$

4.2 Asymptotic confidence intervals

To obtain confidence intervals for the parameters $\underline{\psi} = (\alpha, \beta, \lambda, \theta)$ of AFWE-L distribution, the distributions of the ML estimators $\underline{\hat{\psi}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})$ are needed. Since the ML estimators $\underline{\hat{\psi}} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})$ do not have closed form, so their exact distribution cannot be obtained. Therefore, the ACIs can be derived by using the asymptotic distribution of the ML estimators. The ML estimators are asymptotically normal with mean $(\alpha, \beta, \lambda, \theta)$ and the asymptotic variance-covariance matrix is given by the inverse of the asymptotic Fisher information matrix as

$$\tilde{I}^{-1}\left(\underline{\psi}\right)\Big|_{\underline{\hat{\psi}}} \simeq \begin{pmatrix} \widetilde{var}(\hat{\alpha}) & \widetilde{cov}(\hat{\alpha},\hat{\beta}) & \widetilde{cov}(\hat{\alpha},\hat{\lambda}) & \widetilde{cov}(\hat{\alpha},\hat{\theta}) \\ \widetilde{cov}(\hat{\alpha},\hat{\beta}) & \widetilde{var}(\hat{\beta}) & \widetilde{cov}(\hat{\beta},\hat{\lambda}) & \widetilde{cov}(\hat{\beta},\hat{\theta}) \\ \widetilde{cov}(\hat{\alpha},\hat{\lambda}) & \widetilde{cov}(\hat{\beta},\hat{\lambda}) & \widetilde{var}(\hat{\lambda}) & \widetilde{cov}(\hat{\lambda},\hat{\theta}) \\ \widetilde{cov}(\hat{\alpha},\hat{\theta}) & \widetilde{cov}(\hat{\beta},\hat{\theta}) & \widetilde{cov}(\hat{\lambda},\hat{\theta}) & \widetilde{var}(\hat{\theta}) \end{pmatrix}, \tag{42}$$

where the derivatives of the I_{ij} elements of the asymptotic Fisher information matrix are given in Appendix III.

Therefore, the $(1-\omega)100\%$ bounds of the ACIs of the parameters $\underline{\psi} = (\alpha, \beta, \lambda, \theta)$ are as follows:

$$\hat{\alpha} \pm Z_{\left(1-\frac{\omega}{2}\right)}\sqrt{\widetilde{var}(\hat{\alpha})}, \qquad \hat{\beta} \pm Z_{\left(1-\frac{\omega}{2}\right)}\sqrt{\widetilde{var}(\hat{\beta})},$$

$$\hat{\lambda} \pm Z_{\left(1-\frac{\omega}{2}\right)}\sqrt{\widetilde{var}(\hat{\lambda})} \text{ and } \hat{\theta} \pm Z_{\left(1-\frac{\omega}{2}\right)}\sqrt{\widetilde{var}(\hat{\theta})}, \tag{43}$$

where $Z_{\left(1-\frac{\omega}{2}\right)}$ is the $(1-\omega)100\%$ percentage point of the standard normal distribution.

To obtain the ACIs of the rf and the hrf of AFWE-L distribution, variances of the ML estimators of the rf and hrf are needed. Therefore, the delta method discussed in Greene (2018) and used by EL-Sagheer (2018), Thach and Bris (2021), EL-Sagheer *et al.* (2021), Buzaridah *et al.* (2022) and Thach (2022) can be used to derive the asymptotic variances of $\hat{R}\left(x;\underline{\hat{\psi}}\right)$ and $\hat{h}\left(x;\underline{\hat{\psi}}\right)$.

The asymptotic variances of $\hat{R}\left(x;\underline{\hat{\psi}}\right)$ and $\hat{h}\left(x;\underline{\hat{\psi}}\right)$ can be given, respectively, by:

$$\widetilde{var}\left(\hat{R}\left(x;\underline{\hat{\psi}}\right)\right) = \acute{\xi}\tilde{I}^{-1}\left(\underline{\psi}\right)\xi\Big|_{\underline{\hat{\psi}}},$$

$$\hat{var}\left(\hat{h}\left(x;\hat{\underline{\psi}}\right)\right)=\acute{\eta}\tilde{I}^{-1}\left(\underline{\psi}\right)\eta\Big|_{\hat{\underline{\psi}}}, \tag{44}$$

where

$$\acute{\xi}=\left(R_{\alpha}\left(x;\underline{\psi}\right)\quad R_{\beta}\left(x;\underline{\psi}\right)\quad R_{\lambda}\left(x;\underline{\psi}\right)\quad R_{\theta}\left(x;\underline{\psi}\right)\right),$$

and

$$\acute{\eta}=\left(h_{\alpha}\left(x;\underline{\psi}\right)\quad h_{\beta}\left(x;\underline{\psi}\right)\quad h_{\lambda}\left(x;\underline{\psi}\right)\quad h_{\theta}\left(x;\underline{\psi}\right)\right).$$

For more details about the asymptotic variances of $\hat{R}\left(x;\hat{\underline{\psi}}\right)$ and $\hat{h}\left(x;\hat{\underline{\psi}}\right)$ see Appendix IV.

Thus, the $(1-\omega)100\%$ bounds of the ACIs of the rf and the hrf are:

$$\hat{R}\left(x;\hat{\underline{\psi}}\right)\pm Z_{\left(1-\frac{\omega}{2}\right)}\sqrt{\hat{var}\left(\hat{R}\left(x;\hat{\underline{\psi}}\right)\right)},$$

and

$$\hat{h}\left(x;\hat{\underline{\psi}}\right)\pm Z_{\left(1-\frac{\omega}{2}\right)}\sqrt{\hat{var}\left(\hat{h}\left(x;\hat{\underline{\psi}}\right)\right)}. \tag{45}$$

5. Simulation Study

In this section, a simulation study is conducted to examine the performance of the ML estimates of the parameters, rf and hrf of AFWE-L distribution under Type II censoring scheme, as follows:

a. Conducting various simulations for different samples of size $(n=30,60,100,200,500)$ generated from AFWE-L distribution using different parameter values:

$$I:(\alpha=1.15,\beta=0.3,\lambda=0.15,\theta=0.1),$$
$$II:(\alpha=0.5,\beta=0.25,\lambda=0.15,\theta=0.1),$$

and

$$III:(\alpha=0.8,\beta=0.5,\lambda=0.5,\theta=0.5).$$

b. The simulation study is performed based on two level of censoring $(30\%,0\%)$.
c. The simulation study is conducted using number of replications $NR=1000$ using Mathematica 11.
d. Tables 5 - 7 display The ML averages, the *estimated risks* (ER), the *relative errors* (RE), the variances and the ACIs of the parameters with their lengths, where the ER and the RE are computed as follows:

$$ER=\frac{\sum_{i=1}^{NP}\left(\hat{\psi}_i-\psi\right)^2}{NR}$$

and

$$RE = \frac{\sqrt{ER}}{true\ value}.$$

e. Tables 8 - 10 present the ML averages, ER, RE, variances of the rf, hrf and the ACIs at time $x_0 = 0.4$.

Concluding remarks:

- From Tables 5 - 7, one can observe that the ML averages of the estimates of the parameters of AFWE-L distribution are close to the population parameter values as the sample size n increases and as the level of censoring decreases. Moreover, the ERs, REs and the variances of the ML estimates of the parameters $\underline{\psi} = (\alpha, \beta, \lambda, \theta)$ decrease, in most cases, as the sample size increases and as the level of censoring decreases.
- From Tables 8-10, one can conclude that in most cases, as the sample size increases and as the level of censoring decreases, the ERs, REs and the variances of the ML estimates of the rf and the hrf decrease.
- As the sample size increases and the level of censoring decreases, the length of the ACIs of the parameters, rf and hrf of AFWE-L distribution become narrower, in most cases.

Table 5

ML averages, estimated risks, relative errors, variances and 95% ACIs of the parameters of AFWE-L distribution for different samples of size n and $NR = 1000, (\alpha = 1.15, \beta = 0.3, \lambda = 0.15, \theta = 0.1)$

