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Discrete of a Novel Alpha Power Transformed Exponential Distribution: Estimation and COVID- 19 Application

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Abstract

A new distribution with two parameters named a discrete of the novel alpha power transformed exponential (DNAPTE) distribution is introduced using the survival of discretizing approach. Some of the statistical properties are obtained for the new distribution such as survival, hazard rate, alternative hazard rate functions, moments, and order statistics. Maximum likelihood method is applied under Type II censored samples for estimating the unknown parameters, survival and hazard rate function of the proposed model. A simulation study is carried out to illustrate the theoretical results of the maximum likelihood estimation. Finally, the DNAPTE distribution is adopted for fitting the number of COVID-19 deaths in China and Europe countries.

***Keywords*:** Discrete novel alpha power transformed exponential distribution; Order statistics; Type II censored samples; Maximum likelihood method; A simulation study; COVID-19 data.

1. Introduction

The discrete probability distributions have great importance in modeling real count data in many applied sciences such as public health, medicine, agriculture, epidemiology, and sociology. Several discrete distributions were introduced for modeling count data. However, some traditional discrete models such as Poisson, Geometric distributions have limited applications in reliability, failure times and count. Some real count data show either under-dispersion or over-dispersion. This has motivated several statisticians to explore new discrete models based on classical continuous distributions for modeling discrete failure times and reliability data. In the last two decades. Nowadays, the authors introduced discrete models by the discretization of continuous distribution for example: **Krishna and Pundir (2009)** proposed discrete analogues of the Pareto and Burr distribution, **Gomez-Deniz (2010)** introduced the discrete generalized exponential distribution, and **Jazi et al. (2010)** introduced the discrete inverse Weibull distribution. However, there is still a clear need to construct more flexible discrete distributions to serve several applied areas such as social sciences, economics, and reliability studies to properly suit different types of count data. Furthermore, **Kamari *et al.* (2016)** applied Bayesian approach under two types of loss functions: squared error and absolute error. Also, **Para and Jan (2016)** used the ML estimation of the unknown parameters of discrete Burr Type XII distribution and discrete Lomax distribution. While **AL-Babtain *et al.* (2020a, 2020b)** introduced the new two discrete models named the discrete Poisson - Lindley and discrete Lindley distributions and the natural discrete Lindley distribution, respectively. **Al-Metwally *et al.* (2020)** introduced a new distribution named discrete Marshall Olkin inverse Topp Leone distribution. **Eliwa *et al.* (2020)** proposed a new flexible discrete family of distributions, named discrete Gompertz-G family of distributions. In addition, **El-Morshedy *et al.* (2020a, 2020b)** introduced the discrete Burr-Hatke and exponentiated discrete Lindley distributions, respectively. **Almazah *et al.* (2021)** proposed the transmuted record type Geometric distribution. **Aljohani *et al.* (2021)** introduced the uniform Poisson–Ailamujia model. Also, **EL deep *et al.* (2021)** proposed a new distribution named discrete Ramos Louzada

distribution. While **Shafgat *et al.* (2021)** proposed a new discrete Nadaraiah and Haghighi distribution. A new distribution with two parameters named *discrete inverted Kumaraswamy* (DIK) distribution is introduced by **El-Helbawy *et al.* (2022)**. Also, **EL-Morshedy *et al.* (2022)** proposed a flexible discrete family of distributions named discrete odd Weibull-G family of distributions. Then, **Chesneau *et al.* (2022)** proposed a new distribution with one parameter heavy tailed discrete inverse Burr distribution using the general approach of discretization of continuous distribution.

The rest of the paper is organized as follows: *discrete of a novel alpha power transformed exponential* (DNAPTE) distribution is introduced, and some statistical properties are given in Section 2. While, in Section 3, *maximum likelihood* (ML) estimators are derived of the unknown parameters. The efficiency of the introduced estimation is assessed via simulation study and results are presented in Section 4. Section 5 provides two real applications to COVID -19 data of the DNAPTE distribution. Conclusion is discussed in Section 6.

2. Discretizing a Continuous Distribution

The general approach of discretizing a continuous variable can be used to construct a discrete model by introducing a grouping on the time axis see **Roy (2003, 2004)**. If the **crv,** X has the sf, $S(x) = P(X \geq x)$ and times are grouped into unit intervals so that the **drv** of X denoted$= [X]$; which is the largest integer less than or equal to x, will have the *probability mass function* (pmf).

The *probability mass function* (pmf) is a mathematical function that calculates the probability a discrete random variable will be a specific value. pmf also describe the probability distribution for the full range of values for a discrete variable. A discrete random variable can take on a finite or countable infinite number of possible values, such as the number of heads in a series of coin flips or the number of customers who visit a store on a given day.

$$\mathbf{P(x) = S(x) - S(x+1)} \quad \mathbf{, x = 0, 1, 2, \ldots}$$
(1)

The pmf of the **drv**, dX can be viewed as discrete concentration of pdf of X. So, given any continuous distribution it is possible to construct corresponding discrete distribution using (1).

One of the advantages of applying this approach of discretizing is that the **sf** for discrete distributions has the same functional form of the **sf** for the continuous distributions; as a result, many reliability characteristics and properties remain unchanged. Thus, discretization of a continuous lifetime model according to this approach is an interesting and simple approach to derive a discrete lifetime model corresponding to the continuous one.

