A Mixture of Weibull Distribution and its Inverse: Properties and Estimation

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The methods to construct appropriate new models for lifetime data sets are very popular nowadays among the researchers of this area where existed models in the literature are unsuitable for some situations. Among these methods, mixture distributions which are useful in fitting data that is generated by a complex process. Also inverted distributions are useful in modeling data that variable is inherently the reciprocal of a known variable. The Weibull distribution, having the exponential and Rayleigh as special cases, is a very popular distribution for modeling lifetime data and for modeling phenomena with monotone failure rates. It is one of the best known distributions and has wide applications in diverse disciplines. In this paper we propose anew distribution, which is a mixture of weibull and its inverse (MWIW). The main purpose of this paper is to introduce a new mixture of weibull and its inverse distribution as a new model distribution in order to be applied efficiently in lifetime. Two cases are considered when the mixing proporotion is not related to parameter values and when it depends on parameter values. Some properties of the two models with some graphs of density, comulative, hazard and survival functions are discussed.The model parameters are estimated by the method of maximum likelihood estimation. A simulation study is carried out to illustrate the theoretical results of the maximum likelihood estimation. Finally, applications of the two mixtures are illustrated by real data set.

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1 Introduction:

In Many applications, the available data can be considered as data coming from a mixture population of two or more populations. This idea enables us to mix statistical distributions to get a new distribution carrying the properties of its components. In cases where each of the underlying random variables is continuous, the outcome variable will also be continuous and its probability density function is sometimes referred to as a mixture density.

The number of components in a mixture distribution is often restricted to being finite, although in some cases the components may be countable. Finite mixtures of distributions provide an important tool in modelling a wide range of observed phenomena, which do not normally yield to modelling through classical distributions like normal, gamma, binomial, etc.,. In a finite mixture model, the distribution of random quantity of interest is modelled as a mixture of a finite number of component distributions in varying proportions. A mixture model is, thus, able to model quite complex situations through an appropriate choice of its components to represent accurately the local areas of support of the true distribution. It can handle situations where a single parametric family is unable to provide a satisfactory model for local variation in the observed data.

[?] studied the estimation of the parameters of a mixture of normal distribution, and in this article he gives references to modern works in mixtures distributions that goes back as far as 1894 due to Karl Pearson.

In life testing reliability and quality control problems, mixed failure populations are sometimes encountered. Mixture distributions comprise a finite or infinite number of components, possibly of different distributional types, that can describe different features of data. Some of the references that discussed different types of mixtures of distributions are [?], [?] and [?].

Recent interest in the study of mixtures distribution started when [?] published a paper on estimating the five parameters in a mixture of two normal distributions. Finite mixtures involve a finite number of components. It results from the fact that different causes of failure of a system or production from different sources could lead to different failure distributions, this means that the population under study is non-homogenous. The observed distribution is supposed to be mixture of k different distributions e.g. you can suppose that your data have at K subpopulations.

$$f(y,\lambda) = p_1 f(y,\lambda_1) + p_2 f(y,\lambda_2) + \cdots + p_k f(y,\lambda_k)$$

$$f(y,\lambda) = \sum_{i=1}^{k} p_i f_i(y,\lambda_i) \quad (1)$$

where $x > o, k > 1, 0 \leq p_i \leq 1, i = 1,\ldots,k$ and $\sum_{i=1}^{k} p_i = 1$ where k is the number of components. The parameters $p_1, p_2, \ldots, p_k$ are called the mixing parameters, where p_i represent the probability that a given observation comes from population i with density function $f_1(.), f_2(.), \ldots, f_k(.)$. The special case where $k = 2$, eq(1) can be written as:

$$f(y) = pf_1(y,\lambda_1) + (1-p)f_2(y,\lambda_2) \quad (2)$$

and the cumulative distribution function (CDF) of this mixture distribution is given by:

$$F(y) = pF_1(y,\lambda_1) + (1-p)F_2(y,\lambda_2) \quad (3)$$

The reliability function at time t is given by

$$R(t) = pR_1(t,\lambda_1) + (1-p)R_2(t,\lambda_2) \quad (4)$$

2 A Mixture of Weibull and Inverse Weibull distribution

2.1 The probability density function

The probability density function of the mixture of Weibull distribution(WD) and Inverse Weibull distribution(IWD) has the following form

$$f(x) = \lambda f_1(x;\alpha,\beta) + (1-\lambda)f_2(x;a,b) \quad (5)$$

where λ and $1-\lambda$ are the mixing proportion, $f_1(x,\alpha,\beta)$ is the pdf of WD and $f_2(x,a,b)$ is the pdf of IWD. The mixture of these probability densities is given by;

$f(x;\alpha,\beta,a,b,\lambda) = (\lambda(\beta x\hat{}(\beta-1)\ e\hat{}(-(x/\alpha)\hat{}\beta)))/\alpha\hat{}\beta\ +$ $((1-\lambda)(bx\hat{}(-b-1)\ e\hat{}(-(1/ax)\hat{}b))/a\hat{}b).$ (6)

By choosing some arbitrary values for parameters, we provide a different shapes for the pdf of the MWIW Distribution as figure 1.

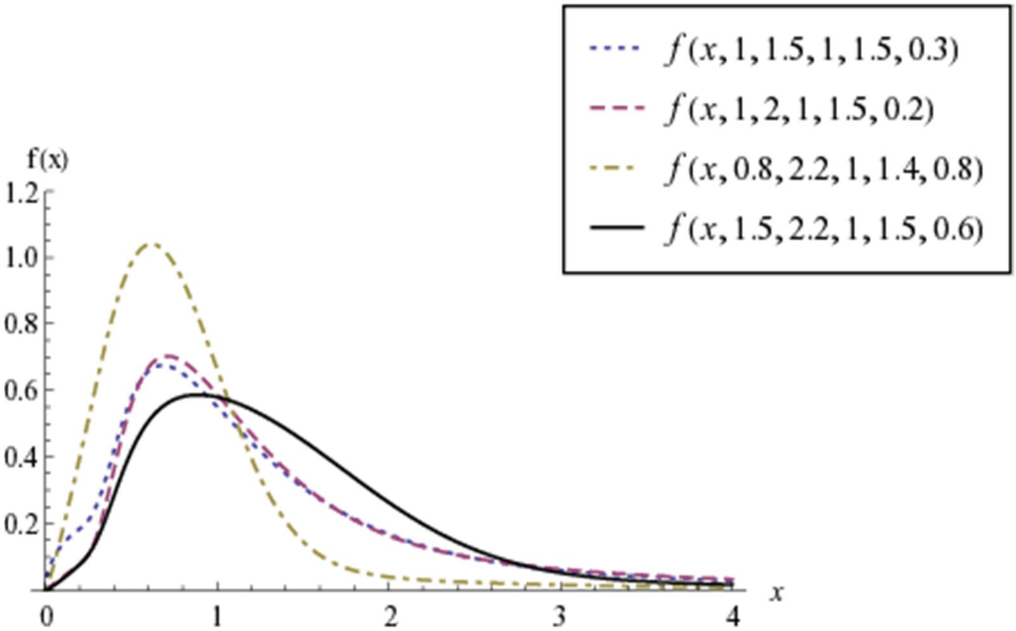

Figure 1: figure of the pdf of MWIW

2.2 Cumulative Distribution Function:

The cumulative distribution function of the mixture of Weibull and inverse weibull distribution has the following form,

$$F(x;\alpha\ \beta,\mathrm{a},\mathrm{b},\lambda) = (1-\lambda)e^{-\left(\frac{1}{ax}\right)^{b}} + \lambda\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right) \quad (7)$$

Where$(x,\beta,a,b,\alpha > 0)$. we provide a different shapes for the cdf of the (MWIW) Distribution as Figure 2:

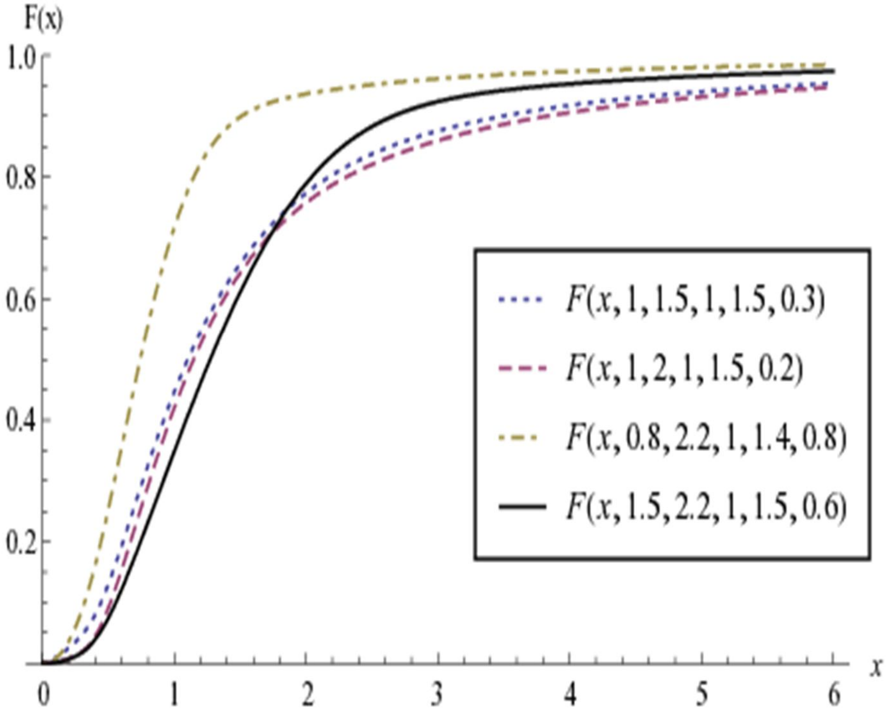

Figure 2: figure of the CDF of MWIW

2.3 Survival Function:

survival Function is also known as reliability function. It defined as the probability that the system will continue to survive beyond the specific time. It is defined mathematically as;

$$S(x) = 1 - F(x)$$

$$S(x;\alpha,\beta,\text{a},\text{b},\lambda) = 1 - \left((1-\lambda)e^{-\left(\frac{1}{ax}\right)^{b}} + \lambda\left(1 - e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)\right) \quad (8)$$

Where $(x,\lambda,a,b,\alpha > 0)$. we provide a diffrent shapes for the survival density for the(MWIW) Distribution as figure(3):

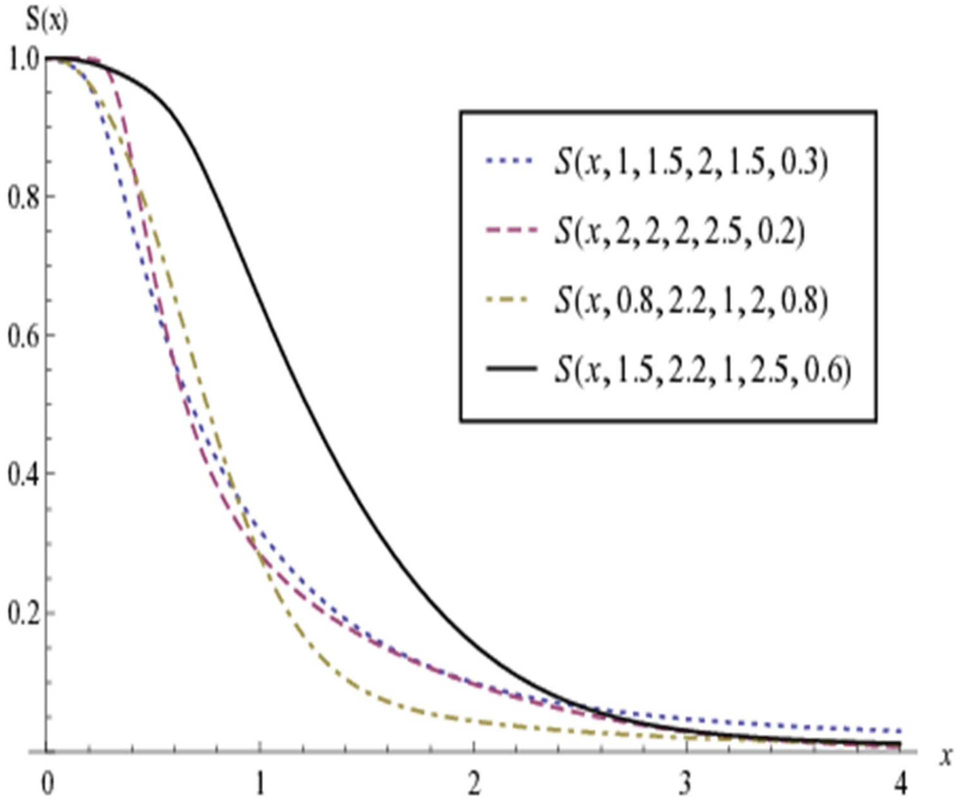

Figure 3: figure of the Survival density of MWIW

2.4 Hazard function:

Hazard function is defined mathematically as the ratio pdf and reliability function and is expressed as;

$$h(x) = \frac{f(x)}{S(x)}$$

so the hazard function of the mixture Weibull inverse Weibull is in the form:

$$h(x;\alpha,\beta,\mathrm{a},\mathrm{b},\lambda) = \frac{\frac{(1-\lambda)\left(bx^{-b-1}e^{-\left(\frac{1}{ax}\right)^{b}}\right)}{a^{b}} + \frac{\lambda\left(\beta x^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)}{\alpha\beta}}{1-\left((1-\lambda)e^{-\left(\frac{1}{ax}\right)^{b}} + \lambda\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)\right)} \quad (9)$$

Where$(x, \lambda, a, b, \alpha > 0)$. we provide a possible shapes for the hazard densities of the mixture model as figure 4:

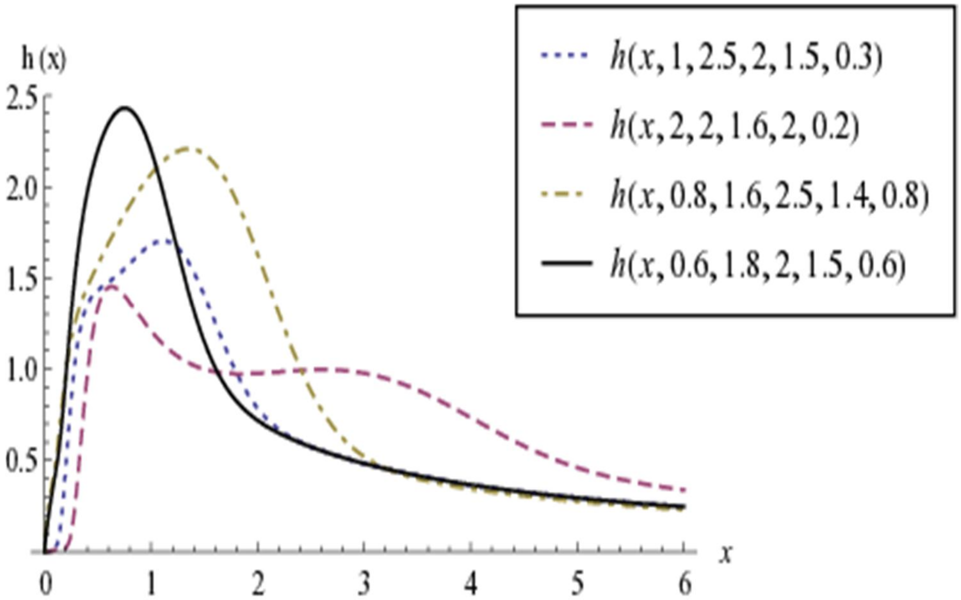

Figure 4: figure of the hazard density of MWIW

2.5 Some Statistical Properties:

Moments

Moments are important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution. Let $E_W(x^r)$ and $E_{IW}(x^r)$ are the r^{th} moments of Weibull and Inverse Weibull distributions, respectively. then the r^{th} moments of two-component mixture of distribution. produced by the mixture between $E_W(x^r)$ and $E_{IW}(x^r)$ is define by:

$$E_{MWIW} = \lambda E_W(x^r) + (1-\lambda)E_{IW}(x^r), r = 1,2,3, \dots, x > 0.$$

since,

$$E_W(x^r) = \int_0^\infty \frac{\left(\beta x^r e^{-\left(\frac{x}{\alpha}\right)^\beta}\left(\frac{x}{\alpha}\right)^{\beta-1}\right)}{\alpha} dx = \alpha^r \Gamma\left(1+\frac{r}{\beta}\right)$$

and for $r < b$

$$E_{IW}(x^r) = \int_0^\infty \frac{x^r\left(bx^{-1-b}e^{-\left(\frac{1}{ax}\right)^b}\right)}{a^b}\,dx = a^{-r}\Gamma\left(1-\frac{r}{b}\right)$$

then

$$E_{MWIW}(x^r) = (1-\lambda)a^{-r}\Gamma\left(1-\frac{r}{b}\right) + \lambda\alpha^r\Gamma\left(1+\frac{r}{\beta}\right)$$

(10)

Thus, the **mean** of the pdf of the MWIWD given in (10) is

$$\mu = (1-\lambda)a^{-1}\Gamma\left(1-\frac{1}{b}\right) + \lambda\alpha\Gamma\left(1+\frac{1}{\beta}\right) \quad (11)$$

and the **variance** of MWIWD is :

$$\sigma^2 = \left(\frac{(1-\lambda)}{a^2}\Gamma(1-\frac{2}{b}) + \lambda\alpha^2\Gamma(1+\frac{2}{\beta})\right) - \left(\frac{(1-\lambda)}{a}\Gamma(1-\frac{1}{b}) + \lambda\alpha\Gamma(1+\frac{1}{\beta})\right)^2 \quad (12)$$