n	r	ψ	Average	ER	RE	Variance	UL	LL	Length
30	21	α	1.1051	0.0993	0.2740	0.0973	1.7164	0.4938	1.2226
		β	0.3381	0.0111	0.3525	0.0097	0.5314	0.1448	0.3867
		λ	0.1922	0.0283	1.1216	0.0265	0.5114	0.0000	0.5114
		θ	0.1076	0.0056	0.7450	0.0055	0.2529	0.0000	0.2529
	30	α	1.2149	0.0439	0.1822	0.0397	1.6054	0.8244	0.7810
		β	0.2960	0.0028	0.1759	0.0028	0.3991	0.1928	0.2063
		λ	0.1752	0.0067	0.5468	0.0061	0.3282	0.0223	0.3059
		θ	0.1035	0.0024	0.4846	0.0023	0.1982	0.0087	0.1895
60	42	α	1.0566	0.0721	0.2335	0.0634	1.5500	0.5633	0.9867
		β	0.3359	0.0092	0.3204	0.0080	0.5106	0.1612	0.3494
		λ	0.1768	0.0083	0.6060	0.0075	0.3471	0.0066	0.3405
		θ	0.0996	0.0035	0.5916	0.0035	0.2156	0.0000	0.2156
	60	α	1.1787	0.0196	0.1216	0.0187	1.4470	0.9104	0.5366
		β	0.2885	0.0015	0.1273	0.0013	0.3599	0.2171	0.1428
		λ	0.1594	0.0054	0.4913	0.0053	0.3027	0.0161	0.2865
		θ	0.0929	0.0014	0.3715	0.0013	0.1644	0.0215	0.1429
100	70	α	1.0440	0.0580	0.2094	0.0467	1.4678	0.6203	0.8475
		β	0.3309	0.0025	0.1650	0.0015	0.4067	0.2551	0.1516
		λ	0.1671	0.0050	0.4727	0.0047	0.3020	0.0321	0.2698
		θ	0.0933	0.0017	0.4153	0.0017	0.1736	0.0130	0.1607
	100	α	1.1638	0.0112	0.0922	0.0110	1.3697	0.9578	0.4119
		β	0.2880	0.0011	0.1094	0.0009	0.3478	0.2281	0.1197
		λ	0.1551	0.0036	0.3986	0.0036	0.2719	0.0384	0.2335
		θ	0.0896	0.0011	0.3359	0.0010	0.1522	0.0270	0.1252
200	140	α	1.0174	0.0577	0.2088	0.0401	1.4098	0.6251	0.7847
		β	0.3309	0.0019	0.1455	0.0010	0.3913	0.2706	0.1207
		λ	0.1648	0.0050	0.4729	0.0048	0.3008	0.0288	0.2720
		θ	0.0907	0.0009	0.3054	0.0009	0.1476	0.0337	0.1140
	200	α	1.1613	0.0056	0.0653	0.0055	1.3067	1.0159	0.2909
		β	0.2866	0.0007	0.0855	0.0005	0.3295	0.2437	0.0858
		λ	0.1410	0.0034	0.3867	0.0033	0.2533	0.0287	0.2246
		θ	0.0831	0.0012	0.3393	0.0009	0.1408	0.0254	0.1154
500	350	α	1.0136	0.0574	0.2083	0.0388	1.3994	0.6277	0.7717
		β	0.3279	0.0014	0.1227	0.0006	0.3750	0.2808	0.0942
		λ	0.1610	0.0022	0.3105	0.0021	0.2497	0.0723	0.1774
		θ	0.0878	0.0005	0.2267	0.0004	0.1252	0.0504	0.0749
	500	α	1.1543	0.0021	0.0401	0.0021	1.2444	1.0643	0.1801
		β	0.2856	0.0004	0.0697	0.0002	0.3152	0.2559	0.0593
		λ	0.1208	0.0032	0.3759	0.0023	0.2153	0.0263	0.1890
		θ	0.0736	0.0014	0.3738	0.0007	0.1253	0.0218	0.1036

Table 6

ML averages, estimated risks, relative errors, variances and 95% ACIs of the parameters of AFWE-L distribution for different samples of size n and $NR = 1000, (\alpha = 0.5, \beta = 0.25, \lambda = 0.15, \theta = 0.1)$

n	r	ψ	Average	ER	RE	Variance	UL	LL	Length
30	21	α	0.4911	0.0294	0.3432	0.0294	0.8269	0.1552	0.6717
		β	0.3133	0.0161	0.5073	0.0121	0.5287	0.0979	0.4308
		λ	0.2040	0.0314	1.1819	0.0285	0.5350	0.0000	0.5350
		θ	0.1190	0.0099	0.9968	0.0096	0.3108	0.0000	0.3108
	30	α	0.5424	0.0127	0.2252	0.0109	0.7469	0.3379	0.4090
		β	0.2840	0.0101	0.4011	0.0089	0.4689	0.0991	0.3698
		λ	0.1798	0.0199	0.9396	0.0190	0.4498	0.0000	0.4498
		θ	0.1180	0.0097	0.9867	0.0094	0.3081	0.0000	0.3081
60	42	α	0.4586	0.0171	0.2618	0.0154	0.7080	0.2153	0.4867
		β	0.3038	0.0084	0.3659	0.0055	0.4488	0.1587	0.2901
		λ	0.2055	0.0225	1.0006	0.0195	0.4788	0.0000	0.4788
		θ	0.1152	0.0078	0.8840	0.0076	0.2859	0.0000	0.2859
	60	α	0.5253	0.0055	0.1480	0.0048	0.6616	0.3890	0.2726
		β	0.2608	0.0040	0.2539	0.0039	0.3834	0.1381	0.2452
		λ	0.1709	0.0159	0.8399	0.0154	0.4144	0.0000	0.4144
		θ	0.1015	0.0065	0.8073	0.0065	0.2597	0.0000	0.2597
100	70	α	0.4549	0.0134	0.2317	0.0114	0.6640	0.2458	0.4182
		β	0.3052	0.0068	0.3288	0.0037	0.4246	0.1859	0.2387
		λ	0.2007	0.0145	0.8014	0.0119	0.4143	0.0000	0.4143
		θ	0.1124	0.0048	0.6951	0.0047	0.2464	0.0000	0.2464
	100	α	0.5217	0.0037	0.1221	0.0033	0.6335	0.4100	0.2236
		β	0.2557	0.0034	0.2321	0.0033	0.3689	0.1426	0.2263
		λ	0.1681	0.0168	0.8646	0.0165	0.4198	0.0000	0.4198
		θ	0.0979	0.0084	0.9176	0.0084	0.2777	0.0000	0.2777
200	140	α	0.4476	0.0122	0.2205	0.0094	0.6377	0.2574	0.3802
		β	0.2975	0.0033	0.2284	0.0010	0.3597	0.2352	0.1245
		λ	0.1967	0.0107	0.6899	0.0085	0.3777	0.0157	0.3620
		θ	0.1085	0.0023	0.4789	0.0022	0.2009	0.0161	0.1848
	200	α	0.5160	0.0018	0.0838	0.0015	0.5919	0.4401	0.1518
		β	0.2491	0.0010	0.1248	0.0010	0.3102	0.1880	0.1222
		λ	0.1638	0.0113	0.7085	0.0111	0.3703	0.0000	0.3703
		θ	0.0904	0.0032	0.5622	0.0031	0.1989	0.0000	0.1989
500	350	α	0.4403	0.0122	0.2212	0.0087	0.6229	0.2578	0.3652
		β	0.2963	0.0028	0.2116	0.0007	0.3466	0.2460	0.1006
		λ	0.1881	0.0072	0.5653	0.0057	0.3366	0.0396	0.2970
		θ	0.1030	0.0015	0.3828	0.0015	0.1778	0.0282	0.1496
	500	α	0.5132	0.0010	0.0641	0.0009	0.5704	0.4560	0.1144
		β	0.2462	0.0006	0.0959	0.0006	0.2926	0.1998	0.0928
		λ	0.1614	0.0202	0.9465	0.0200	0.4387	0.0000	0.4387
		θ	0.0875	0.0042	0.6509	0.0041	0.2127	0.0000	0.2127

Table 7

ML averages, estimated risks, relative errors, variances and 95% ACIs of the parameters of AFWE-L distribution for different samples of size n and $NR = 1000, (\alpha = 0.8, \beta = 0.5, \lambda = 0.5, \theta = 0.5)$

n	r	ψ	Average	ER	RE	Variance	UL	LL	Length
30	21	α	0.7220	0.0519	0.2848	0.0458	1.1415	0.3025	0.8391
		β	0.4186	0.0200	0.2829	0.0134	0.6454	0.1918	0.4536
		λ	0.5490	0.0865	0.5882	0.0841	1.1174	0.0000	1.1174
		θ	0.5033	0.0835	0.5778	0.0835	1.0695	0.0000	1.0695
	30	α	0.8029	0.0343	0.2315	0.0343	1.1659	0.4400	0.7259
		β	0.3785	0.0254	0.3189	0.0107	0.5808	0.1762	0.4046
		λ	0.5115	0.0592	0.4865	0.0590	0.9878	0.0353	0.9525
		θ	0.4964	0.0724	0.5382	0.0724	1.0238	0.0000	1.0238
60	42	α	0.6906	0.0431	0.2595	0.0311	1.0364	0.3447	0.6916
		β	0.4128	0.0125	0.2238	0.0049	0.5502	0.2754	0.2747
		λ	0.5291	0.0429	0.4143	0.0421	0.9311	0.1270	0.8041
		θ	0.4798	0.0287	0.3391	0.0283	0.8097	0.1499	0.6598
	60	α	0.7898	0.0079	0.1108	0.0078	0.9623	0.6173	0.3450
		β	0.3791	0.0203	0.2850	0.0057	0.5270	0.2312	0.2958
		λ	0.4756	0.0239	0.3089	0.0233	0.7745	0.1766	0.5979
		θ	0.4628	0.0283	0.3365	0.0269	0.7843	0.1412	0.6431
100	70	α	0.6811	0.0386	0.2456	0.0245	0.9877	0.3745	0.6133
		β	0.4095	0.0105	0.2047	0.0023	0.5035	0.3156	0.1878
		λ	0.5217	0.0148	0.2432	0.0143	0.7562	0.2872	0.4690
		θ	0.4695	0.0138	0.2349	0.0129	0.6918	0.2472	0.4446
	100	α	0.7866	0.0045	0.0834	0.0043	0.9147	0.6586	0.2561
		β	0.3740	0.0177	0.2658	0.0018	0.4569	0.2911	0.1658
		λ	0.4536	0.0141	0.2377	0.0120	0.6682	0.2391	0.4291
		θ	0.4353	0.0134	0.2318	0.0092	0.6236	0.2469	0.3768
200	140	α	0.6762	0.0349	0.2335	0.0196	0.9504	0.4021	0.5483
		β	0.4081	0.0097	0.1971	0.0013	0.4780	0.3382	0.1398
		λ	0.5217	0.0050	0.1419	0.0046	0.6541	0.3894	0.2647
		θ	0.4657	0.0041	0.1283	0.0029	0.5721	0.3594	0.2127
	200	α	0.7804	0.0026	0.0641	0.0022	0.8732	0.6876	0.1856
		β	0.3698	0.0175	0.2647	0.0006	0.4161	0.3235	0.0925
		λ	0.4331	0.0087	0.1864	0.0042	0.5601	0.3060	0.2542
		θ	0.4113	0.0104	0.2040	0.0025	0.5102	0.3125	0.1977
500	350	α	0.6677	0.0243	0.2314	0.0168	0.9215	0.4139	0.5076
		β	0.4063	0.0096	0.1960	0.0008	0.4608	0.3513	0.1094
		λ	0.5198	0.0028	0.1051	0.0024	0.6152	0.4244	0.1909
		θ	0.4622	0.0025	0.1006	0.0011	0.5273	0.3971	0.1302
	500	α	0.7770	0.0014	0.0472	0.0009	0.8358	0.7182	0.1176
		β	0.3695	0.0172	0.2625	0.0002	0.3961	0.3428	0.0533
		λ	0.4307	0.0064	0.1596	0.0016	0.5080	0.3533	0.1548
		θ	0.4052	0.0098	0.1975	0.0008	0.4592	0.3512	0.1081