2.1 Construction of discrete a novel alpha transformed exponential distribution.

Mashwani et al. (2021) proposed a new flexible family of distributions, named New Alpha Power Transformed NAPT family of distributions. A New Alpha Power Transformed Exponential NAPTE distribution is introduced as a special case of this family. They obtained some of the statistical properties for the NAPTE distribution. The model parameters have been estimated by the ML method. The pdf of A NAPTE distribution is given by

$$\boldsymbol{g(x;\alpha,\beta) = \frac{\ln(\alpha)\beta\alpha^{\ln\left(1-e^{-\beta x}\right)}}{e^{\beta x}-1}}, \qquad \boldsymbol{x > 0} \quad , \quad \boldsymbol{\alpha,\beta > 0} \qquad \textbf{(2)}$$

where are $\alpha \; and \; \beta$ shape parameters and should be positive.
The corresponding cdf and sf are, respectively, given by

$$\boldsymbol{G(x;\alpha,\beta) = \alpha^{\ln\left(1-e^{-\beta x}\right)}}, \qquad \boldsymbol{x > 0} \quad , \quad \boldsymbol{\alpha,\beta > 0} \qquad \textbf{(3)}$$

and

$$S(x) = 1 - \alpha^{\ln(1-e^{-\beta x})}, \quad x > 0 \quad , \quad \alpha, \beta > 0 \qquad (4)$$

Using (1) dX can be viewed as the discrete analogue to the continuous NAPTE variable X, and is commonly said to follow DNAPTE distribution with two parameters α and β, denoted by DNAPTE (α, β) distribution, where the corresponding pmf of dX can be written as

$$p(x) = \alpha^{\ln(1-e^{-\beta(x+1)})} - \alpha^{\ln(1-e^{-\beta x})} \quad , \quad x = 0, 1, 2, \ldots \qquad (5)$$

and the cdf, sf and hrf are as follows:

$$F(x) = 1 - S(x) + P(x) = \alpha^{\ln(1-e^{-\beta(x+1)})}, \quad x = 0, 1, 2, \ldots \qquad (6)$$

$$S(x) = 1 - F(x) + P(x) = 1 - \alpha^{\ln(1-e^{-\beta x})}, \quad x = 0, 1, 2, \ldots \qquad (7)$$

and

$$h(x) = \frac{P(x)}{S(x)} = \frac{\alpha^{\ln(1-e^{-\beta(x+1)})} - \alpha^{\ln(1-e^{-\beta x})}}{1-\alpha^{\ln(1-e^{-\beta x})}} \quad , x = 0, 1, 2, \ldots; \ \alpha, \beta > 0 \qquad (8)$$

There are some problems associated with the definition of $h(x)$, three of the more notable ones are given below:

a. $h(x)$is not additive for series system.
b. The cumulative hrf, $H(x) = \sum h(x) \neq -\ln S(x)$.
c. $h(x) \leq 1$ and it has the interpretation of a probability. [For more details, see **Xie *et al.* (2002) and Lai (2013) and (2014)**].

Therefore, it was necessary to find an alternative definition that is consistent with its continuous counterpart. **Roy and Gupta (1992)** provide an excellent alternative definition of a discrete hrf denoted by $h_1(x)$:

$$h_1(x) = \ln\left[\frac{s(x)}{s(x+1)}\right] = \ln\left[\frac{1-\alpha^{\ln(1-e^{-\beta x})}}{1-\alpha^{\ln(1-e^{-\beta(x+1)})}}\right] \quad , x = 0, 1, 2, \ldots; \ \alpha, \beta > 0 \quad (9)$$

There is a relationship between $h_1(x)$ and $h(x)$, given by:

$$h(x) = 1 - e^{-h_1(x)} \quad (10)$$

Plots of **pmf** and **hrf** of DNAPTE distribution are presented, respectively, in Figures 1 to 2, for some selected values of the parameters.