2.6 Maximum likelihood estimations of the parameters

The estimation of the parameters of Weibull inverse weibull mixture distribution is achieved using the method of maximum likelihood estimation. Let $x_1, x_2, \ldots, x_n$ be a random sample from the Mixture of weibull and inverse weibull Distribution with unknown parameter vector

$$\boldsymbol{\theta} = (\alpha, \beta, a, b, \lambda)$$

The likelihood and its logarithm are given by

$$L = \prod_{i=1}^n \left(b(1-\lambda)a^{-b}x_i^{-b-1}e^{-\left(\frac{1}{ax_i}\right)^b} + \frac{\beta\lambda e^{-\left(\frac{x_i}{\alpha}\right)^\beta}\left(\frac{x_i}{\alpha}\right)^{\beta-1}}{\alpha} \right) \quad (13)$$

$$LogL = \sum_{i=1}^n \log\left(b(1-\lambda)a^{-b}x_i^{-b-1}e^{-a^{-b}x_i^{-b}} + \beta\lambda\alpha^{-\beta}x_i^{\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}\right) \quad (14)$$

Differentiating (Log L) partially with respect to each of the parameter (α,β,a,b,λ) and setting the results equal to zero gives the maximum likelihood estimates of the respective parameters. The partial derivatives of (Log L) with respect to each parameter or the score function is given by:

$$\frac{\partial LogL}{\partial\alpha} = \sum_{i=1}^n k * \left(\beta^2\lambda\alpha^{-2\beta-1}x_i^{2\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}} - \beta^2\lambda\alpha^{-\beta-1}x_i^{\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}\right) \quad (15)$$

$$\frac{\partial LogL}{\partial \beta} = \sum_{i=1}^{n} k * \Bigg(\left(\lambda \alpha^{-\beta} x_i^{\beta-1} e^{-\alpha^{-\beta} x_i^{\beta}}\right) - \left(\beta\lambda\alpha^{-\beta}\log(\alpha) x_i^{\beta-1} e^{-\alpha^{-\beta} x_i^{\beta}}\right) + \left(\beta\lambda\alpha^{-\beta} x_i^{\beta-1} \log(x_i) e^{-\alpha^{-\beta} x_i^{\beta}}\right) + \left(\beta\lambda\alpha^{-\beta} x_i^{\beta-1} e^{-\alpha^{-\beta} x_i^{\beta}} \left(\alpha^{-\beta}\log(\alpha) x_i^{\beta} - \alpha^{-\beta} x_i^{\beta} \log(x_i)\right)\right)\Bigg) \quad (16)$$

$$\frac{\partial LogL}{\partial a} = \sum_{i=1}^{n} k * \left(b^2(1-\lambda) a^{-2b-1} x_i^{-2b-1} e^{-a^{-b} x_i^{-b}} - b^2(1-\lambda) a^{-b-1} x_i^{-b-1} e^{-a^{-b} x_i^{-b}}\right). \quad (17)$$

$$\frac{\partial LogL}{\partial b} = \sum_{i=1}^{n} k * \Big((1-\lambda) a^{-b} x_i^{-b-1} e^{-a^{-b} x_i^{-b}} - b(1-\lambda) a^{-b}\log(a) x_i^{-b-1} e^{-a^{-b} x_i^{-b}} - b(1-\lambda) a^{-b} x_i^{-b-1}\log(x_i) e^{-a^{-b} x_i^{-b}} + b(1-\lambda) a^{-b} x_i^{-b-1} e^{-a^{-b} x_i^{-b}} \left(a^{-b}\log(a) x_i^{-b} + a^{-b} x_i^{-b}\log(x_i)\right)\Big). \quad (18)$$

$$\frac{\partial LogL}{\partial \lambda} = \sum_{i=1}^{n} k * \left(\beta\alpha^{-\beta} x_i^{\beta-1} e^{-\alpha^{-\beta} x_i^{\beta}} - b a^{-b} x_i^{-b-1} e^{-a^{-b} x_i^{-b}}\right). \quad (19)$$

where

$$k = \frac{1}{b(1-\lambda) a^{-b} x_i^{-b-1} e^{-a^{-b} x_i^{-b}} + \beta\lambda\alpha^{-\beta} x_i^{\beta-1} e^{-\alpha^{-\beta} x_i^{\beta}}}$$

Hence, the MLE will obtained by solving this nonlinear system of equations. Solving this system of nonlinear equations is complicated, we can therefore use computational software to solve the equations numerically.

2.7 Maximum likelihood estimations of the reliability and the hazard

The invariance property of MLEs enables us to obtain the MLEs of rf and hrf by replacing the parameters by their MLEs in(12) and (13) respectively as follows:

$$\hat{R}(x) = 1 - \left((1-\hat{\lambda}) e^{-\left(\frac{1}{\hat{a}x}\right)^{\hat{b}}} + \hat{\lambda}\left(1 - e^{-\left(\frac{x}{\hat{\alpha}}\right)^{\hat{\beta}}}\right)\right) \quad (20)$$

and

$$\hat{h}(x) = \frac{\frac{(1-\hat{\lambda})\left(\hat{b}x^{-\hat{b}-1}e^{-\left(\frac{1}{\hat{a}x}\right)^{b}}\right)}{a^{b}} + \frac{\lambda\left(\beta x^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)}{\alpha^{\beta}}}{1-\left((1-\hat{\lambda})e^{-\left(\frac{1}{\hat{a}x}\right)^{\hat{b}}}+\hat{\lambda}\left(1-e^{-\left(\frac{x}{\hat{\alpha}}\right)^{\hat{\beta}}}\right)\right)} \quad (21)$$

where $\theta = (\hat{\alpha}, \hat{\beta}, \hat{a}, \hat{b}, \hat{\lambda})$ are the MLEs of $\theta = (\alpha, \beta, a, b, \lambda)$.

2.8 Fisher Information Matrix

The **observed Fisher information matrix** $I_{ij}(\theta)$ for the MLEs of the parameters θ is the 5*5 symmetric matrix of the negative second partial derivatives of log-LF, given by (18) with respect to the parameters . That is

$$I_n(\theta_{ij}) = -E\left[\frac{\partial^2 lnf(x;\theta)}{\partial\theta_i\partial\theta_j}\right], i, j = 1,2,..,5$$

The inverse of $I_{ij}(\theta)$ is the estimate V of the asymptotic variance covariance matrix of $\hat{\theta}$. That is

$$V = [I_{ij}(\theta)]^{-1} = \sigma_{ij} = cov(\hat{\theta}_i, \hat{\theta}_j), i, j = 1,2,..,5 \quad (22)$$

The observed Fisher information matrix enables us to construct confidence intervals for the parameters, where $V(\hat{\theta}_i) = \sigma_{ii}$ the i^{th} diagonal element of the matrix V, given by (22).

2.9 A numerical Illustration

This section aims to investigate the precision of the theoretical results of estimation on basis of simulated and real data

2.9.1 A simulation study

a simulation study is conducted to study the performance of the presented ML estimates on the basis of generated data from the MWIW distribution.A simulation study are performed using Mathematica 9 for illustrating the obtained results.

The steps of the simulation procedure is as follows:

1. Specify initial values for the parameters θ.
2. Specify the sample size n.
3. Random samples of size n are generated from the MWIW distribution.
4. Get the MLE for the parameter based on the samples.
5. Repeat all the previous steps N times where N represents a fixed number of simulated samples.

6. Calculate the Mean, variance, bias and relative bias for the estimated parameters.
7. calculate the mean square error (MSE) and the root mean square error for each estimator.

Simulation results of ML estimates are displayed in Table[1] and [2], where N = 1000 is the number of repetitions, (n=30, 80 , 120), are the sample sizes and with different initial values for the parameters (α, β, a , b and λ). From the table, it is observed that the MSE and the Bias for the estimates α, β, a, b and λ are decreasing when the sample size n is increasing.