Table 8

ML averages, estimated risks, relative errors, variances and 95% ACIs of the rf and the hrf of AFWE-L distribution for different samples of size n and $NR = 1000, (\alpha = 1.15, \beta = 0.3, \lambda = 0.15, \theta = 0.1, x_0 = 0.4)$

n	r	rf and hrf	Average	ER	RE	Variance	UL	LL	Length
30	21	$R\left(x_0;\underline{\psi}\right)$	0.4517	0.0051	0.1723	0.0038	0.5728	0.3305	0.2424
		$h\left(x_0;\underline{\psi}\right)$	1.8342	0.5425	0.4566	0.4937	3.2114	0.4570	2.7544
	30	$R\left(x_0;\underline{\psi}\right)$	0.4071	0.0021	0.1094	0.0020	0.4947	0.3196	0.1751
		$h\left(x_0;\underline{\psi}\right)$	1.5923	0.1119	0.2074	0.1115	2.2467	0.9378	1.3089
60	42	$R\left(x_0;\underline{\psi}\right)$	0.4581	0.0039	0.1505	0.0021	0.5479	0.3683	0.1796
		$h\left(x_0;\underline{\psi}\right)$	1.8082	0.4848	0.4316	0.4468	3.1183	0.4981	2.6203
	60	$R\left(x_0;\underline{\psi}\right)$	0.4089	0.0009	0.0731	0.0009	0.4669	0.3508	0.1161
		$h\left(x_0;\underline{\psi}\right)$	1.5352	0.0547	0.1450	0.0486	1.9673	1.1031	0.8642
100	70	$R\left(x_0;\underline{\psi}\right)$	0.4591	0.0032	0.1367	0.0013	0.5306	0.3876	0.1430
		$h\left(x_0;\underline{\psi}\right)$	1.7627	0.0806	0.1760	0.0583	2.2358	1.2897	0.9461
	100	$R\left(x_0;\underline{\psi}\right)$	0.4113	0.0007	0.0614	0.0006	0.4606	0.3620	0.0986
		$h\left(x_0;\underline{\psi}\right)$	1.5257	0.0411	0.1257	0.0335	1.8841	1.1672	0.7170
200	140	$R\left(x_0;\underline{\psi}\right)$	0.4634	0.0032	0.1364	0.0009	0.5230	0.4037	0.1193
		$h\left(x_0;\underline{\psi}\right)$	1.7563	0.0568	0.1477	0.0363	2.1296	1.3829	0.7468
	200	$R\left(x_0;\underline{\psi}\right)$	0.4117	0.0003	0.0442	0.0003	0.4468	0.3765	0.0703
		$h\left(x_0;\underline{\psi}\right)$	1.5091	0.0290	0.1056	0.0182	1.7734	1.2449	0.5285
500	350	$R\left(x_0;\underline{\psi}\right)$	0.4625	0.0029	0.1294	0.0007	0.5138	0.4112	0.1026
		$h\left(x_0;\underline{\psi}\right)$	1.7332	0.0361	0.1177	0.0217	2.0218	1.4447	0.5771
	500	$R\left(x_0;\underline{\psi}\right)$	0.4136	0.0001	0.0286	0.0001	0.4365	0.3906	0.0459
		$h\left(x_0;\underline{\psi}\right)$	1.4905	0.0252	0.0985	0.0102	1.6882	1.2927	0.3954

Table 9

ML averages, estimated risks, relative errors, variances and 95% ACIs of the rf and the hrf of AFWE-L distribution for different samples of size n and $NR = 1000, (\alpha = 0.5, \beta = 0.25, \lambda = 0.15, \theta = 0.1, x_0 = 0.4)$

n	r	rf and hrf	Average	ER	RE	Variance	UL	LL	Length
30	21	$R(x_0;\underline{\psi})$	0.5012	0.0047	0.1492	0.0027	0.6025	0.3999	0.2026
		$h(x_0;\underline{\psi})$	1.6132	0.6751	0.6550	0.5454	3.0620	0.1645	2.8975
	30	$R(x_0;\underline{\psi})$	0.4716	0.0024	0.1079	0.0022	0.5637	0.3796	0.1842
		$h(x_0;\underline{\psi})$	1.4716	0.4666	0.5445	0.4194	2.7412	0.2026	2.5385
60	42	$R(x_0;\underline{\psi})$	0.5031	0.0036	0.1315	0.0015	0.5779	0.4284	0.1495
		$h(x_0;\underline{\psi})$	1.5364	0.3065	0.4413	0.2270	2.4702	0.6027	1.8676
	60	$R(x_0;\underline{\psi})$	0.4664	0.0011	0.0715	0.0010	0.5275	0.4053	0.1221
		$h(x_0;\underline{\psi})$	1.3107	0.1841	0.3421	0.1810	2.1445	0.4770	1.6676
100	70	$R(x_0;\underline{\psi})$	0.5054	0.0032	0.1244	0.0009	0.5629	0.4479	0.1150
		$h(x_0;\underline{\psi})$	1.5387	0.2490	0.3978	0.1683	2.3427	0.7349	1.6079
	100	$R(x_0;\underline{\psi})$	0.4656	0.0008	0.0632	0.0008	0.5194	0.4118	0.1076
		$h(x_0;\underline{\psi})$	1.2752	0.1546	0.3134	0.0008	2.0448	0.5056	1.5392
200	140	$R(x_0;\underline{\psi})$	0.5029	0.0026	0.1118	0.0005	0.5455	0.4603	0.0852
		$h(x_0;\underline{\psi})$	1.4852	0.0879	0.2364	0.0347	1.8502	1.1202	0.7300
	200	$R(x_0;\underline{\psi})$	0.4641	0.0004	0.0421	0.0003	0.4989	0.4292	0.0697
		$h(x_0;\underline{\psi})$	1.2275	0.0408	0.1610	0.0401	1.6198	0.8353	0.7845
500	350	$R(x_0;\underline{\psi})$	0.5045	0.0026	0.1119	0.0003	0.5401	0.4689	0.0712
		$h(x_0;\underline{\psi})$	1.4700	0.0682	0.2081	0.0219	1.7600	1.1795	0.5801
	500	$R(x_0;\underline{\psi})$	0.4637	0.0002	0.0317	0.0002	0.4886	0.4388	0.0498
		$h(x_0;\underline{\psi})$	1.2047	0.0274	0.1319	0.0249	1.5141	0.8953	0.6188

Table 10

ML averages, estimated risks, relative errors, variances and 95% ACIs of the rf and the hrf of AFWE-L distribution for different samples of size n and $NR = 1000, (\alpha = 0.8, \beta = 0.5, \lambda = 0.5, \theta = 0.5, x_0 = 0.4)$

n	r	rf and hrf	Average	ER	RE	Variance	UL	LL	Length
30	21	$R(x_0; \underline{\psi})$	0.4704	0.0036	0.1187	0.0025	0.5692	0.3717	0.1698
		$h(x_0; \underline{\psi})$	2.6328	0.9767	0.3087	0.6589	4.2177	1.0479	3.1698
	30	$R(x_0; \underline{\psi})$	0.4381	0.0058	0.1520	0.0017	0.5187	0.3574	0.1613
		$h(x_0; \underline{\psi})$	2.3984	1.1910	0.3409	0.5470	3.8481	0.9488	2.8993
60	42	$R(x_0; \underline{\psi})$	0.4752	0.0022	0.0925	0.0014	0.5491	0.4013	0.1478
		$h(x_0; \underline{\psi})$	2.5664	0.6280	0.2476	0.2254	3.4969	1.6359	1.8610
	60	$R(x_0; \underline{\psi})$	0.4415	0.0045	0.1341	0.0008	0.4981	0.3850	0.1131
		$h(x_0; \underline{\psi})$	2.3880	0.9760	0.3086	0.3151	3.4882	1.2878	2.2004
100	70	$R(x_0; \underline{\psi})$	0.4768	0.0015	0.0771	0.0009	0.5339	0.4197	0.1141
		$h(x_0; \underline{\psi})$	2.5322	0.5460	0.2308	0.0988	3.1484	1.9163	1.2319
	100	$R(x_0; \underline{\psi})$	0.4428	0.0040	0.1261	0.0005	0.4853	0.4004	0.0849
		$h(x_0; \underline{\psi})$	2.3374	0.8374	0.2859	0.0917	2.9309	1.7498	1.1871
200	140	$R(x_0; \underline{\psi})$	0.4777	0.0013	0.0703	0.0006	0.5273	0.4281	0.0992
		$h(x_0; \underline{\psi})$	2.5170	0.5187	0.2250	0.0510	2.9594	2.0746	0.8848
	200	$R(x_0; \underline{\psi})$	0.4441	0.0036	0.1196	0.0002	0.4726	0.4155	0.0571
		$h(x_0; \underline{\psi})$	2.2917	0.8521	0.2884	0.0253	2.6033	1.9801	0.6232
500	350	$R(x_0; \underline{\psi})$	0.4781	0.0010	0.0636	0.0004	0.5188	0.4373	0.0814
		$h(x_0; \underline{\psi})$	2.4986	0.5243	0.2262	0.0311	2.8443	2.1530	0.6913
	500	$R(x_0; \underline{\psi})$	0.4456	0.0033	0.1142	0.0001	0.4624	0.4289	0.0334
		$h(x_0; \underline{\psi})$	2.2832	0.8500	0.2880	0.0077	2.4552	2.1112	0.3440