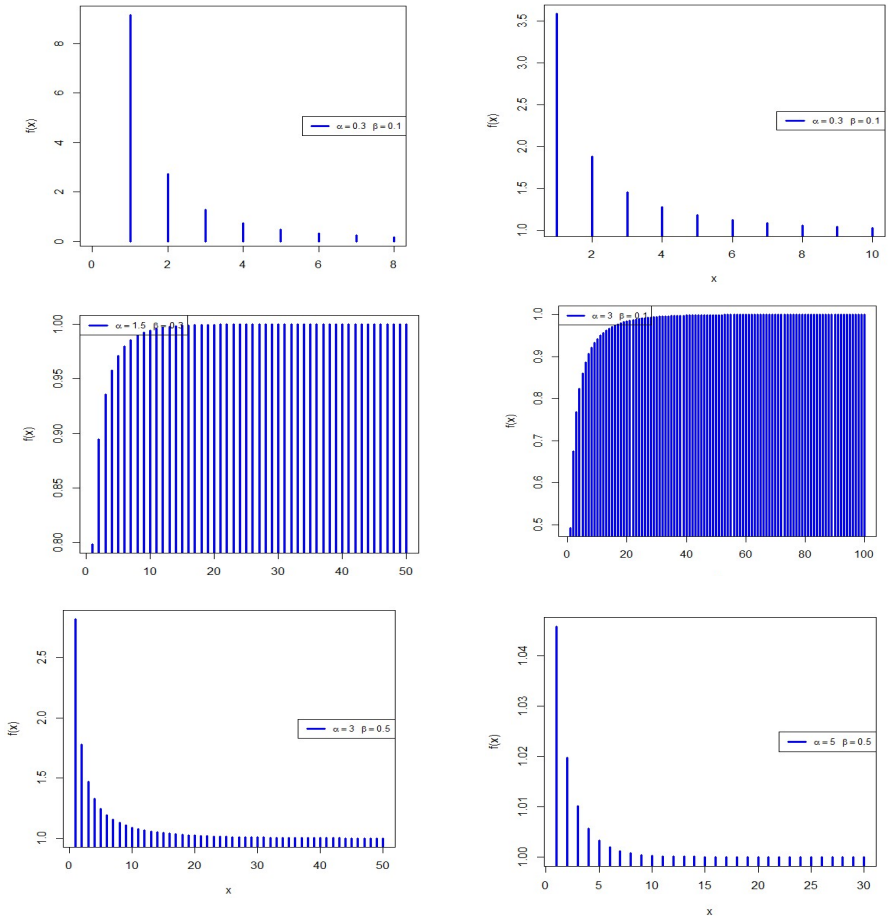

Figure 1: The plots of the probability mass function

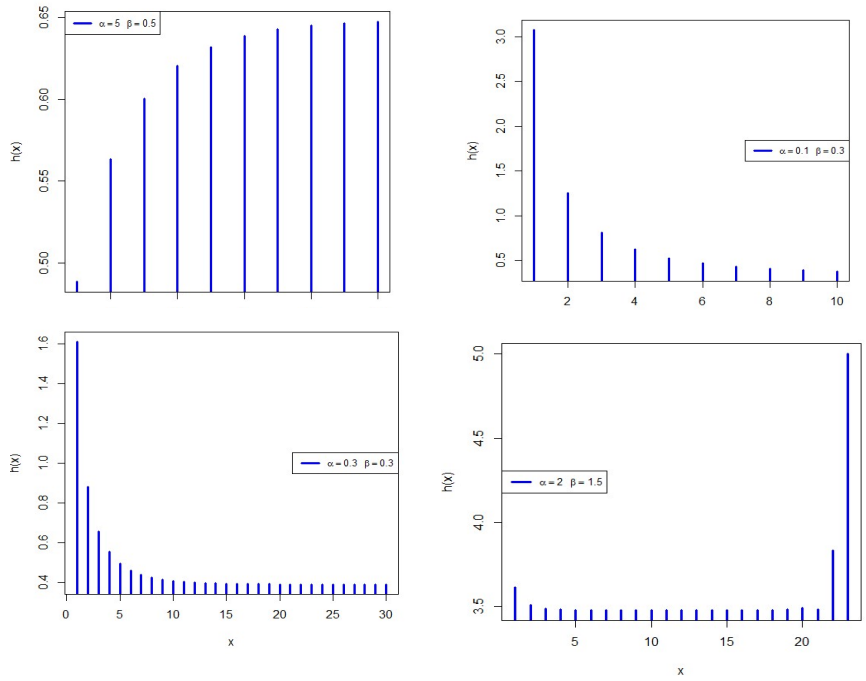

Figure 2: The plots of the hazard rate function

Figure 1 shows that the pmf of DNAPTE distribution can be decreasing and increasing according to the selected values of the parameters. While Figure 2 the hrf of DNAPTE (α, β) distribution is increasing, decreasing and bathtub so the DNAPTE (α, β) distribution provides a good fit to several data in literature.

2.2 Some properties of discrete a novel alpha Power transformed exponential distribution

This section is devoted to obtain some important statistical properties of DNAPTE (α, β) distribution, such as the mode, r^{th} moments and order statistics.

2.2.1 The moments of discrete a novel alpha power transformed exponential Distribution

In this subsection, non-central, central and standard moments are obtained.

a. The non-central moments of the discrete a novel alpha power transformed exponential distribution

The non-central moments of DNAPTE distribution can be obtained using (5) as follows:

$$\mu'_r = E(x^r) = \sum_{x=0}^{\infty} x^r\, p(x)$$

$$= \sum_{x=0}^{\infty} x^r \left[\alpha^{\ln(1-e^{-\beta(x+1)})} - \alpha^{\ln(1-e^{-\beta x})}\right], r = 1,2,3,4 \qquad (11)$$

In particular, the mean (μ) of DNAPTE distribution is given by

$$\mu'_1 = \mu = \sum_{x=0}^{\infty} x \left[\alpha^{\ln(1-e^{-\beta(x+1)})} - \alpha^{\ln(1-e^{-\beta x})}\right]. \qquad (12)$$

b. The central moments of the discrete a novel alpha power transformed exponential distribution

The central moments can be derived using the relation between the central and non-central moments as given below

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \mu^j \mu'_{r-j}, \quad r=1,2,\ldots \qquad (13)$$

thus, the *variance* (var) of DNAPTE distribution is

$$\mu_2 = \sum_{x=0}^{\infty} x^2 \left[\alpha^{\ln(1-e^{-\beta(x+1)})} - \alpha^{\ln(1-e^{-\beta x})}\right] - \left\{\sum_{x=0}^{\infty} x \left[\alpha^{\ln(1-e^{-\beta(x+1)})} - \alpha^{\ln(1-e^{-\beta x})}\right]\right\}^2. \qquad (14)$$

c. The standard moments of the discrete a novel alpha power transformed exponential distribution

The r^{th} standard moments can be obtained as follows:

$$\boldsymbol{a_r = E(\frac{x-\mu}{\sigma})^r.} \quad \textbf{(15)}$$

The skewness and kurtosis of the DNAPTE distribution are given by, respectively,

$\alpha_3 = \frac{\mu_3}{\mu_2^{1.5}}$ and $\alpha_4 = \frac{\mu_4}{\mu_2^2}$, Where μ_r, is given by (18) and r = 1, 2, ….