Table 1: averages, biases, relative biases, mean square error, root mean square error and variances of ML estimators from MWIW distribution for different sizes n and replications NR=1000(α=2, β=1.5 , a=0.6 , b=1.2, λ=0.8)

n	Parameter	Average	Variance	Bias ^{2}	RAB	MES	RootMSE
	α	1.985	0.292	0.005	0.007	0.293	0.086
	β	1.545	0.125	0.002	0.030	0.127	0.016
30	a	0.681	0.210	0.006	0.135	0.217	0.074
	b	1.690	0.820	0.240	0.408	1.061	1.126
	λ	0.772	0.071	0.0007	0.034	0.071	0.005
	α	1.932	0.129	0.0045	0.004	0.134	0.018
	β	1.510	0.045	0.001	0.007	0.045	0.002
80	a	0.577	0.056	0.005	0.037	0.056	0.003
	b	1.454	0.29	0.064	0.212	0.36	0.132
	λ	0.754	0.042	0.0005	0.057	0.043	0.002
	α	1.933	0.060	0.004	0.003	0.064	0.004
	β	1.525	0.021	0.0006	0.006	0.021	0.0004
120	a	0.547	0.037	0.002	0.008	0.040	0.0016
	b	1.38	0.169	0.035	0.17	0.205	0.042
	λ	0.777	0.019	0.0002	0.028	0.020	0.0004

Table 2: averages, biases, relative biases, mean square error, root mean square error and variances of ML estimators from MWIW distribution for different sizes n and replications NR=1000(α=1.2, β=2 , a=0.6 , b=1.2, λ=0.6)

n	Parameter	Average	Variance	Bias ^{2}	RBias	MES	RMSE
	α	1.015	0.114	0.034	0.153	0.131	0.017
	β	1.824	0.397	0.030	0.086	0.397	0.158
30	a	0.780	0.128	0.034	0.308	0.163	0.026
	b	1.63	0.275	0.186	0.360	0.462	0.213
	λ	0.383	0.096	0.048	0.367	0.143	0.021
	α	1.020	0.056	0.032	0.149	0.088	0.007
	β	1.832	0.148	0.028	0.084	0.177	0.031
80	a	0.739	0.036	0.019	0.233	0.056	0.003
	b	1.448	0.061	0.062	0.207	0.123	0.015
	λ	0.379	0.054	0.047	0.361	0.103	0.011
	α	1.071	0.046	0.016	0.107	0.080	0.006
	β	1.995	0.116	0.0001	0.002	0.164	0.021
120	a	0.727	0.036	0.016	0.212	0.052	0.002
	b	1.422	0.043	0.049	0.185	0.092	0.008
	λ	0.412	0.049	0.039	0.331	0.089	0.007

Table 3: averages, biases, relative biases, mean square error, root mean square error and variances of ML estimators from MWIW distribution for different sizes n and replications NR=1000(α=2.9, β=2.3 , a=0.5 , b=5.7, λ=0.8)

n	Parameter	Average	Variance	Bias ^{2}	RBias	MES	RMSE
	α	2.903	0.087	0.0001	0.0012	0.0875	0.0076
	β	2.530	0.256	0.053	0.101	0.309	0.096
30	a	0.505	0.004	0.0002	0.010	0.005	0.0002
	b	5.508	13.879	0.037	0.033	13.916	193.658
	λ	0.864	0.042	0.004	0.082	0.046	0.002
	α	2.899	0.057	0.0002	0.001	0.057	0.0032
	β	2.438	0.111	0.0192	0.060	0.1299	0.016
80	a	0.506	0.0073	0.0004	0.013	0.0072	0.00005
	b	5.789	9.814	0.0079	0.017	9.820	96.481
	λ	0.801	0.029	0.000002	0.0006	0.0029	0.00085
	α	2.861	0.0308	0.00011	0.0011	0.0319	0.001
	β	2.362	0.0680	0.0044	0.029	0.0725	0.0052
120	a	0.508	0.0028	0.00006	0.0161	0.0029	0.000008
	b	5.904	5.939	0.0867	0.0516	6.021	36.311
	λ	0.791	0.0193	0.000001	0.0014	0.0193	0.00037

Concluding remarks

It is noticed, from Tables, that the ML averages are very close to the initial values of the parameters as the sample size increases. Also, Rbias and MES are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the population parameter values as the sample size increases.

2.9.2 Application to a real data set

In this section, we use a real data set reported in ([?]), to show that the MWIW distribution could be used, to estimate it's parameter and apply the kolmogorov-smirnov test and chi-square goodness of fit test for the distribution. The data represented the failure times of the windshields on a specific model of aircrafts. These failures do not result in damage to the aircraft but do result in replacement of the windshield. The data is reproduced in Table ??: (The unit for measurement is 1000 h)

Table 4: data

0.04	1.652	2.300	3.376	0.301	1.652	2.324	3.443
0.309	1.77	2.85	3.467	0.7	1.866	2.481	3.48
0.610	1.876	2.610	3.578	0.650	1.899	2.625	3.595
0.720	1.9	2.632	3.699	0.840	1.912	2.646	3.779
0.943	1.914	2.661	3.924	1.070	1.981	2.688	4.035
1.124	2.010	2.823	4.121	1.248	2.038	2.890	4.167
1.281	2.085	2.902	4.240	1.281	2.089	2.934	4.255
1.303	2.097	2.962	4.278	1.432	2.135	2.964	4.305
1.480	2.154	3.000	4.376	1.505	2.190	3.103	4.449
1.506	2.194	3.114	4.485	1.568	2.223	3.117	4.570
1.615	2.224	3.166	4.602	1.619	2.229	3.344	4.663

After forming the likelihood function and the normal equation, the normal equations ware solved using Mathematica software to estimate the parameters $(\alpha, \beta, a, b, \lambda)$. The following results were obtained $(\alpha = 2.9, \beta = 2.3, a = .5, b = 5.7, \lambda = .8)$. To assess the results, the empirical and fitted Survival functions from the MWIWD and the windsheild data were drawn together as shown in figure (5). it seems from the figure that the fit is a good one.

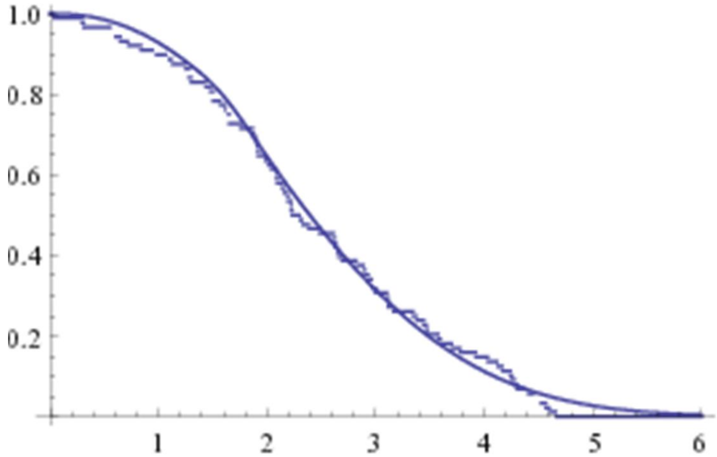

Figure 5: Empirical and fitted survival functions from the MWIW distribution and windsheild data

2.10 Goodness of fit Test

2.10.1 The Kolmogorove-Smirnov Test:

The Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model. It is based on the empirical cumulative distribution function. The Kolmogorove-Smirnov(K-S) statistic (D_n) is based on the largest vertical difference between the CDF of the MWIW distribution F(x) and the empirical CDF of the Data $F_n(x)$. It is defined as

$$D_n = \sup|F(x) - F_n(x)|$$

H_0: The data follow the MWIW distribution.

H_1: The data do not follow the MWIW distribution.

The hypothesis regarding the distributional form is rejected at the chosen significance level (α) if the test statistic ,D_n, is greater than the critical value obtained from a table.

K – SStatistic	p – Value
0.052	0.968

The Kolmogorove-Smirnov test for testing the fitting of the

MWIW to the given data through the Mathematica package produced those result, D_n = 0.052, and the Critical value at α=0.05 is (0.968). Therefore the null hypothesis will be accepted that the data came from the mixture of weibull and inverse weibull distribution.

Again, the p-value of (0.968) is much larger than 0.05 leading to the acceptance (non rejection) of the null hypothesis that the data came from the Mixture Weibull and inverse Weibull distribution.

2.10.2 Chi-Square Goodness of Fit Test:

A goodness-of-fit statistic tests the following hypothesis:

H_0 : the data comes from MWIW distribution,

H_1: the data did not come from MWIW distribution.

Pearson and deviance test statistics which is the sum of differences between observed and expected outcome frequencies; each squared and divided by the expectation:

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_I)^2}{E_i}$$

where:

O_i = an observed frequency for bin i

E_i = an expected (theoretical) frequency for bin i, asserted by the null hypothesis.

The expected frequency is calculated by:

$$E_i = (F(Y_u) - F(Y_1))N$$

where:

F = the cumulative Distribution function for the distribution being tested.

Y_u= the upper limit for class i,

Y_1 = the lower limit for class i, and

N = the sample size

The data was classified in 11 class intervals of length 0.5 starting from 0 to 5 as shown in table

class interval.	O_i.	E_i.	$\frac{(O_i - E_I)^2}{E_i}$
0.0 -	3	1.253	2.43
0.5 -	2	4.804	1.63
1.0 -	8	8.6725	0.052
1.5 -	15	15.066	0.002
2.0 -	16	15.199	0.042
2.5 -	12	12.576	0.026
3.0 -	11	9.892	0.124
3.5 -	5	7.089	0.616
4.0 -	9	4.551	4.350
4.5	3	2.603	0.060
Sum			9.341

Table 5: Chi-square goodness of fit test

The Calculated statistic value of Chi-square test (9.341) is less than the tabulated statistic value $X^2_{(0.95,4)}$ =11.075 and the corresponding p-value =0.963 is greater than 0.05. Therefore the null hypothesis does not reject that the data came from the Mixture of weibull and inverse Weibull distribution.