6. Applications

This section is devoted to exhibit the applicability and flexibility of AFWE-L distribution for data modeling. Three applications on COVID-19 data in some countries is used to demonstrate the superiority of AFWE-L distribution over some known distributions namely, L-W, NMW, AW, FWE and L distributions. ML estimates of the parameters, rf and the hrf based on two level of Type II censoring $(30\%, 0\%)$ and their *standard errors* (SE), *Kolmogorov-Smirnov* (K-S) statistic and its corresponding p-value, the

-2*log likelihood statistic* $(-2\mathcal{L})$, *Akaike information criterion* (AIC), *Bayesian information criterion* (BIC) and *corrected Akaike information criterion* (CAIC) are used to compare the fit of the competitor distributions, where

$$AIC = 2m - 2\mathcal{L}, \qquad BIC = m\ln(n) - 2\mathcal{L}$$

and

$$CAIC = AIC + 2\left(\frac{m(m+1)}{n-m-1}\right),$$

where

$\mathcal{L}$ is the natural logarithm of the value of the likelihood function evaluated at the ML estimates,

n is the number of the observations and m is the number of the estimated parameters.

The best distribution corresponds to the lowest values of AIC, BIC and CAIC, also the highest p-value.

6.1 Application 1

This application is given by Mubarak and Almetwally (2021). The application represents COVID-19 data which belong to the United Kingdom of 76 days, from 15 April to 30 June 2020. The data are formed of drought mortality rates. The data are: 0.0587, 0.0863, 0.1165, 0.1247, 0.1277, 0.1303, 0.1652, 0.2079, 0.2395, 0.2751, 0.2845, 0.2992, 0.3188, 0.3317, 0.3446, 0.3553, 0.3622, 0.3926, 0.3926, 0.4110, 0.4633, 0.4690, 0.4954, 0.5139, 0.5696, 0.5837, 0.6197, 0.6365, 0.7096, 0.7193, 0.7444, 0.8590, 1.0438, 1.0602, 1.1305, 1.1468, 1.1533, 1.2260, 1.2707, 1.3423, 1.4149, 1.5709, 1.6017, 1.6083, 1.6324, 1.6998, 1.8164, 1.8392, 1.8721, 1.9844, 2.1360, 2.3987, 2.4153, 2.5225, 2.7087, 2.7946, 3.3609, 3.3715, 3.7840, 3.9042, 4.1969, 4.3451, 4.4627, 4.6477, 5.3664, 5.4500, 5.7522, 6.4241, 7.0657, 7.4456, 8.2307, 9.6315, 10.1870, 11.1429, 11.2019 and 11.4584.

Figure 4 displays the plot of the empirical scaled TTT-transform of COVID-19 data of the United Kingdom, which implies that this data has a modified bathtub hazard function, boxplot and the histogram of the data. One can notice that this data is right-skewed. P-P plot, Q-Q plot and the fitted AFWE-L distribution plots indicate that AFWE-L distribution provides a better fit to this data.

Table 11 displays the K-S statistic and its corresponding p-value, $-2\mathcal{L}$ statistic, AIC, BIC and CAIC and Table 12 presents the ML estimates of the parameters, rf and hrf along with their SEs, under 0% and 30% levels of Type II censoring, for COVID-19 data of the United Kingdom.

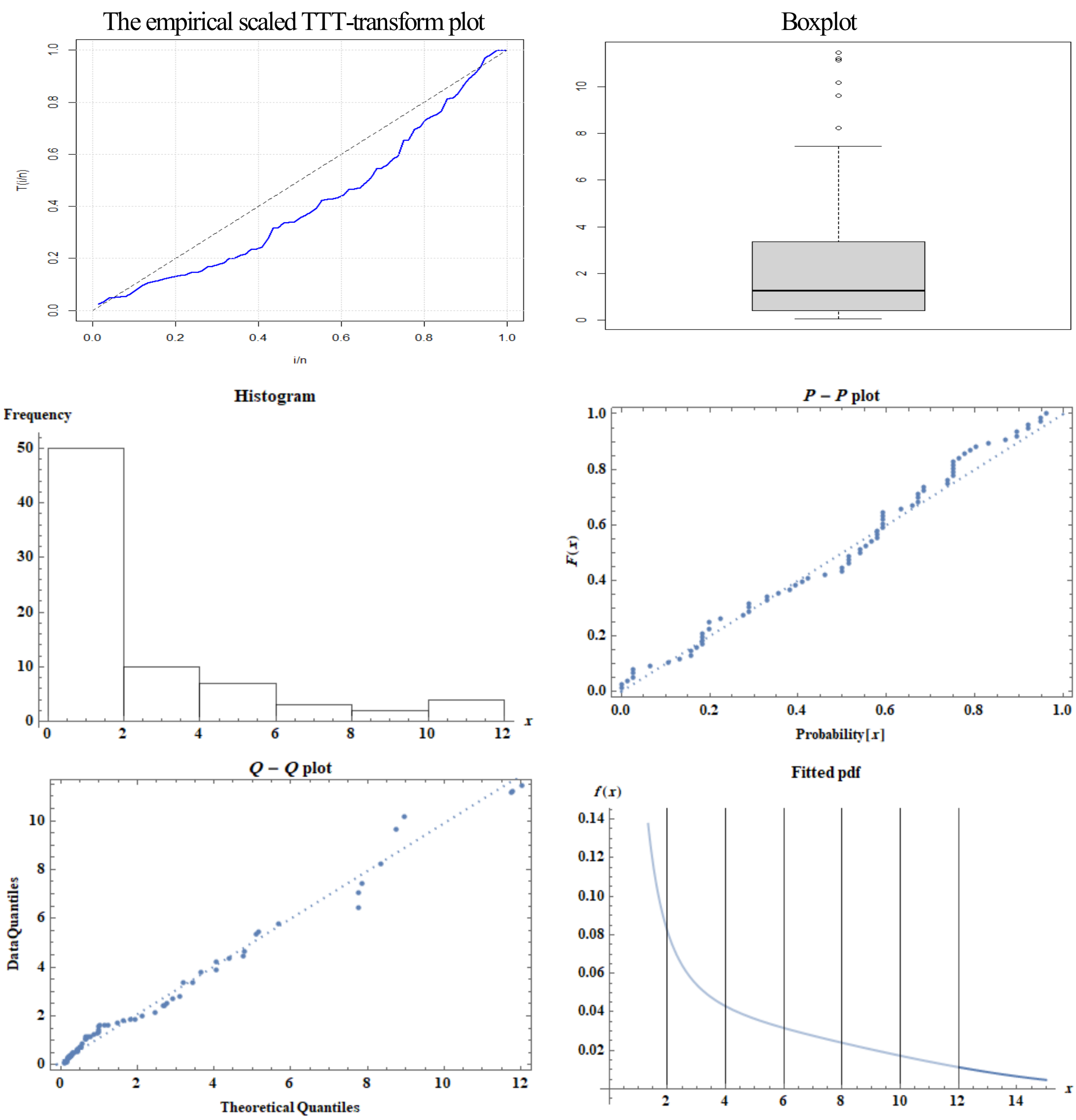

Figure 4: The empirical scaled TTT-transform plot, boxplot, histogram, P-P plot, Q-Q plot and the fitted pdf for COVID-19 data of the United Kingdom.

Table 11

K-S statistics, P-values, $-2\mathcal{L}$, AIC, BIC and CAIC of the fitted models for COVID-19 data of the United Kingdom

Model	K-S	P-value	$-2\mathcal{L}$	AIC	BIC	CAIC
AFWE-L	0.0790	0.9735	280.4346	288.4346	297.7575	288.9980
L-W	0.1316	0.5291	285.7349	293.7349	303.0579	294.2983
NMW	0.1184	0.6643	338.8597	348.8597	360.5134	349.7169
AW	0.1579	0.3012	415.7225	423.7225	433.0455	424.2859
FWE	0.1447	0.4050	334.5277	338.5277	343.1892	338.6921
L	0.1447	0.4057	297.1085	301.1085	305.7700	301.2729

Table 12

ML estimates and their relevant SEs of the fitted models for COVID-19 data of the United Kingdom

Level of censoring	$\underline{\psi}$, rf and hrf	MLE	SE
0%	α	0.1034	4.5277e-5
	β	0.5030	3.945e-5
	λ	0.2190	4.1068e-4
	θ	0.0434	8.7687e-5
	$R\left(x;\underline{\psi}\right)$	0.7107	4.2349e-5
	$h\left(x;\underline{\psi}\right)$	2.4842	1.8367e-4
30%	α	0.0985	2.7943e-5
	β	0.6212	0.0023
	λ	0.6901	0.0084
	θ	0.1553	0.0020
	$R\left(x;\underline{\psi}\right)$	0.7474	7.5962e-4
	$h\left(x;\underline{\psi}\right)$	3.3368	0.0165

6.2 Application 2

This application is provided by Mubarak and Almetwally (2021). The application represents COVID-19 data which belong to Japan of 38 days, from 4 September to 12 October 2020. The data is formed of drought mortality rates. The data are: 0.1596, 0.2733, 0.1142, 0.0851, 0.1976, 0.2243, 0.1810, 0.0828, 0.1504, 0.2169, 0.0404, 0.1208, 0.1334, 0.1589, 0.1184, 0.1698, 0.0648, 0.1027, 0.0511, 0.1019, 0.1520, 0.1006, 0.0624, 0.0372, 0.1112, 0.0859, 0.0854, 0.0847, 0.1443, 0.0836, 0.0238, 0.0355,0.0353, 0.0937, 0.0349, 0.0924, 0.0344 and 0.0228.