Table 1: The mean, standard deviation (SD), skewness (SK), Kurtosis (KT) and coefficient of variation (CV) of DNAPTE distribution

α	β	$\grave{\mu}_1$	SD	SK	KT	CV
0.25	0.25	0.2125	3.7876	9.15294	3.16235	17.8241
	0.5	0.15	4.48424	12.9667	4.48	29.8949
	0.75	0.0875	5.56234	22.2286	7.68	63.5684
	1.25	0.3704	2.67634	5.25106	1.814	7.2255
	1.5	0.19711	3.93794	9.86737	3.409	19.978
0.5	0.25	0.3000	3.09795	6.4833	2.24	10.3265
	0.5	0.425	2.40133	4.57647	1.58118	5.6501
	0.75	0.3469	2.8073	5.60647	1.93704	8.09211
	1.25	0.43843	2.33912	4.4363	1.53275	5.33524
	1.5	0.44829	2.29464	4.3387	1.49904	5.1187
0.75	0.25	0.6375	1.5904	3.05098	1.05412	2.49475
	0.5	0.45	2.2870	4.32222	1.49333	5.08226
	0.75	0.2625	3.36501	7.40952	2.56	12.8191
	1.25	0.6576	1.52819	2.95753	1.02183	2.32374
	1.5	0.5209	1.99439	3.73391	1.29007	3.82873
1.25	0.25	1.0625	0.56875	1.83059	0.63247	0.53529
	0.5	0.75	1.26536	2.5933	0.896	1.687145
	0.75	0.4375	2.34336	4.44571	1.536	5.35624
	1.25	0.86817	0.97274	2.24035	0.77404	1.12045
	1.5	2.355	0.6081	1.86696	0.64504	0.583696

2.2.2 Entropy

The average quantity of "information," "surprise," or "uncertainty" present in a random variable's potential outcomes is measured by a random variable's entropy in accordance with information theory. Renyi entropy (RE), (see Renyi (1961)), is a fundamental entropy. It is a key sign of complexity and ambiguity in many fields, such as statistical inference, physics, econometrics, and pattern recognition in computer science. You could enter ($\rho > 0, \rho \neq 1$, as the RE specification for the DNAPTE distribution.

$$RE(\rho) = \frac{1}{1-\rho} log \sum_{x=0}^{\infty} p_x^{\rho}(x),$$

$$= \frac{\rho}{1-\rho} log \sum_{x=0}^{\infty} \left\{ \alpha^{\ln\left(1-e^{-\beta(x+1)}\right)} - \alpha^{\ln\left(1-e^{-\beta x}\right)} \right\}.$$

Shannon entropy (SE), another well-known entropy, can be obtained as a special case of RE as $\rho \to 1$, where $SE = -E[log\, p(x)]$.

2.2.3 The order statistic of the discrete a novel alpha power transformed exponential distribution

Let $F(x; \alpha, \beta)$; the cdf of the i^{th} order statistic for a random sample $X_1, X_2, \dots, X_n$, from the DNAPTE (α, β) distribution, is given by

$$\boldsymbol{F_i(x;\ \alpha, \beta) = \sum_{r=i}^{n} \binom{n}{r} [F(x;\ \beta, \alpha)]^r [1 - F(x;\ \beta, \alpha)]^{n-r}.} \quad \textbf{(16)}$$

Using the binomial expansion for $[1 - F_i(x; \alpha, \beta)]^{n-r}$ and substituting (6) in (21), where

$$\boldsymbol{F_i(x;\ \alpha, \beta) = \sum_{r=i}^{n} \binom{n}{r} [F(x;\ \alpha, \beta)]^r \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j [F(x;\ \alpha, \beta)]^j =}$$

$$\boldsymbol{\sum_{r=i}^{n} \binom{n}{r} \sum_{j=0}^{n-r} \binom{n-r}{j} (-1)^j \left[\alpha^{\ln\left(1-e^{-\beta(x+1)}\right)}\right]^{r+j}.} \quad \textbf{(17)}$$

Special cases

Case I: If i=1 in (22) one can obtain the distribution function of the first order statistic, as given below

$$\boldsymbol{F_1(x;\ \alpha,\beta) = 1 - [1 - F(x;\ \alpha,\beta\ )]^n = 1 - \left[1 - \left(\alpha^{\ln(1-e^{-\beta(x+1)})}\right)\right]^n.}$$

(18)

Case II: If i = n in (22) the distribution function of the largest order statistic, as follows:

$$\boldsymbol{F_n(x;\ \alpha,\beta) = [F(x;\ \alpha,\beta\ )]^n = \left[\alpha^{\ln(1-e^{-\beta(x+1)})}\right]^n,} \quad \textbf{(19)}$$

which is the cdf of DNAPTE (α, β), and the sf of DNAPTE (n, β) is

$$\boldsymbol{S(x) = 1 - \left(\alpha^{\ln(1-e^{-\beta(x+1)})}\right)^n.} \quad \textbf{(20)}$$

Suppose that $X_1, X_2, X_3, ..., X_n$ is a random sample from the DNAPTE distribution with two parameters α and β. Let $X_{1:n}, X_{2:n}, X_{3:n}, ... , X_{n:n}$ denote the corresponding order statistics. Then, the pmf of $X_{i:n}$, is defined by:

$$\boldsymbol{P(X_{i:n} = x) = \frac{n!}{(i-1)!(n-i)!}\int_{F(x-1)}^{F(x)} v^{i-1}(1-v)^{n-i}dv.} \quad \textbf{(21)}$$

Using the binomial expansion for$(1-v)^{n-i}$, then the pmf in (26).

$$P(X_{i:n} = x) = \frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\int_{F(x-1)}^{F(x)} v^{i+v-1}dv =$$

$$\frac{n!}{(i-1)!(n-i)!}\sum_{j=0}^{n-i}\binom{n-i}{j}(-1)^j\left(\frac{1}{i+j}\right)$$

$$\times\left[\left[\alpha^{\ln(1-e^{-\beta(x+1)})}\right]^{i+j} - \left[\alpha^{\ln(1-e^{-\beta x})}\right]^{i+j}\right]. \quad \textbf{(22)}$$