2.10.3 The Information Matrix

Fisher information is a key concept in the theory of statistical inference and is defined in the following manner:

Let $x = (x_1, \dots, x_n)$ be a random sample, and let $f(x;\theta)$ denote the probability density function for some model of the data. which has parameter vector $\theta = (\theta_1, \dots, \theta_k)$.

Then the Fisher information matrix $I_n(\theta)$ of sample size n is given by the (k*k) a symmetric matrix whose ij^{th} element is given by the covariance between first partial derivatives of the log-likelihood,

$$I_n(\theta_{ij}) = Cov\left[\frac{\partial f(x;\theta)}{\theta_i}, \frac{\partial f(x;\theta)}{\theta_j}\right]$$

An alternative, but equivalent, definition for the Fisher information matrix is based on the expected values of the second partial derivatives, and is given by

$$I_n(\theta_{ij}) = -E\left[\frac{\partial^2 lnf(x;\theta)}{\partial\theta_i \partial\theta_j}\right]$$

Strictly, this definition corresponds to the expected Fisher information. In other word the Fisher information matrix can be expressed in terms of the second derivative of the log likelihood function under the regularity conditions. for windshield data Fisher information matrix can be obtained as follow:

$I_n(\theta_{ij})$ =
(0.03730.02850.01070.08470.00370.02850.08340.0181 – 0.08490.00870.01070.01810.01090.08670.00650.0847 – 0.08490.08676.0520 – 0.28100.00370.00870.0065 – 0.28100.0010)

the variances of the estimates are generally small, with $(\hat{b})$having the largest variance (6.05).

2.11 Related Distribution(special cases)

1. In equation(6) and (7),putting b=1 we get Mixture of Weibull and Inverse Exponential(MWIE)) distribution. Then its pdf,cdf,hazard and survival function are respectively given by

$$f(x;\alpha,\beta,a,\lambda) = \frac{(1-\lambda)e^{-\frac{1}{ax}}}{ax^2} + \frac{\lambda\left(\beta e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\left(\frac{x}{\alpha}\right)^{\beta-1}\right)}{\alpha} \quad (23)$$

$$F(x;\alpha,\beta,a,\lambda) = (1-\lambda)e^{-\frac{1}{ax}} + \lambda\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right) \quad (24)$$

$$h(x;\alpha,\beta,a,\lambda) = \frac{\frac{(1-\lambda)e^{-\frac{1}{ax}}}{ax^2}+\frac{\lambda\left(\beta e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\left(\frac{x}{\alpha}\right)^{\beta-1}\right)}{\alpha}}{1-\left((1-\lambda)e^{-\frac{1}{ax}}+\lambda\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)\right)} \quad (25)$$

and

$$S(x;\alpha,a,b,\lambda) = 1-\left((1-\lambda)e^{-\frac{1}{ax}} + \lambda\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)\right) \quad (26)$$

2. In equation(6) and (7), putting β = 1, we get Mixture of Exponential and Inverse Weibull(MEIW) distribution. Then its pdf, cdf, hazard and survival function are respectively given by:

$$f(x;\alpha,a,b,\lambda) = \frac{(1-\lambda)\left(bx^{-b-1}e^{-\left(\frac{1}{ax}\right)^{b}}\right)}{a^{b}} + \frac{\lambda e^{-\frac{x}{\alpha}}}{\alpha} \quad (27)$$

$$F(x;\alpha,a,b,\lambda) = (1-\lambda)e^{-\left(\frac{1}{ax}\right)^{b}} + \lambda\left(1-e^{-\frac{x}{\alpha}}\right) \quad (28)$$

$$h(x;\alpha,a,b,\lambda) = \frac{\frac{(1-\lambda)\left(bx^{-b-1}e^{-\left(\frac{1}{ax}\right)^{b}}\right)}{a^{b}} + \frac{\lambda e^{-\frac{x}{\alpha}}}{\alpha}}{1-\left((1-\lambda)e^{-\left(\frac{1}{ax}\right)^{b}} + \lambda\left(1-e^{-\frac{x}{\alpha}}\right)\right)} \quad (29)$$

and

$$S(x;\alpha,a,b,\lambda) = 1-\left((1-\lambda)e^{-\left(\frac{1}{ax}\right)^{b}} + \lambda\left(1-e^{-\frac{x}{\alpha}}\right)\right) \quad (30)$$

3. In equation(6) and (7), Putting (b= β = 1), we get a Mixture of Exponential and Inverse Exponential (MEIE) distribution. Then its pdf,cdf,hazard and survival function are respectively given by

$$f(x;\alpha,a,\lambda) = \frac{(1-\lambda)e^{-\frac{1}{ax}}}{ax^{2}} + \frac{\lambda e^{-\frac{x}{\alpha}}}{\alpha} \quad (31)$$

$$F(x;\alpha,a,\lambda) = (1-\lambda)e^{-\frac{1}{ax}} + \lambda\left(1-e^{-\frac{x}{\alpha}}\right) (32)$$

$$h(x;\alpha,a,\lambda) = \frac{\frac{(1-\lambda)e^{-\frac{1}{ax}}}{ax^{2}} + \frac{\lambda e^{-\frac{x}{\alpha}}}{\alpha}}{1-\left((1-\lambda)e^{-\frac{1}{ax}} + \lambda\left(1-e^{-\frac{x}{\alpha}}\right)\right)} \quad (33)$$

and

$$S(x;\alpha,a,\lambda) = 1-\left((1-\lambda)e^{-\frac{1}{ax}} + \lambda\left(1-e^{-\frac{x}{\alpha}}\right)\right) \quad (34)$$

4. In equation(6) and (7)), Putting (b= β = 2), we get Mixture of Rayleigh and Inverse Rayleigh (MRIR) distribution.

5. In equation(6) and (7), Putting (β = 2), we get Mixture of Rayleigh and Inverse Weibull (MRIW) distribution.

6. In equation((6) and (7)), Putting (b= 2), we get Mixture of Weibull and Inverse Rayleigh (MWIR) distribution.

3. Mixtures with mixing proportion a function in the parameters

In some applications, the mixing proportion may depend on one or more of the parameter of the distribution. While this doesn't effect the general formula of the distribution, it has it's effect on the estimation of the parameter and the fitted of the distribution to data. In our mixture we used that technique to get anew mixture model called New Mixture of Weibull and Inverse Weibull (NMWIW) distribution.

3.1 The probability density function

In the following, we shall consider the case where the mixing proportion $\lambda=\frac{\alpha}{\alpha+a}$,and (1-λ)=$\frac{a}{\alpha+a}$, then the New Mixture of Weibull and Inverse Weibull distribution (NWIWD) would be as follow:

$$f(x) = \frac{\alpha}{\alpha+a} f_1(x,\alpha,\beta) + \frac{a}{\alpha+a} f_2(x,a,b) \quad (35)$$

Where $f_1(x,\alpha,\beta)$ is the pdf of WD and $f_2(x,a,b)$ is the pdf of IWD. The mixture of these probability densities is given by;

$$f(x;\alpha,\beta,a,b) = \frac{a\left(bx^{-b-1}e^{-\left(\frac{1}{ax}\right)^{b}}\right)}{(a+\alpha)a^{b}} + \frac{\alpha\left(\beta x^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)}{(a+\alpha)\alpha^{\beta}} \quad (36)$$

Different values for parameters result in different shapes for the pdf of the NWIW Distribution as figure (6).