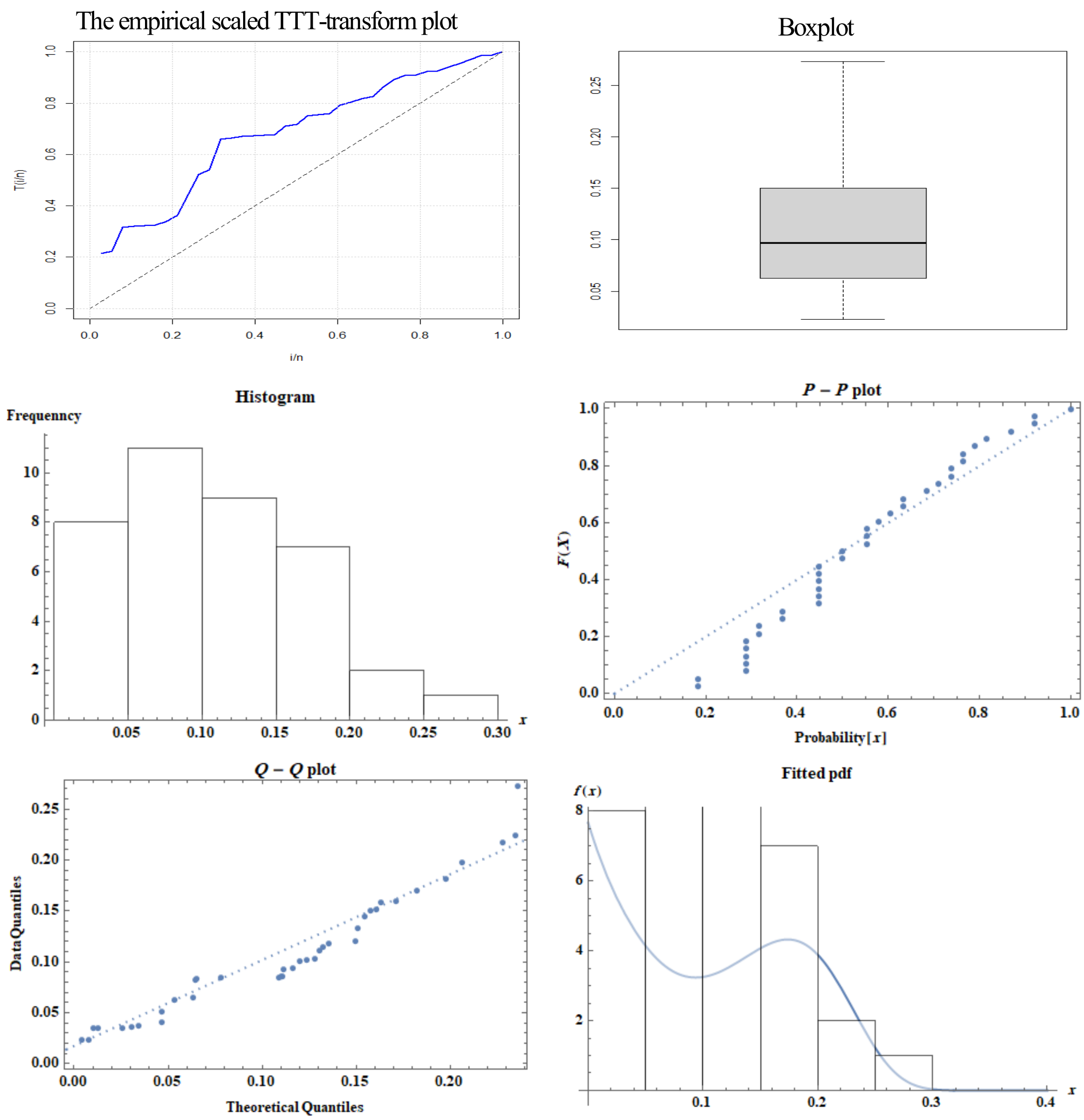

Figure 5: The empirical scaled TTT-transform plot, boxplot, histogram, P-P plot, Q-Q plot and the fitted pdf for COVID-19 data of Japan.

Figure 5 shows the plot of the empirical scaled TTT-transform of COVID-19 data of Japan, which indicates that this data has an increasing hazard function, the boxplot implies that this data is right skewed, the histogram of the data shows that this data is unimodal. The P-P plot, Q-Q plot and the fitted AFWE-L distribution plots indicate that AFWE-L distribution gives better fit for this data.

Tables 13 presents the K-S statistic and its corresponding p-value, $-2\mathcal{L}$ statistic, AIC, BIC and CAIC and Table 14 presents the ML estimates of the parameters, rf and hrf along with their SEs, under 0% and 30% levels of Type II censoring, for COVID-19 data of Japan.

Table 13

K-S statistics, P-values, $-2\mathcal{L}$, AIC, BIC and CAIC of the fitted models for COVID-19 data of Japan

Model	K-S	P-value	$-2\mathcal{L}$	AIC	BIC	CAIC
AFWE-L	0.1316	0.9033	-95.2786	-87.2786	-80.7283	-86.0665
L-W	0.1579	0.7379	-82.0023	-74.0023	-67.4519	-72.7902
NMW	0.1842	0.5453	-78.0422	-68.0422	-59.8543	-66.1692
AW	0.2105	0.3727	-65.0143	-57.0143	-50.4640	-55.8022
FWE	0.3684	0.0109	-58.9937	-54.9937	-51.7185	-54.6508
L	0.2632	0.1445	-77.4367	-73.4367	-70.1615	-73.0938

Table 14

ML estimates and their relevant SEs of the fitted models for COVID-19 data of Japan

Level of censoring	$\underline{\psi}$, rf and hrf	MLE	SE
0%	α	14.4886	0.0664
	β	0.7426	0.0065
	λ	0.2609	0.0030
	θ	2.2023	0.0283
	$R\left(x;\underline{\psi}\right)$	0.1825	7.9325e-4
	$h\left(x;\underline{\psi}\right)$	26.0360	0.3424
30%	α	17.2038	0.1964
	β	0.7113	0.0080
	λ	0.2473	0.0037
	θ	1.9928	0.0318
	$R\left(x;\underline{\psi}\right)$	0.1260	0.0010

	$h\left(x;\underline{\psi}\right)$	18.8154	0.2120

6.3 Application 3

This application is given by Liu *et al.* (2021). In this application the survival times of patients suffering from the COVID-19 epidemic in China are considered. The data represent the survival times of patients from the time admitted to the hospital until death. Among them, a group of 53 COVID-19 patients were found in critical condition in hospital from January to February 2020. Among them, 37 patients (70%) were men and 16 women (30%), 40 patients (75%) were diagnosed with chronic diseases, especially including high blood pressure, and diabetes, 47 patients (88%) had common clinical symptoms of the flu, 42 patients (81%) were coughing, 37 (69%) were short of breath, and 28 patients (53%) had fatigue. 50 (95%) patients had bilateral pneumonia showed by the chest computed tomographic scans.

The data are: 0.054, 0.064, 0.704, 0.816, 0.235, 0.976, 0.865, 0.364, 0.479, 0.568, 0.352, 0.978, 0.787, 0.976, 0.087, 0.548, 0.796, 0.458, 0.087, 0.437, 0.421, 1.978, 1.756, 2.089, 2.643, 2.869, 3.867, 3.890, 3.543, 3.079, 3.646, 3.348, 4.093, 4.092, 4.190, 4.237, 5.028, 5.083, 6.174, 6.743, 7.274, 7.058, 8.273, 9.324, 10.827, 11.282, 13.324, 14.278, 15.287, 16.978, 17.209, 19.092 and 20.083.

Figure 6 presents the plot of the empirical scaled TTT-transform of COVID-19 data of China, which indicates that this data has a bathtub hazard function, boxplot and the histogram of the data show that this data is right-skewed. The P-P plot, Q-Q plot and the fitted AFWE-L distribution plots implies that AFWE-L distribution presents better fit for this data.

Tables 15 displays the K-S statistic and its corresponding p-value, $-2\mathcal{L}$ statistic, AIC, BIC and CAIC and Table 16 presents the ML estimates of the parameters, rf and hrf of AFWE-L distribution along with their SEs, under 0% and 30% levels of Type II censoring, for COVID-19 data of China.