The pmf of the smallest order statistic is obtained by substituting i=1 in (27) as follows:

$$P(X_{1:n}=x)=n\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\left(\frac{1}{1+j}\right)\times\left[\left[\alpha^{\ln(1-e^{-\beta(x+1)})}\right]^{1+j}-\left[\alpha^{\ln(1-e^{-\beta x})}\right]^{1+j}\right]. \quad (23)$$

And, the pmf of largest order statistic is obtained by substituting $i=n$ in (27) as follows:

$$P(X_{n:n}=x)=\left[\alpha^{\ln(1-e^{-\beta(x+1)})}\right]^{n}-\left[\alpha^{\ln(1-e^{-\beta x})}\right]^{n}. \quad (24)$$

Also, (22) can be used to obtain the pmf of the DNAPTE (α,β) distribution, (see Arnold *et al*. (2008)).

3. Estimation of the Parameters of Discrete a Novel Alpha Power Transformed Exponential Distribution

In this section, methods of moments and ML are used to derive the estimators of the parameters for the DNAPTE distribution.

3.1 Method of moments

In this subsection, method of moments is applied to estimate the unknown parameters of the DNAPTE distribution. The method of moments is based on equating the population moments; which are functions of the parameters to the corresponding sample moments and subsequently solving the two equations simultaneously. The first the second population and sample moments, respectively, are

$$\mu(\alpha,\beta)=\mu=\sum_{x=0}^{\infty}x\left[\alpha^{\ln(1-e^{-\beta(x+1)})}-\alpha^{\ln(1-e^{-\beta x})}\right], \quad (25)$$

$$\mu_2(\alpha,\beta)=\mu_2=\sum_{x=0}^{\infty}x^2\left[\alpha^{\ln(1-e^{-\beta(x+1)})}-\alpha^{\ln(1-e^{-\beta x})}\right], \quad (26)$$

$$M_1=\frac{1}{n}\sum_{i=1}^{n}x_i \text{ and } M_2=\frac{1}{n}\sum_{i=1}^{n}x^2. \quad (27)$$

$$\text{Then equating } \mu(\tilde{\alpha},\tilde{\beta})=M_1 \text{ and } \mu_2(\tilde{\alpha},\tilde{\beta})=M_2, \quad (28)$$

where $\tilde{\alpha}$ *and* $\tilde{\beta}$ are the estimators of α *and* β

Since the moments of DNAPTE distribution cannot be obtained in closed forms and (33) cannot be solved via ordinary techniques, therefore the estimates can be obtained numerically.

3.2 Method of maximum likelihood

In this section, method of ML is used to derive the estimators of the parameters for the DNAPTE distribution.

The method of ML is used to estimate the vector of two parameters,
$\underline{\varphi} = (\alpha, \lambda)$ sf, hrf, and ahrf of the DNAPTE (α, β) distribution. Based on Type II censored samples, also confidence interval of the parameters (α, β) sf, hrf, and ahrf are derived. Suppose that $X_1, X_2, \ldots, X_r$ is a Type II censored sample of size r obtained from a life test on n items whose lifetimes have a DNAPTE (α, β) distribution. Then the likelihood function is

$$\boldsymbol{L\left(\underline{\varphi}, \underline{x}\right) \propto \{\prod_{i=1}^{r} p(x_i)\}[S(x_r)]^{n-r},} \qquad \textbf{(29)}$$

where p(x) and S(x) are given, respectively, by (5) and (7). The $X_{(i)}$'s are ordered times for $i = 1, 2, \ldots r$

$$\boldsymbol{L\left(\underline{\varphi}; \underline{x}\right) \propto \left\{\prod_{i=1}^{r} \alpha^{\ln\left(1-e^{-\beta(x_i+1)}\right)} - \alpha^{\ln\left(1-e^{-\beta x_i}\right)}\right\} \times \left[1 - \alpha^{\ln\left(1-e^{-\beta x_r}\right)}\right]^{n-r}.} \qquad \textbf{(30)}$$

The natural logarithm of the likelihood function is given by

$$\ell = lnL\left(\underline{\varphi}; \underline{x}\right) \propto ln \prod_{i=1}^{r} \left[\alpha^{\ln\left(1-e^{-\beta(x_i+1)}\right)} - \alpha^{\ln\left(1-e^{-\beta x_i}\right)}\right] + (n-r) \ln\left[1 - \alpha^{\ln\left(1-e^{-\beta x_r}\right)}\right] \qquad (31)$$

$$\boldsymbol{\ell = \sum_{i=1}^{r} ln\left[\alpha^{\ln\left(1-e^{-\beta(x_i+1)}\right)} - \alpha^{\ln\left(1-e^{-\beta x_i}\right)}\right] + (n-r) \ln\left[1 - \alpha^{\ln\left(1-e^{-\beta x_r}\right)}\right].} \qquad \textbf{(32)}$$