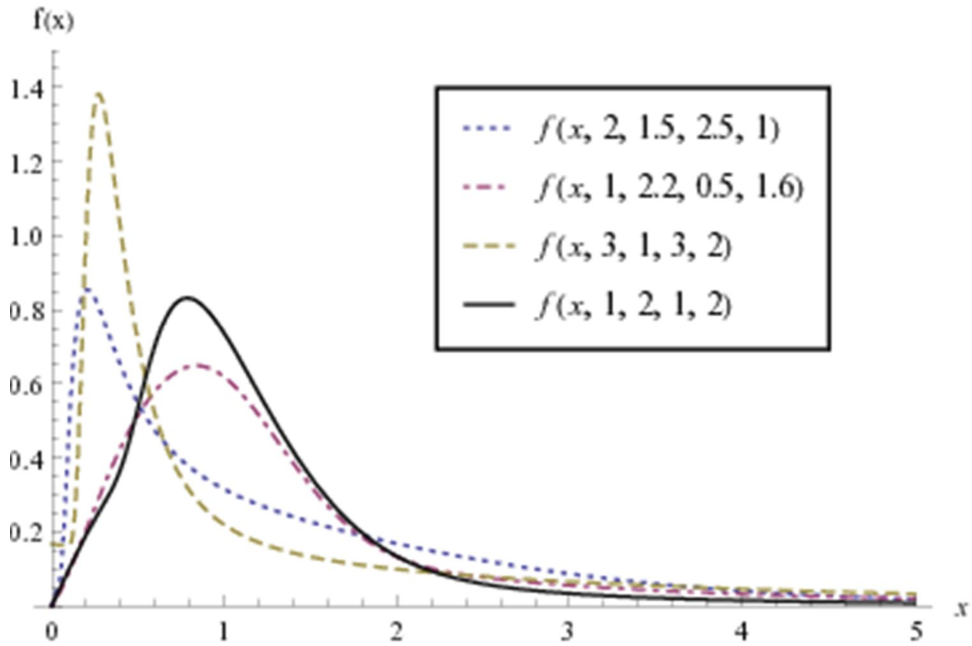

Figure 6: figure of the pdf of NWIW

3.2 Cumulative Distribution Function:

The cumulative distribution function of the New Wiebull inverse weibull distribution has the following form,

$$F(x;\alpha\ \beta,\text{a},\text{b},) = \frac{ae^{-\left(\frac{1}{ax}\right)^{b}}}{a+\alpha} + \frac{\alpha\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)}{a+\alpha} \quad (37)$$

Where$(x,\beta,a,b,\alpha > 0)$. Possible shapes for the cdf of the (NWIW) Distribution as Figure (7):

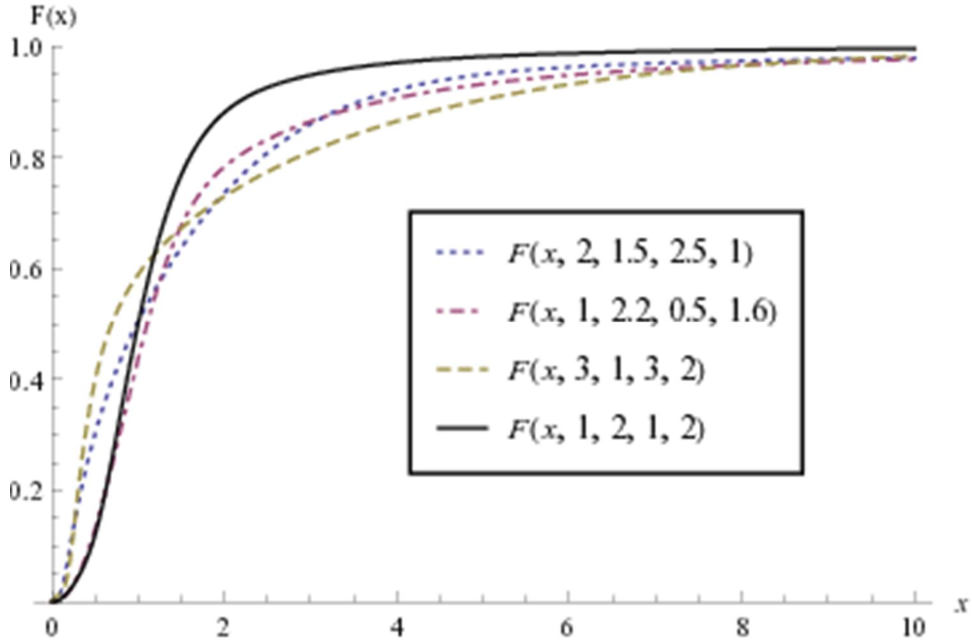

Figure 7: figure of the CDF of NWIW

3.3 Survival Function:

survival Function is also known as reliability function. It defined as the probability that the system will continue to survive beyond the specific time. It is defined mathematically as;

$$S(x) = 1 - F(x)$$

$$S(x;\alpha,\beta,a,b) = 1 - \left(\frac{ae^{-\left(\frac{1}{ax}\right)^{b}}}{a+\alpha} + \frac{\alpha\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)}{a+\alpha} \right) \tag{38}$$

Where $(x, \lambda, a, b, \alpha > 0)$. we provide a possible shapes for the survival density for the(NWIW) Distribution as figure(8):

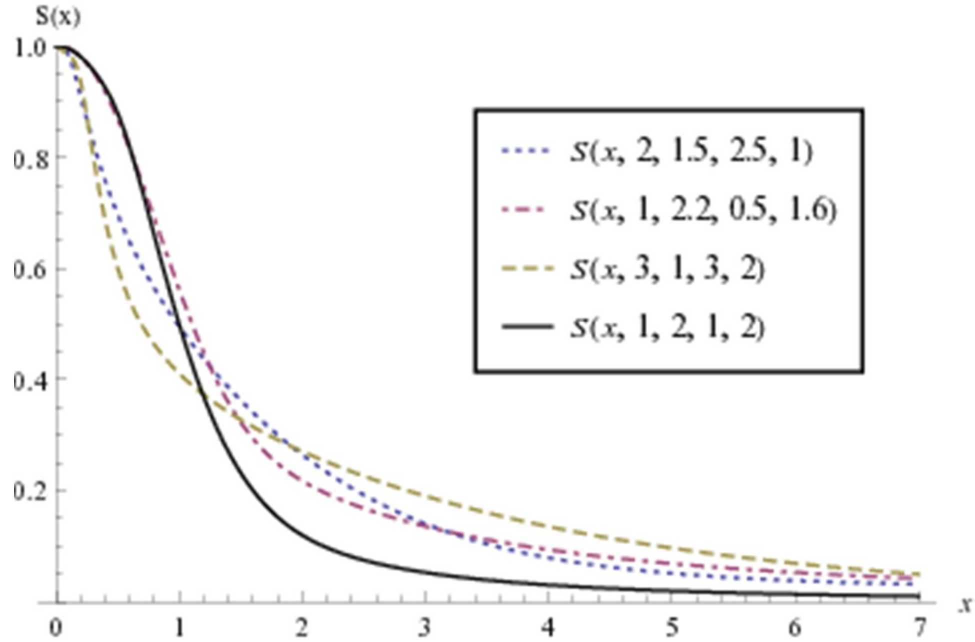

Figure 8: figure of the Survival density of NWIW

3.4 Hazard function:

Hazard function is defined mathematically as the ratio pdf and reliability function and is expressed as;

$$h(x) = \frac{f(x)}{S(x)}$$

so the hazard function of the New Weibull inverse Weibull is in the form:

$$h(x;\alpha,\beta,a,b) = \frac{\frac{a\left(bx^{-b-1}e^{-\left(\frac{1}{ax}\right)^{b}}\right)}{(a+\alpha)a^{b}}+\frac{\alpha\left(\beta x^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)}{(a+\alpha)\alpha^{\beta}}}{1-\left(\frac{ae^{-\left(\frac{1}{ax}\right)^{b}}}{a+\alpha}+\frac{\alpha\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)}{a+\alpha}\right)} \qquad (39)$$

Where $(x, \lambda, a, b, \alpha > 0)$. we provide a possible shapes for the hazard densities of the New weibull inverse weibull Distribution as figure (9):

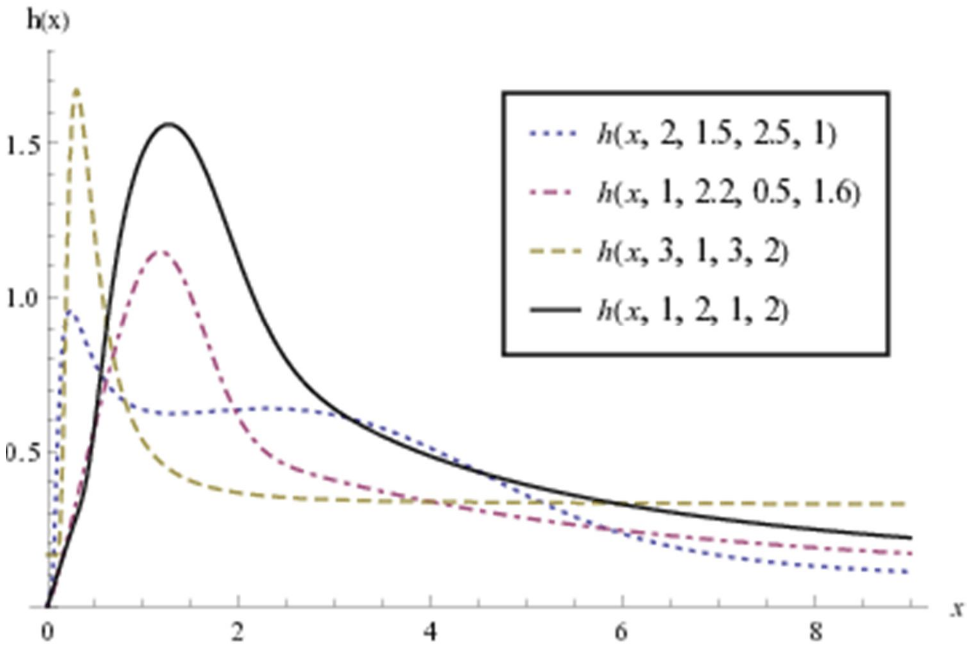

Figure 9: figure of the hazard density of NWIW

3.5 Statistical Properties:

Moments

$$E_{NWIW} = (\frac{\alpha}{\alpha+a})E_W(x^r) + (\frac{a}{\alpha+a})E_{IW}(x^r), r = 1,2,3,\ldots, x > 0.$$

then

$$E_{NWIW} = \frac{\alpha^{r+1}\Gamma\left(1+\frac{r}{\beta}\right)+a^{1-r}\Gamma\left(1-\frac{r}{b}\right)}{\alpha+a} \qquad (40)$$