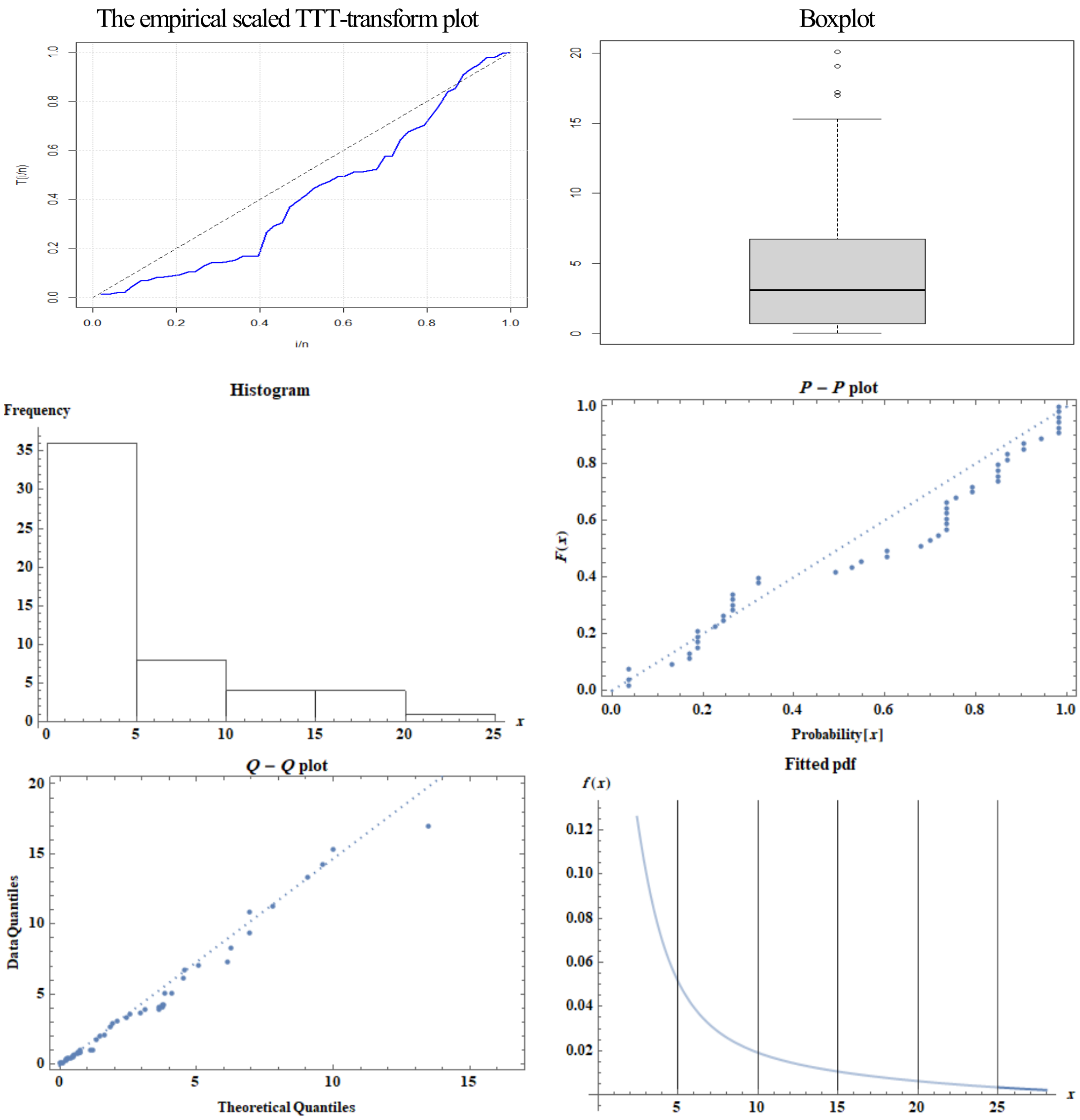

Figure 6: The empirical scaled TTT-transform plot, Boxplot, P-P plot, Q-Q plot and the histogram and the fitted AFWE-L distribution for COVID-19 data of the China

Table 15
K-S statistics, P-values, $-2\mathcal{L}$, AIC, BIC and CAIC of the fitted models for COVID-19 data of China

Model	K-S	P-value	$-2\mathcal{L}$	AIC	BIC	CAIC
AFWE-L	0.0943	0.9747	268.5411	276.5411	284.4223	277.3745
L-W	0.1509	0.5861	279.0068	287.0068	294.8880	287.8401
NMW	0.2264	0.1317	302.7820	312.7820	322.6334	314.0586
AW	0.1132	0.8898	313.3504	321.3504	329.2316	322.1838
FWE	0.1132	0.8907	279.2810	283.2810	287.2216	283.5210
L	0.2453	0.0815	295.8015	299.8015	303.7420	300.0415

Table 16
ML estimates and their relevant SEs of the fitted models for COVID-19 data of China

Level of censoring	$\underline{\psi}$, rf and hrf	MLE	SE
0%	α	0.0526	5.0006e-5
	β	3.6678	0.0063
	λ	0.3888	0.0021
	θ	0.2198	5.7627e-4
	$R\left(x;\underline{\psi}\right)$	0.8559	1.4027e-4
	$h\left(x;\underline{\psi}\right)$	23.2526	0.0395
30%	α	0.0465	9.5686e-5
	β	3.5952	0.0111
	λ	0.4269	0.0020
	θ	0.2291	5.7361e-4
	$R\left(x;\underline{\psi}\right)$	0.8594	1.0782e-4
	$h\left(x;\underline{\psi}\right)$	22.7908	0.0695

Concluding remarks:

- AFWE-L distribution has the lowest K-S values and the highest p-values for the three applications. Thus, it provides the best fit for these data compared to the other competitors of distributions.
- Moreover, the AFWE-L distribution has the smallest values of the $-2\mathcal{L}$ statistic, AIC, BIC and CAIC, which imply that the proposed model is the best among the other competitors of distributions (L-W, NMW, AW, FWE and L).

- The ML estimates of the parameters, rf and hrf of AFWE-L distribution have smaller SEs for the case of the complete samples (0% level of censoring) comparing to the case of censored samples (30% level of censoring). This returns to the amount of lost information through the censoring.

7. Conclusion

In this paper, a new four-parameter competing risks model, called AFWE-L distribution is introduced by combining the FWE distribution and the L distribution in a series system. AFWE-L distribution has high flexibility and diversity in the shapes of the pdf as well as the hrf. Several statistical properties of the proposed model are derived. The ML method is used to estimate the model parameters, rf and hrf based on Type II censored samples. Moreover, simulation study is conducted to evaluate the performance of the ML estimates of AFWE-L distribution parameters, rf and hrf. AFWE-L distribution is the best fitting among many known distributions to three real applications on COVID-19 data in some countries.

8. References

Abba, B., Wang, H. and Bakouch, H. S. (2022). A reliability and survival model for one and two failure modes system with applications to complete and censored datasets. *Reliability Engineering and System Safety,* Vol. 223, No. 108460, pp. 1-17. https://doi.org/10.1016/j.ress.2022.108460.

Almalki, S. J. and Yuan, J. (2013). A new modified Weibull distribution. Reliability *Engineering and System Safety,* Vol. 111, pp. 164-170.

Bebbington, M., Lai, C. D. and Zitikis, R. (2007a). A flexible Weibull extension. *Reliability Engineering and System Safety,* Vol. 92, No. 6, pp. 719-726. https://doi.org/10.1016/j.ress.2006.03.004.

Bebbington, M., Lai, C. D. and Zitikis, R. (2007b). Modeling human mortality using mixtures of bathtub shaped failure distributions. *Journal of Theoretical Biology,* Vol. 245, No. 3, pp. 528-538. https://doi.org/10.1016/j.jtbi.2006.11.011.

Bousquet, N. and Bertholon, H. (2006). An alternative competing risk model to the Weibull distribution for modelling aging in lifetime data analysis. *Lifetime Data Analysis,* Vol. 12, No. 4, pp. 481-504. https://doi.org/10.1007/s10985-006-9019-8.

Buzaridah, M. M., Ramadan, D. A. and El-Desouky, B. S. (2022). Estimation of some lifetime parameters of flexible reduced logarithmic-inverse Lomax distribution under progressive Type-II censored data. *Journal of Mathematics*, Vol. 2022, No. 1690458, pp. 1-13. https://doi.org/10.1155/2022/1690458.

Choquet, R., Guedon, Y., Bensnard, A., Guillemain, M. and Pradel, R. (2013). Estimating stop over duration in the presence of trap-effects. *Ecological Modelling,* Vol. 250, pp. 111-118. https://doi.org/10.1016/j.ecolmodel.2012.11.002.

Cordeiro, G. M., Ortega, E. M. and Lemonte, A. (2013). The exponential-Weibull distribution. *Journal of Statistical Computation and Simulation,* Vol. 84, No. 12, pp. 2592-2606. https://doi.org/10.1080/00949655.2013.797982.

EL-Sagheer, R. M. (2018). Estimation of parameters of Weibull–Gamma distribution based on progressively censored data. *Statistical Papers*, Vol. 59, No. 2, pp. 725-757. https://doi.org/10.1007/s00362-016-0787-2.

EL-Sagheer, R. M., Shokr, E. M., Mohamed A. W. Mahmoud, M. A. W. and El-Desouky, B. S. (2021). Inferences for Weibull Fréchet Distribution Using a Bayesian and Non-Bayesian Methods on Gastric Cancer Survival Times. *Computational and Mathematical Methods in Medicine*, Vol. 2021, No. 9965856, pp. 1-12. https://doi.org/10.1155/2021/9965856.

Greene, W. H. (2018). *Econometric analysis*, 8th edition. Pearson Education India.

Gurvich, M. R., Dibenedetto, A. T. and Ranade, S. V. (1997). A new statistical distribution for characterizing the random strength of brittle materials. *Journal of Materials Science,* Vo. 32, pp. 2559-2564. https://doi.org/10.1023/A:1018594215963.

He, B., Cui, W. and Du, X. (2016). An additive modified Weibull distribution. *Reliability Engineering and System Safety,* Vol. 145, pp. 28-37. https://doi.org/10.1016/j.ress.2015.08.010.

Kamal, R. M. and Ismail, M. A. (2020). The flexible Weibull extension-Burr XII distribution: model, properties and applications. *Pakistan Journal of*

Statistics and Operation Research, Vo.16, No. 3, pp. 447-460. https://doi.org/10.18187/pjsor.v16i3.2957.

Khalil, A., Ijaz, M., Ali, K., Mashwani, W. K., Shafiq, M., Humam, P. and Kumam, W. (2021). A novel flexible additive Weibull distribution with real-life applications. *Communications in Statistics—Theory and Methods,* Vol. 50, No. 7, pp. 1557-1572. https://doi.org/10.1080/03610926.2020.1732658.