Considering the two parameters, α and β are unknown and differentiating the log likelihood function in (37), with respect to α and β, one obtains

$$\frac{\partial \ell}{\partial \alpha} = \sum_{i=1}^{r} \frac{\left[\ln\left(1-e^{-\beta(x_i+1)}\right)\alpha^{\left[\ln\left(1-e^{-\beta(x_i+1)}\right)-1\right]}\right] - \left[\ln\left(1-e^{-\beta x_i}\right)\alpha^{\left[\ln\left(1-e^{-\beta x_i}\right)-1\right]}\right]}{\alpha^{\ln\left(1-e^{-\beta(x_i+1)}\right)} - \alpha^{\ln\left(1-e^{-\beta x_i}\right)}}$$

$$-(n-r)\frac{ln\left(1-e^{-\beta x_r}\right)\alpha^{\left[\ln\left(1-e^{-\beta x_r}\right)-1\right]}}{\left[1-\alpha^{\ln\left(1-e^{-\beta x_r}\right)}\right]}. \qquad (33)$$

and

$$\frac{\partial \ell}{\partial \beta} = \sum_{i=1}^{r} \frac{\left\{\alpha^{\ln\left(1-e^{-\beta(x_i+1)}\right)}\left[\frac{(x_i+1)e^{-\beta(x_i+1)}}{1-e^{-\beta(x_i+1)}}\right]\ln\alpha\right\} - \left\{\alpha^{\ln\left(1-e^{-\beta x_i}\right)}\left[\frac{x_i e^{-\beta x_i}}{1-e^{-\beta x_i}}\right]\ln\alpha\right\}}{\alpha^{\ln\left(1-e^{-\beta(x_i+1)}\right)} - \alpha^{\ln\left(1-e^{-\beta x_i}\right)}}$$

$$-(n-r)\frac{\alpha^{\ln\left(1-e^{-\beta x_r}\right)}\left(x_i e^{-\beta x_r}\right)\ln\alpha}{\left[1-e^{-\beta x_r}\right]\left[1-\alpha^{\ln\left(1-e^{-\beta x_r}\right)}\right]}. \qquad (34)$$

Then the ML estimators of the parameters, denoted by $\hat{\alpha}$ and $\hat{\beta}$ are derived by equating the two nonlinear likelihood (38) and (39) to zeros and solving numerically.

Depending on the invariance property, the ML estimators of S(x), h(x) and h_1(x) can be obtained by replacing α and β with their corresponding ML estimators $\hat{\alpha}$ and $\hat{\beta}$, respectively, in (7), (8) and (9) as given below

$$\widehat{S}_{ML}(x) = 1 - \hat{\alpha}^{\ln\left(1-e^{-\hat{\beta}x}\right)}, \qquad x = 0, 1, 2, \ldots \qquad (35)$$

$$\widehat{h}_{ML}(x) = \frac{\hat{\alpha}^{\ln\left(1-e^{-\hat{\beta}(x+1)}\right)} - \hat{\alpha}^{\ln\left(1-e^{-\hat{\beta}x}\right)}}{1-\hat{\alpha}^{\ln\left(1-e^{-\hat{\beta}x}\right)}}, \qquad x = 0, 1, 2, \ldots \qquad (36)$$

And

$$\hat{h}_{1_{ML}}(x) = \ln\left[\frac{1-\hat{\alpha}^{\ln\left(1-e^{-\hat{\beta}x}\right)}}{1-\hat{\alpha}^{\ln\left(1-e^{-\hat{\beta}(x+1)}\right)}}\right], \qquad x = 0,1,2, \ldots \qquad (37)$$

When the sample size is large and the regularity conditions are satisfied, see **(Lehmann and Casella (1998))**, the asymptotic distribution of the ML estimators is

$\underline{\varphi} \sim$ Bivariate Normal $\left(\underline{\varphi}, \mathrm{I}^{-1}\underline{x}\left(\underline{\varphi}\right)\right)$, where $\underline{\varphi} = (\alpha, \beta)$, $\hat{\varphi} = (\hat{\alpha}, \hat{\beta})$, and $I^{-1}(\varphi)$.

The asymptotic variance-covariance matrix of the ML estimators α and β ,which is the inverse of the observed Fisher information matrix. The asymptotic observed Fisher information matrix can be obtained as follows:

$$\boldsymbol{I}_{\underline{x}}\left(\underline{\boldsymbol{\varphi}}\right) \approx \begin{bmatrix} -\left(\frac{\partial^2 \ell}{\partial \alpha^2}\right) & -\left(\frac{\partial^2 \ell}{\partial \alpha\, \partial \lambda}\right) \\ -\left(\frac{\partial^2 \ell}{\partial \alpha\, \partial \lambda}\right) & -\left(\frac{\partial^2 \ell}{\partial \lambda^2}\right) \end{bmatrix}_{(\hat{\alpha},\hat{\beta})} . \quad \textbf{(38)}$$

The asymptotic $100(1-\alpha)$ confidence interval for α, λ, $S_{ML}(x)$, $h_{ML}(x)$ and $h_{1_{ML}}(x)$ are given, respectively by:

$$\boldsymbol{L_\omega = \hat{\omega} - Z_{\frac{\alpha}{2}}\sigma_{\hat{\omega}} \quad and \quad U_\omega = \hat{\omega} - Z_{\frac{\alpha}{2}}\sigma_{\hat{\omega}}},$$

(39)

where L_ω and U_ω are the lower and upper bound $\hat{\omega}$ is $\hat{\alpha}$, $\hat{\lambda}$, $\hat{S}(x)$, $\hat{h}(x)$ or $\hat{h}_1(x)$, Z is the $100(1-\frac{\alpha}{2})\%$ the standard normal percentile, $(1-\alpha)\%$ is the confidence coefficient, $\sigma_{\hat{w}}$ is the standard deviation and length $= U_\omega - L_\omega$.

4. Numerical Results

This section aims to investigate the precision of the theoretical results based on simulated and real data, by evaluating *relative absolute biases* (RABs) and *relative errors* (REs).

4.1 Simulation study

In this subsection, a simulation study is presented to illustrate the application of the various theoretical results developed in the previous section on the basis of generated. Data from DNAPTE (α, β) distribution, for different sample sizes (*n*=30, 50 and 100) and using number of replications *N*=1000. The computations are performed

using R package. The numerical procedures are performed according to the following algorithm.