Thus, the **mean** of the pdf of the NMWIW given in (36) is

$$\mu = \frac{\alpha^2\Gamma\left(1+\frac{1}{\beta}\right)+\Gamma\left(1-\frac{1}{b}\right)}{\alpha+a} \qquad (41)$$

and the **variance** of NMWIW is :

$$\sigma^2 = \left(\frac{\alpha^3\Gamma\left(1+\frac{2}{\beta}\right)+a^{-1}\Gamma\left(1-\frac{2}{b}\right)}{\alpha+a}\right) - \left(\frac{\alpha^2\Gamma\left(1+\frac{1}{\beta}\right)+\Gamma\left(1-\frac{1}{b}\right)}{\alpha+a}\right)^2 \qquad (42)$$

The **mode** for the NMWIW distribution can be calculated by differentiating f(x) with respect to x as follows

$$f(x) = \frac{a\left(bx^{-b-1}e^{-\left(\frac{1}{ax}\right)^{b}}\right)}{(a+\alpha)a^{b}} + \frac{\alpha\left(\beta x^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)}{(a+\alpha)\alpha^{\beta}}$$

$$\frac{\partial f(x)}{\partial x} = \frac{b^{2}a^{-b}x^{-b-3}e^{-\left(\frac{1}{ax}\right)^{b}}\left(\frac{1}{ax}\right)^{b-1}}{a+\alpha} + \frac{(-b-1)ba^{1-b}x^{-b-2}e^{-\left(\frac{1}{ax}\right)^{b}}}{a+\alpha} - \frac{\beta^{2}\alpha^{-\beta}x^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\left(\frac{x}{\alpha}\right)^{\beta-1}}{a+\alpha} + \frac{(\beta-1)\beta\alpha^{1-\beta}x^{\beta-2}e^{-\left(\frac{x}{\alpha}\right)^{\beta}}}{a+\alpha}$$

By equating the previous with zero, we get the mode of the NWIWD. and the **median** of the NMWIW distributon can be obtained by solving that non linear equation

$$\frac{ae^{-\left(\frac{1}{ax}\right)^{b}}}{a+\alpha} + \frac{\alpha\left(1-e^{-\left(\frac{x}{\alpha}\right)^{\beta}}\right)}{a+\alpha} = 0.5$$

3.6 Maximum likelihood estimations for the parameters

Let $x_1, x_2, \ldots, x_n$ be a random sample from the New Mixture of weibull and inverse weibull Distribution with unknown parameter vector

$$\theta = (\alpha, \beta, a, b)^{T}$$

The log-likelihood function for θ is obtained from f(x) as follows:

$$L = \prod_{i=1}^{n}\left(\frac{ba^{1-b}x_i^{-b-1}e^{-\left(\frac{1}{ax_i}\right)^{b}}}{a+\alpha} + \frac{\beta e^{-\left(\frac{x_i}{\alpha}\right)^{\beta}}\left(\frac{x_i}{\alpha}\right)^{\beta-1}}{a+\alpha}\right) \quad (43)$$

$$LogL = \sum_{i=1}^{n}\log\left(\frac{ba^{1-b}x_i^{-b-1}e^{-a^{-b}x_i^{-b}}}{a+\alpha} + \frac{\beta\alpha^{1-\beta}x_i^{\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}}{a+\alpha}\right) \quad (44)$$

So, differentiating (Log L) partially with respect to each of the parameter (α,a,β,b) and setting the results equal to zero gives the maximum likelihood estimates of the respective parameters. The partial derivatives of (Log L) with respect to each parameter or the score function is given by:

$$\frac{\partial LogL}{\partial \alpha}=\sum_{i=1}^{n} T*\left(-\frac{ba^{1-b}x_i^{-b-1}e^{-a^{-b}x_i^{-b}}}{(a+\alpha)^2}+\frac{\beta^2\alpha^{-2\beta}x_i^{2\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}}{a+\alpha}+\frac{(1-\beta)\beta\alpha^{-\beta}x_i^{\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}}{a+\alpha}-\frac{\beta\alpha^{1-\beta}x_i^{\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}}{(a+\alpha)^2}\right). \quad (45)$$

$$\frac{\partial LogL}{\partial \beta}=\sum_{i=1}^{n} T*\left(\frac{\beta\alpha^{1-\beta}x_i^{\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}\left(\alpha^{-\beta}\log(\alpha)x_i^{\beta}-\alpha^{-\beta}x_i^{\beta}\log(x_i)\right)}{a+\alpha}+\frac{\beta\alpha^{1-\beta}x_i^{\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}\log(x_i)}{a+\alpha}-\frac{\beta\alpha^{1-\beta}x_i^{\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}\log(\alpha)}{a+\alpha}+\frac{\alpha^{1-\beta}x_i^{\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}}{a+\alpha}\right) \quad (46)$$

$$\frac{\partial LogL}{\partial b}=\sum_{i=1}^{n} T*\left(\frac{a^{1-b}x_i^{-b-1}e^{-a^{-b}x_i^{-b}}}{a+\alpha}+\frac{ba^{1-b}x_i^{-b-1}e^{-a^{-b}x_i^{-b}}\left(a^{-b}\log(a)x_i^{-b}+a^{-b}x_i^{-b}\log(x_i)\right)}{a+\alpha}-\frac{ba^{1-b}\log(a)x_i^{-b-1}e^{-a^{-b}x_i^{-b}}}{a+\alpha}-\frac{ba^{1-b}x_i^{-b-1}\log(x_i)e^{-a^{-b}x_i^{-b}}}{a+\alpha}\right) \quad (47)$$

$$\frac{\partial LogL}{\partial a}=\sum_{i=1}^{n} T*\left(\frac{b^2a^{-2b}x_i^{-2b-1}e^{-a^{-b}x_i^{-b}}}{a+\alpha}+\frac{(1-b)ba^{-b}x_i^{-b-1}e^{-a^{-b}x_i^{-b}}}{a+\alpha}-\frac{ba^{1-b}x_i^{-b-1}e^{-a^{-b}x_i^{-b}}}{(a+\alpha)^2}-\frac{\beta\alpha^{1-\beta}x_i^{\beta-1}e^{-\alpha^{-\beta}x_i^{\beta}}}{(a+\alpha)^2}\right) \quad (48)$$

Since

$$T = \frac{1}{\frac{b x_i^{-b-1} e^{-a^{-b} x_i^{-b}} a^{1-b}}{a+\alpha} + \frac{\beta \alpha^{1-\beta} x_i^{\beta-1} e^{-\alpha^{-\beta} x_i^{\beta}}}{a+\alpha}}$$

Hence, the MLE is obtained by solving this nonlinear system of equations. Solving this system of nonlinear equations is complicated, we can therefore use statistical software to solve the equations numerically.

3.7 A Simulation study

In this subsection, a simulation study is presented to illustrate the application of the various theoretical results developed in the previous section on the basis of generated data from A new Mixture of Weibull and Inverse Weibul (NMWIW) distribution, for different sample sizes (n=30, 60 and 120) and using number of replications NR=1500. The computations are performed using Mathematica 9.

The results of the simulation study are given in Table 5.

Table 6: averages, biases, relative biases, mean square error, root mean square error and variances of ML estimators from NMWIW distribution for different sizes n and replications NR=1500(α=1.5, β=2 , a=1.2 and b=2)

n	Parameter	Average	Variance	Bias ^{2}	RBias	MES	RMSE
	α	1.629	0.233	0.0168	0.086	0.249	0.062
	β	1.815	1.132	0.034	0.092	1.317	1.735
30	a	0.795	0.162	0.167	0.341	0.330	0.103
	b	1.706	0.361	0.085	0.146	0.447	0.199
	α	1.592	0.173	0.008	0.061	0.181	0.032
	β	1.874	1.086	0.015	0.062	1.121	1.225
60	a	0.833	0.696	0.134	0.303	0.831	0.691
	b	1.737	0.321	0.068	0.131	0.391	0.152
	α	1.586	0.155	0.007	0.057	0.162	0.026
	β	1.993	0.966	0.004	0.003	0.928	0.964
120	a	0.895	0.177	0.092	0.253	0.270	0.073
	b	1.756	0.306	0.057	0.119	0.364	0.132

Concluding remarks

It is noticed, from Tables, that the ML averages are very close to the initial values of the parameters as the sample size increases. Also, Rbias, MSE and RMES are decreasing when the sample size is increasing. This is indicative of the fact that the estimates are consistent and approaches the population parameter values as the sample size increases.