Lai, C. D. (2013). Constructions and applications of lifetime distributions. *Applied Stochastic Models in Business and Industry,* Vol. 29, No. 2, pp. 127-129.

Lai, C. D., Xie, M. and Murthy, D. N. P. (2003). A modified Weibull distribution. *IEEE Transactions on Reliability,* Vol. 52, No. 1, pp. 33-37.

Liu, X., Ahmed, Z., Gemeay, A. M., Abdulrahman, A. T., Hafez, E. H. and Khalil. N. (2021). Modeling the survival times of the COVID-19 patients with a new statistical model: A case study from China. *Plos one,* Vol. 16, No. 7, pp. 1-31. https://doi.org/10.1371/journal.pone.0254999.

Lomax, K. S. (1954). Business failures: another example of the analysis of failure data. *Journal of the American Statistical Association,* Vol. 49, No. 268, pp. 847-852.

Makubate, B., Oluyede, B. and Gabanakgosi, M. (2021). A new Lindley-Burr XII distribution: Model, Properties and Applications. *International Journal of Statistics and Probability,* Vol. 10, No. 4, pp. 33-51. https://doi.org/10.5539/ijsp.v10n4p33.

Mdlongwa, P., Oluyede, B. O., Amey, A. and Huang, S. (2017). The Burr XII modified Weibull distribution: Model, Properties and Applications. *Electronic Journal of Applied Statistical Analysis,* Vol. 10, No. 1, pp. 118-145.

Mubarak, A. E. and Almetwally, E. M. (2021). A new extension exponential distribution with applications of COVID-19 data. *Journal of Financial and Business Research,* Vol. 22, No. 1, pp. 444-460.

Oluyede, B. O., Foya, S., Warahena-Liyanage, G. and Huang, S. (2016). The log-logistic Weibull distribution with applications to lifetime data. *Austrian Journal of Statistics,* Vol. 45, No. 3, pp. 43-69. https://doi.org/10.17713/ajs.v45i3.107.

Osagie, S. A. and Osemwenkhae, J. E. (2020). Lomax-Weibull distribution with properties and applications in lifetime analysis. *International Journal of Mathematical Analysis and Optimization: Theory and Applications,* Vol. 2020, No. 1, pp. 718-732.

Rényi, A. (1961). On measures of entropy and information. *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability*, Vol. 1, pp. 547–561. https://projecteuclid.org/euclid.bsmsp/1200512181.

Shakhatreh, M. K., Lemonte, A. J. and Moreno-Arenas, G. (2019). The log-normal modified Weibull distribution and its reliability implications. *Reliability Engineering and System Safety,* Vol. 188, pp. 6-22. https://doi.org/10.1016/j.ress.2019.03.014.

Singh, B. (2016). An additive Perks-Weibull model with bathtub-shaped hazard rate function. *Communications in Mathematics and Statistics,* Vol. 4, pp. 473-493.

Tarvirdizade, B. and Ahmadpour, M. (2019). A new extension of Chen distribution with applications to lifetime data. *Communications in Mathematics and Statistics,* Vol. 9, pp. 23-38. https://doi.org/10.1007/s40304-019-00185-4.

Thach, T. T. (2022). A Three-Component Additive Weibull Distribution and Its Reliability Implications. *Symmetry*, Vol. 14, No. 7:1455, pp. 1-21. https://doi.org/10.3390/sym14071455.

Thach, T. T. and Bris, R. (2021). An additive Chen-Weibull distribution and its applications in reliability modeling. *Quality and Reliability Engineering International,* Vol. 37, No. 1, pp. 352-373. https://doi.org/10.1002/qre.2740.

Tsallis, C. (1988). Possible generalization of Boltzmann-Gibbs statistics. *Journal of Statistical Physics*, Vol. 52, No. 1/2, pp. 479-487.

Wang, F. K. (2000). A new model with bathtub-shaped failure rate using an additive Burr XII distribution. *Reliability Engineering and System Safety,* Vol. 70, No. 3, pp. 305-312.

Xavier, T., Jose, J. K. and Nadarajah, S. (2022). An additive power□transformed half□logistic model and its applications in reliability. *Quality and Reliability Engineering International*, pp. 1-18. https://doi.org/10.1002/qre.3119.

Xie, M. and Lai, C. D. (1995). Reliability analysis using an additive Weibull model with bathtub-shaped failure rate function. *Reliability Engineering and System Safety*, Vol. 52, No. 1, pp. 87-93. https://doi.org/10.1016/0951-8320(95)00149-2.

9. Appendix

I. The mode

The mode of AFWE-L distribution can be obtained by differentiating the pdf in (7) with respect to x and equating to zero as follows:

$$\acute{f}\left(x_0;\underline{\psi}\right)=0.$$

Since

$$f\left(x;\underline{\psi}\right) = h\left(x;\underline{\psi}\right)R\left(x;\underline{\psi}\right) = h\left(x;\underline{\psi}\right)e^{-H\left(x;\underline{\psi}\right)},$$

where $H\left(x;\underline{\psi}\right)$ is the chrf defined in (10), then,

$$\acute{f}\left(x;\underline{\psi}\right) = h\left(x;\underline{\psi}\right)\left\{\left[-\frac{\partial}{\partial x}H\left(x;\underline{\psi}\right)\right]R\left(x;\underline{\psi}\right)\right\} + \left[\frac{\partial}{\partial x}h\left(x;\underline{\psi}\right)\right]R\left(x;\underline{\psi}\right), \qquad (A1)$$

where

$$\frac{\partial}{\partial x}H\left(x;\underline{\psi}\right) = h\left(x;\underline{\psi}\right),$$

and

$$\frac{\partial}{\partial x}h\left(x;\underline{\psi}\right) = \acute{h}\left(x;\underline{\psi}\right).$$

Hence, $(A1)$ can be written as

$$\acute{f}\left(x;\underline{\psi}\right) = \left[\acute{h}\left(x;\underline{\psi}\right) - h^2\left(x;\underline{\psi}\right)\right]R\left(x;\underline{\psi}\right), \qquad (A2)$$

where

$$\acute{h}\left(x;\underline{\psi}\right) = \left[\left(\alpha+\frac{\beta}{x^2}\right)^2 - \frac{2\beta}{x^3}\right]e^{\alpha x-\frac{\beta}{x}} - \frac{\theta}{\lambda^2}\left(1+\frac{x}{\lambda}\right)^{-2},$$

$$h^2\left(x;\underline{\psi}\right) = \left(\alpha+\frac{\beta}{x^2}\right)^2 e^{2\left(\alpha x-\frac{\beta}{x}\right)} + \frac{\theta^2}{\lambda^2}\left(1+\frac{x}{\lambda}\right)^{-2} - \frac{2\theta}{\lambda}\left(\alpha+\frac{\beta}{x^2}\right)\left(1+\frac{x}{\lambda}\right)^{-1}e^{\alpha x-\frac{\beta}{x}}.$$

Therefore, equating $(A2)$ to zero, one can obtain the following nonlinear equation

$$\left[\left(\alpha+\frac{\beta}{x_0^2}\right)^2\left(1-e^{\alpha x_0-\frac{\beta}{x_0}}\right) - \frac{2\beta}{x_0^3} + \frac{2\theta}{\lambda}\left(\alpha+\frac{\beta}{x_0^2}\right)\left(1+\frac{x_0}{\lambda}\right)^{-1}\right] \times e^{\alpha x_0-\frac{\beta}{x_0}} - (\theta+1)\frac{\theta}{\lambda^2}\left(1+\frac{x_0}{\lambda}\right)^{-2} = 0. \qquad (A3)$$

Equation $(A3)$ is a nonlinear equation, which can be solved numerically to obtain the mode of AFWE-L distribution.

II. The r^{th} non-central moment

Since

$$\mu_r' = \int_0^\infty x^r f\left(x;\underline{\psi}\right)dx = -\int_0^\infty x^r dR\left(x;\underline{\psi}\right).$$

Using integration by parts, then

$$\mu_r' = \int_0^\infty r x^{r-1} R\left(x;\underline{\psi}\right) dx = \int_0^\infty r x^{r-1} e^{-e^{\alpha x-\frac{\beta}{x}}} \left(1+\frac{x}{\lambda}\right)^{-\theta} dx\,.$$

Since the power series expansion of $e^{-e^{\alpha x-\frac{\beta}{x}}}$ is as follows:

$$\mu_r' = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \int_0^\infty r x^{r-1} e^{i\left(\alpha x-\frac{\beta}{x}\right)} \left(1+\frac{x}{\lambda}\right)^{-\theta} dx\,,$$

By using the power series expansion of $e^{i\alpha x}$ and $e^{-\frac{i\beta}{x}}$, then

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+k}\, i^{j+k}\, \alpha^j\, \beta^k\, r}{i!\,j!\,k!} \int_0^\infty x^{r+j-k-1} \left(1+\frac{x}{\lambda}\right)^{-\theta} dx\,.$$