Step 1: a random sample$X_1, X_2, \dots, X_n$ of sizes (n=30, 50,100) these random samples are generated from DNAPTE distribution using the following transformation: $x_i = -\frac{1}{\beta}\ln\left[1 - e^{\frac{\ln u}{\ln \alpha}}\right] - 1, i = 1,2, \dots, n$ and u_i are random sample from uniform $(0,1)$ and then taking the ceiling.

Step 2: two different set values of the parameters are selected as, Set 1$(\alpha = 3, \beta = 0.5)$ and Set 2 $\alpha = 0.5,\ \beta = 5$.

Step 3: For each model parameters and for each sample size, the ML estimates are computed.

Step 4: Steps from 1 to 3 are repeated 1000 times for each sample size and for selected sets of the parameters. Then the averages, RABs, REs and variances of the estimates of the unknown parameters are computed.

The results of the simulation study are given in Tables 2 and 3. The RABs and REs of ML estimates of the parameters, sf and hrf are computed at t_0=0.4, as follows:

1) Average $= \frac{\sum_{i=1}^{N}(estimate_i)}{N}$,

2) RAB (estimate) $= \frac{|bias(estimate)|}{true\ value}$,

3) Relative error (estimate) $= \frac{ER(estimate)}{true\ value}$,

4) Estimated risk (estimate) $= \frac{\sum_{i=1}^{N}(estimate_i - tru\ \ value)}{N}$.

Table 2 shows the averages, RABs, REs, variances for the parameters, sf and hrf estimates, also 95% confidence intervals where the initial values for the parameters are α=3, β=0.5 under three levels of $\frac{r}{n} \times 100$ percentage of uncensored observations. Type II censoring 80% and 100%. Table 3 displays the same computational results, but for different initial values of the parameters α=0.5, β=5 , at the same mission time t_0 from the DNAPTE distribution for different sample sizes where (n=30, 50 and 100) and also level of Type II censoring 80% and 100% and number of replications, N = 1000.

Table 2: RABs, REs of ML estimates, 95% confidence intervals of the parameters,
survival and hazard rate functions from DNAPTE distribution for different sample sizes n,
censoring level r and the replications N= 1000, $\alpha = 3, \beta = 0.5$, t_0=0.9

n	r	parameters	estimates	RABs	REs	LL	UL	Length
30	24	α	3.2673	0.0891	0.0727	2.7281	3.8067	1.0786
		β	0.5409	0.0818	0.0029	0.4335	0.6482	0.2147
		$R(t_0)$	0.6721	0.0398	0.0101	0.4752	0.1934	0.4028
		$h(t_0)$	0.3817	0.0066	0.0104	0.6003	0.8781	0.1676
	30	α	3.1859	0.0619	0.0358	2.8075	3.5641	0.7565
		β	0.5283	0.0565	0.0021	0.4386	0.6179	0.1793
		$R(t_0)$	0.6769	0.0052	0.0101	0.4744	0.8771	0.4026
		$h(t_0)$	0.3608	0.0271	0.0102	0.1898	0.5943	0.4045
50	40	α	3.1911	0.0636	0.0377	2.8027	3.5793	0.7766
		β	0.5309	0.0619	0.0022	0.4378	0.6241	0.1863
		$R(t_0)$	0.7468	0.0038	0.0101	0.4734	0.8759	0.4025
		$h(t_0)$	0.3349	0.0310	0.0103	0.1909	0.5961	0.0625
	50	α	3.1523	0.0509	0.0245	2.8395	3.4659	0.6264
		β	0.5197	0.0394	0.0016	0.4397	0.5997	0.1599
		$R(t_0)$	0.7916	0.0074	0.0101	0.4757	0.8786	0.4029
		$h(t_0)$	0.2474	0.0166	0.0102	0.4064	0.5896	0.4032
100	80	α	3.0482	0.1461	0.0031	2.9319	3.1557	0.2237
		β	0.5058	0.0115	0.0012	0.4352	0.5763	0.1411
		$R(t_0)$	0.8349	0.0023	0.0101	0.4724	0.8749	0.4025
		$h(t_0)$	0.1938	0.0048	0.0101	0.1823	0.5848	0.4024
	100	α	3.0196	0.0065	0.0016	2.9398	3.0995	0.1598
		β	0.5035	0.0069	0.0012	0.4336	0.5734	0.1398
		$R(t_0)$	0.8869	0.0027	0.0101	0.4711	0.8735	0.4024
		$h(t_0)$	0.0984	0.0036	0.1012	0.1818	0.5842	0.4025

Table 3: RABs, REs of ML estimates, 95% confidence intervals of the parameters,
survival and hazard rate functions from DNAPTE distribution for different sample sizes n,
censoring level r and the replications N= 1000, $\alpha = 0.5, \beta = 5$, t_0=0.9