3.8 Application to real data

In this section, we use a real data set reported in ([?]), to show that the NMWIW distribution could be used. to estimate it's parameter and apply the goodness of fit test for the distribution.The data consist of measurements made on patients with malignant melanoma. Each patient had their tumour removed by surgery at the Department of Plastic Surgery, University Hospital of Odense, Denmark during the period 1962 to 1977. The surgery consisted of complete removal of the tumour together with about 2.5cm of the surrounding skin. Among the measurements taken were the thickness of the tumour and whether it was ulcerated or not. These are thought to be important prognostic variables in that patients with a thick and/or ulcerated tumour have an increased chance of death from melanoma. Patients were followed until the end of 1977.The data is reproduced as follow.

0.027778, 0.51389, 0.56667, 0.58333, 0.64444, 0.64444, 0.77500, 0.81944, 0.98611, 1.07222, 1.18333, 1.30278, 1.36944, 1.46944, 1.72500, 1.74722, 1.83056, 1.85278, 1.99444, 2.08889, 2.16389, 2.20278, 2.29444, 2.31389, 2.68611, 2.71389, 2.72778, 2.93056, 2.95000, 2.98611, 3.21111, 3.41111, 3.47778, 3.53056, 3.64444, 4.18333, 4.18889, 4.21111, 4.23611, 4.50278, 4.63056, 4.79444, 4.84722, 4.94167, 4.96389, 5.03333, 5.10833, 5.15000, 5.16667, 5.33333, 5.35278, 5.43056, 5.47222, 5.58611, 5.63333, 5.66111, 5.71944, 5.76389, 5.85556, 5.86667, 6.18611, 6.26667, 6.28889, 6.63056, 6.67500, 6.73889, 6.85278, 6.92500, 7.00278, 7.06111, 7.10833, 7.12500, 7.40556, 7.43333, 8.45, 8.96667, 9.15833, 9.25000, 9.27222, 9.40000, 9.41111, 9.60833, 10.4889, 10.4889, 10.7556, 10.8583, 11.4556, 12.1944, 12.4417, 12.4778

After forming the likelihood function and the normal equation, the normal equations ware solved using Mathematica software to estimate the parameters (α, β, a, b) . The following results were obtained (α = 5.6, β = 1.5, a = 0.4, b = 1.4) . To assess the results, the empirical and fitted survival functions from the NMWIWD and malignant melanoma data were drawn together as shown in figure (10). it seems from the figure that the fit is a good one.

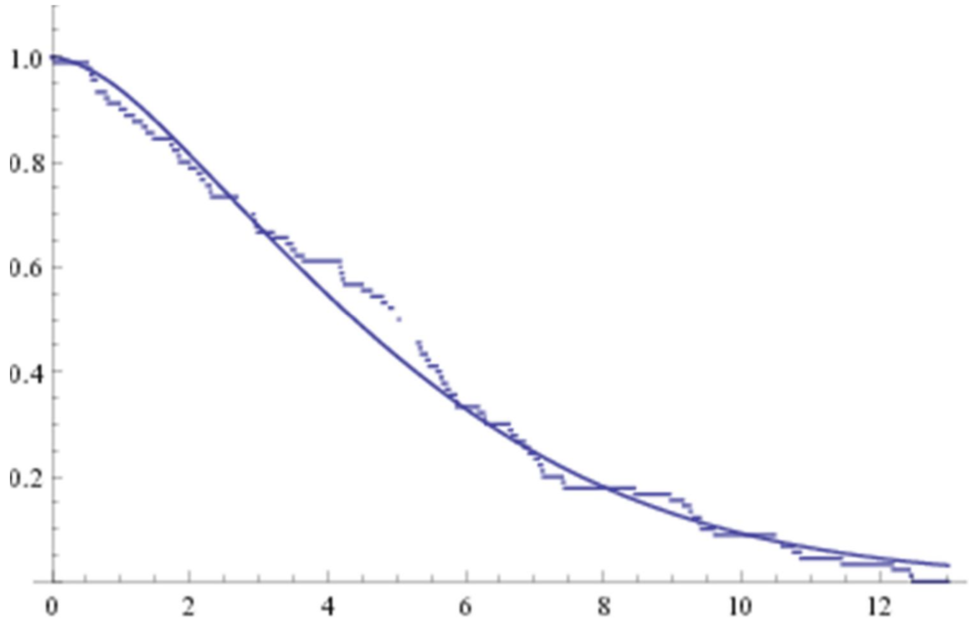

Figure 10: Empirical and fitted survival functions from the NWIW distribution and malignant melanoma data

3.8.1 Goodness of fit Test

The Kolmogorove-Smirnov Test:

The Kolmogorov–Smirnov goodness of fit test is applied to check the validity of the fitted model.It is based on the empirical cumulative distribution function. The Kolmogorove-Smirnov statistic (D_n) is based on the largest vertical difference between the CDF of the MWIW distribution F(x) and the empirical CDF of the Data $F_n(x)$. It is defined as

$$D_n = \sup|F(x) - F_n(x)|$$

H_0: The data follow the NMWIW distribution.

H_1: The data do not follow the NMWIW distribution.

The hypothesis regarding the distributional form is rejected at the chosen significance level (α) if the test statistic ,D_n, is greater than the critical value obtained from a table.

K – SStatistic	p – Value
0.092	0.410

The Kolmogorove-Smirnov test for testing the fitting of the NMWIW to the given data through the Mathematica package produced those result,D_n = 0.092, and the Critical value at α=0.05 is (0.41). Therefore the null hypothesis will be accepted that the data came from the mixture of Anew weibull and inverse weibull distribution.

Chi-Square Goodness of Fit Test:

A goodness-of-fit statistic tests the following hypothesis:

H_0 : the data comes from Anew mixture of weibull and inverse weibull distribution.

H_1: the data did not come from Anew mixture of weibull and inverse weibull distribution.

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_I)^2}{E_i}$$

where:

O_i = an observed frequency for bin i

E_i = an expected (theoretical) frequency for bin i, asserted by the null hypothesis.

The data was classified in 13 class intervals as shown in table(6)

class interval.	O_i.	E_i.	$\frac{(O_i - E_I)^2}{E_i}$
0 -	9	5.662	1.967
1 -	10	11.095	0.108
2 -	11	12.249	0.126
3 -	5	11.744	3.873
4 -	10	10.568	0.030
5 -	15	9.061	3.891
6 -	8	7.459	0.0391
7 -	6	5.926	0.0009
8 -	2	4.561	1.438
9 -	6	3.412	1.961
10 -	4	2.488	0.917
11-13	4	3.009	0.326
Sum			14.681

Table 7: Chi-square goodness of fit test

The Calculated statistic value of Chi-square test statistic 14.681 is less than the tabulated statistic value $\chi^2_{(0.95,8)}$ =15.5073. Therefore the null hypothesis does not reject that the data came from Anew Mixture of weibull and inverse Weibull distribution.

The Information Matrix

The fisher information matrix can be expressed in terms of the second derivative of the log likelihood function under the regularity conditions.

$$I_n(\theta_{ij}) = Cov\left[\frac{\partial f(x;\theta)}{\theta_i}, \frac{\partial f(x;\theta)}{\theta_j}\right]$$

$$I_n(\theta_{ij}) = -E\left[\frac{\partial^2 lnf(x;\theta)}{\partial\theta_i\partial\theta_j}\right]$$

For the data from patients with malignant melanoma ,the fisher information matrix can be obtained as follow:

$$I_n(\theta_{ij}) = \begin{pmatrix} 0.183 & 0.015 & 0.011 & 0.136 \\ 0.015 & 0.018 & -0.002 & 0.025 \\ 0.011 & -0.002 & 0.016 & 0.185 \\ 0.136 & 0.025 & 0.185 & 0.537 \end{pmatrix}$$

the variances of the estimates are generally small, with ($\hat{b}$)having the largest variance (0.537).

Summery and Conclusion

Recently, many researchers hope to derive new distributions that better represent the real set(s) of data. There are many ideas that have been suggested in this regard. One of the most important of these ideas was the mixture of distributions. In this paper, two cases of mixture between two different distribution are investigated.The examples showes that the proposed distributions are flexible and can be fitted to a variety of data and the maximum likelihood estimations are good with the small bias and small mean square error. This asserts that the proposed distribution could be very usefull in fitting data.