Using integration by substitution, then,

$$\mu_r' = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{i+k}\, i^{j+k}\, \alpha^j\, \beta^k}{i!\,j!\,k!} \Big[r\, \lambda^{r+j-k} \times \mathbf{B}\big(r+j-k, \theta-(r+j-k)\big)\Big]\,,$$

where

$$0 < r+j-k < \theta.$$

III. The asymptotic Fisher information matrix

The asymptotic Fisher information of AFWE-L distribution is given by

$$\tilde{I}\left(\underline{\psi}\right) \simeq [I_{ij}], \qquad i,j = 1,2,3,4, \qquad \text{(A4)}$$

where

$$I_{11} = -\frac{\partial^2 \ell}{\partial \alpha^2} = -\sum_{i=1}^{r}\left(\frac{h\left(x_{(i)};\underline{\psi}\right) h_{\alpha\alpha}\left(x_{(i)};\underline{\psi}\right) - h_\alpha^2\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\psi}\right)}\right) + \sum_{i=1}^{r} x_{(i)}^2 e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}} + (n-r) x_{(r)}^2 e^{\alpha x_{(r)}-\frac{\beta}{x_{(r)}}},$$

$$I_{12} = -\frac{\partial^2 \ell}{\partial \alpha \partial \beta} = -\sum_{i=1}^{r}\left(\frac{h\left(x_{(i)};\underline{\psi}\right) h_{\alpha\beta}\left(x_{(i)};\underline{\psi}\right) - h_\beta^2\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\psi}\right)}\right) - \sum_{i=1}^{r} e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}} - (n-r) e^{\alpha x_{(r)}-\frac{\beta}{x_{(r)}}},$$

$$I_{13}=-\frac{\partial^2\ell}{\partial\alpha\partial\lambda}=\sum_{i=1}^{r}\frac{h_\alpha\left(x_{(i)};\underline{\psi}\right)h_\lambda\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\psi}\right)},$$

$$I_{14}=-\frac{\partial^2\ell}{\partial\alpha\partial\theta}=\sum_{i=1}^{r}\frac{h_\alpha\left(x_{(i)};\underline{\psi}\right)h_\theta\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\psi}\right)},$$

$$I_{22}=-\frac{\partial^2\ell}{\partial\beta^2}=-\sum_{i=1}^{r}\left(\frac{h\left(x_{(i)};\underline{\psi}\right)h_{\beta\beta}\left(x_{(i)};\underline{\psi}\right)-h_\beta^2\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\psi}\right)}\right)$$
$$+\sum_{i=1}^{r}\frac{1}{x_{(i)}^2}e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}+\frac{(n-r)}{x_{(r)}^2}e^{\alpha x_{(r)}-\frac{\beta}{x_{(r)}}},$$

$$I_{23}=-\frac{\partial^2\ell}{\partial\beta\partial\lambda}=\sum_{i=1}^{r}\frac{h_\beta\left(x_{(i)};\underline{\psi}\right)h_\lambda\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\psi}\right)},$$

$$I_{24}=-\frac{\partial^2\ell}{\partial\beta\partial\theta}=\sum_{i=1}^{r}\frac{h_\beta\left(x_{(i)};\underline{\psi}\right)h_\theta\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\psi}\right)},$$

$$I_{33}=-\frac{\partial^2\ell}{\partial\lambda^2}=-\sum_{i=1}^{r}\left(\frac{h\left(x_{(i)};\underline{\psi}\right)h_{\lambda\lambda}\left(x_{(i)};\underline{\psi}\right)-h_\lambda^2\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\psi}\right)}\right)$$
$$+\frac{\theta}{\lambda^3}\sum_{i=1}^{r}\frac{x_{(i)}\left(2+\frac{x_{(i)}}{\lambda}\right)}{\left(1+\frac{x_{(i)}}{\lambda}\right)^2}+\frac{\theta(n-r)}{\lambda^3}\frac{x_{(r)}\left(2+\frac{x_{(r)}}{\lambda}\right)}{\left(1+\frac{x_{(r)}}{\lambda}\right)^2},$$

$$I_{34}=-\frac{\partial^2\ell}{\partial\lambda\partial\theta}=-\sum_{i=1}^{r}\left(\frac{h\left(x_{(i)};\underline{\psi}\right)h_{\theta\theta}\left(x_{(i)};\underline{\psi}\right)-h_\theta^2\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\psi}\right)}\right)$$
$$-\frac{1}{\lambda^2}\sum_{i=1}^{r}\frac{x_{(i)}}{\left(1+\frac{x_{(i)}}{\lambda}\right)}-\frac{(n-r)}{\lambda^2}\frac{x_{(r)}}{\left(1+\frac{x_{(r)}}{\lambda}\right)},$$

and

$$I_{44}=-\frac{\partial^2\ell}{\partial\theta^2}=\sum_{i=1}^{r}\frac{h_\theta^2\left(x_{(i)};\underline{\psi}\right)}{h^2\left(x_{(i)};\underline{\psi}\right)},$$

where

$$h_{\alpha\alpha}\left(x_{(i)};\underline{\psi}\right)=\frac{\partial^2 h\left(x_{(i)};\underline{\psi}\right)}{\partial\alpha^2}=\frac{\partial h_\alpha\left(x_{(i)};\underline{\psi}\right)}{\partial\alpha}$$
$$=x_{(i)}^2\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}+2x_{(i)}e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}},$$

$$h_{\alpha\beta}\left(x_{(i)};\underline{\psi}\right)=\frac{\partial h_\alpha\left(x_{(i)};\underline{\psi}\right)}{\partial\beta}=-\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}},$$

$$h_{\beta\beta}\left(x_{(i)};\underline{\psi}\right)=\frac{\partial^2 h\left(x_{(i)};\underline{\psi}\right)}{\partial\beta^2}=\frac{\partial h_\beta\left(x_{(i)};\underline{\psi}\right)}{\partial\beta}$$
$$=\frac{1}{x_{(i)}^2}\left(\alpha+\frac{\beta}{x_{(i)}^2}\right)e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}}-\frac{2}{x_{(i)}^3}e^{\alpha x_{(i)}-\frac{\beta}{x_{(i)}}},$$

$$h_{\lambda\lambda}\left(x_{(i)};\underline{\psi}\right)=\frac{\partial^2 h\left(x_{(i)};\underline{\psi}\right)}{\partial\lambda^2}=\frac{\partial h_\lambda\left(x_{(i)};\underline{\psi}\right)}{\partial\lambda}=\frac{2\theta}{\lambda^3\left(1+\frac{x_{(i)}}{\lambda}\right)^3},$$

and

$$h_{\lambda\lambda}\left(x_{(i)};\underline{\psi}\right)=\frac{\partial h_\lambda\left(x_{(i)};\underline{\psi}\right)}{\partial\theta}=\frac{-1}{\lambda^2\left(1+\frac{x_{(i)}}{\lambda}\right)^2}.$$

IV. The delta method

Using the delta method, the asymptotic variances of $\hat{R}\left(x;\underline{\hat{\psi}}\right)$ and $\hat{h}\left(x;\underline{\hat{\psi}}\right)$ can be derived , respectively, by:

$$\widehat{var}\left(\hat{R}\left(x;\underline{\hat{\psi}}\right)\right)=\acute{\xi}\tilde{I}^{-1}\left(\underline{\psi}\right)\xi\Big|_{\underline{\hat{\psi}}},$$
$$\widehat{var}\left(\hat{h}\left(x;\underline{\hat{\psi}}\right)\right)=\acute{\eta}\tilde{I}^{-1}\left(\underline{\psi}\right)\eta\Big|_{\underline{\hat{\psi}}},$$

where

$\acute{\xi}=\left(R_\alpha\left(x;\underline{\psi}\right)\quad R_\beta\left(x;\underline{\psi}\right)\quad R_\lambda\left(x;\underline{\psi}\right)\quad R_\theta\left(x;\underline{\psi}\right)\right)$ is the first partial differentiation of the rf with respect to α,β,λ and θ and

$\acute{\eta}=\left(h_\alpha\left(x;\underline{\psi}\right)\quad h_\beta\left(x;\underline{\psi}\right)\quad h_\lambda\left(x;\underline{\psi}\right)\quad h_\theta\left(x;\underline{\psi}\right)\right)$ is the first partial differentiation of the rf with respect to α,β,λ and θ, where

$$R_\alpha\left(x;\underline{\psi}\right)=\frac{\partial R\left(x;\underline{\psi}\right)}{\partial\alpha}=-xe^{\alpha x}e^{-e^{\alpha x-\frac{\beta}{x}}},$$

$$R_\beta\left(x;\underline{\psi}\right)=\frac{\partial R\left(x;\underline{\psi}\right)}{\partial\beta}=\frac{1}{x}e^{-\frac{\beta}{x}}e^{-e^{\alpha x-\frac{\beta}{x}}},$$

$$R_\lambda\left(x;\underline{\psi}\right)=\frac{\partial R\left(x;\underline{\psi}\right)}{\partial\lambda}=-\frac{\theta x}{\lambda^2}\left(1+\frac{x}{\lambda}\right)^{-(\theta+1)},$$

$$R_\theta\left(x;\underline{\psi}\right)=\frac{\partial R\left(x;\underline{\psi}\right)}{\partial\lambda}=-\left(1+\frac{x}{\lambda}\right)^{-\theta}\ln\left(1+\frac{x}{\lambda}\right),$$

$$h_\alpha\left(x;\underline{\psi}\right)=\frac{\partial h\left(x;\underline{\psi}\right)}{\partial\alpha}=x\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}}+e^{\alpha x-\frac{\beta}{x}},$$

$$h_\beta\left(x;\underline{\psi}\right)=\frac{\partial h\left(x;\underline{\psi}\right)}{\partial\beta}=\frac{1}{x^2}e^{\alpha x-\frac{\beta}{x}}-\frac{1}{x}\left(\alpha+\frac{\beta}{x^2}\right)e^{\alpha x-\frac{\beta}{x}},$$

$$h_\lambda\left(x;\underline{\psi}\right)=\frac{\partial h\left(x;\underline{\psi}\right)}{\partial\lambda}=-\frac{\theta}{\lambda^2\left(1+\frac{x}{\lambda}\right)^2},$$

and

$$h_\theta\left(x;\underline{\psi}\right)=\frac{\partial h\left(x;\underline{\psi}\right)}{\partial\theta}=\frac{1}{\lambda\left(1+\frac{x}{\lambda}\right)}.$$