n	r	parameters	estimates	RABs	Res	LL	UL	Length
30	24	α	0.4188	0.1624	0.0078	0.2421	0.5955	0.3534
		β	5.4938	0.0988	0.2450	4.5037	6.4839	1.9801
		$R(t_0)$	0.0078	0.1971	0.0101	0	0.1949	0.1949
		$h(t_0)$	0.9933	0.0026	0.0011	0.7946	1.1972	0.4025
	30	α	0.4392	0.1268	0.0049	0.2989	0.5793	0.8032
		β	5.2683	0.0535	0.0726	4.7201	5.8064	1.0782
		$R(t_0)$	0.0898	0.0681	0.0103	0.1939	0.2085	0.4024
		$h(t_0)$	0.8644	0.0015	0.0101	1.1961	0.8937	0.4024
50	40	α	0.4392	0.1217	0.0049	0.2989	0.5732	0.2803
		β	5.2673	0.0535	0.0766	4.7282	5.8064	1.0782
		$R(t_0)$	0.1349	0.0681	0.0101	0.0939	0.1939	0.1000
		$h(t_0)$	0.7264	0.0011	0.0113	0.9369	1.2984	0.3616
	50	α	0.4795	0.0411	0.0016	0.3987	0.5602	0.1615
		β	5.1624	0.0325	0.027	4.8302	5.4947	0.6645
		$R(t_0)$	0.2948	0.0848	0.0101	0	0.1941	0.1941
		$h(t_0)$	0.8394	0.0011	0.1012.	0.9581	1.1684	0.2103
100	80	α	0.4916	0.0168	0.0013	0.4201	0.5632	0.1431
		β	5.1025	0.0205	0.0177	4.8861	5.3189	0.4328
		$R(t_0)$	0.4098	0.0065	0.0101	0.2085	0.4983	0.2899
		$h(t_0)$	0.8895	0.0006	0.0102	0.7927	0.9219	0.1291
	100	α	0.4968	0.0063	0.0012	0.4269	0.5667	0.3709
		β	5.0859	0.0172	0.0086	4.9005	5.2714	0.3709
		$5.R(t_0)$	0.6981	0.0663	0.0101	0.2085	0.7294	0.5209
		$h(t_0)$	0.9235	0.0005	0.0101	0.7927	0.9586	0.1659

5. **Applications of COVID-19 data**

The DNAPTE distribution's flexibility is demonstrated using two real-world COVID-19 data sets. The first set of data is the number of COVID-19 daily deaths in China from January 23 to March. (https://www.worldometers.info/coronavirus/country/china/).

Table 4: The observations are listed below in ascending order.

3	3	4	5	5	6	6	7	7	7	8	8	9	10	11	11	13
3	14	15	16	17	22	22	24	26	26	27	28	29	30	31	31	35
3	3	4	4	4	4	4	4	5	5	6	6	7	7	7	8	8
8	8	2	3	4	5	6	7	2	7	4	5	1	3	3	6	9
9	9	9	9	1	1	1	1	1	1	1	1	1	1	1		
7	7	7	8	0	0	0	1	1	2	3	4	4	4	5		
				5	8	9	4	8	1	6	2	3	5	0		

The second set of data shows the number of COVID-19 daily deaths in Europe from March 1 to March 31 (https://covid19.who.int/). The observations are as follows:

Table 5: The observations are listed below in ascending order

6	18	28	29	44	47	55	116	118	129	150
184	219	236	237	336	421	434	612	648	706	838
1129	1393	1540	1941	2175	2278	2667	2803	2824		

Some descriptive measures of both data sets are reported in Table 6.

Table 6: The descriptive measures of the two data sets in China and Europe

Data	Min	Mean	Median	Var	Skewness	kurtosis	DI	Q3	Max
I	3	49.74	33	1924.8	0.8365	2.4502	38.696	83	150
II	6	818	336	868739.6	1.0167	2.6308	1062.1	1407	2824

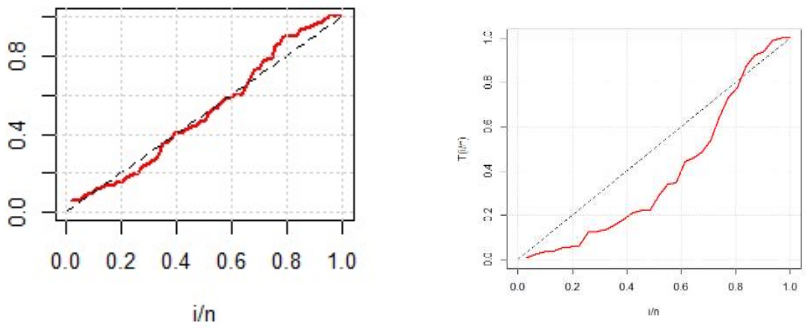

Figure 3. The TTT plot of the DNAPTE model for number of deaths in China and Europe

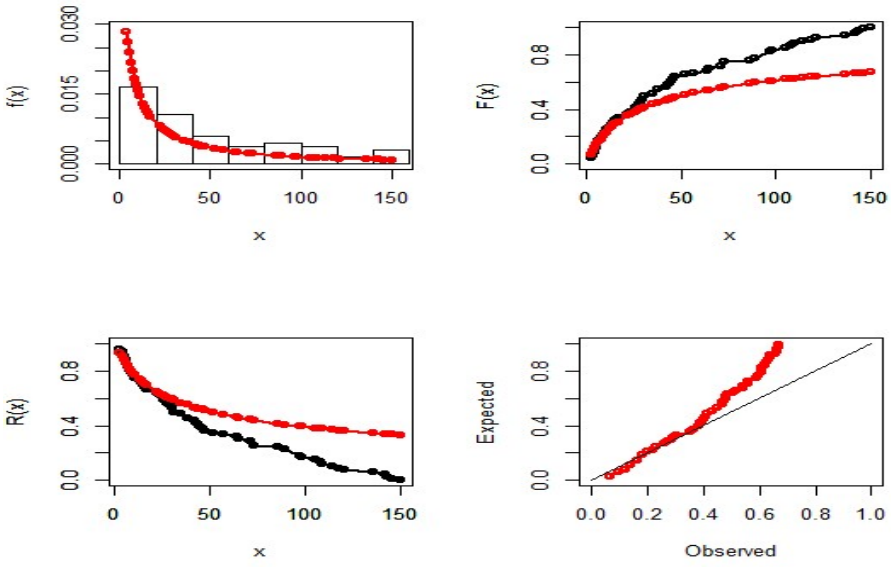

Figure 4: The Histogram, pdf, empirical cdf, empirical sf and the P-P plots of the DNAPTE model for number of deaths in China.

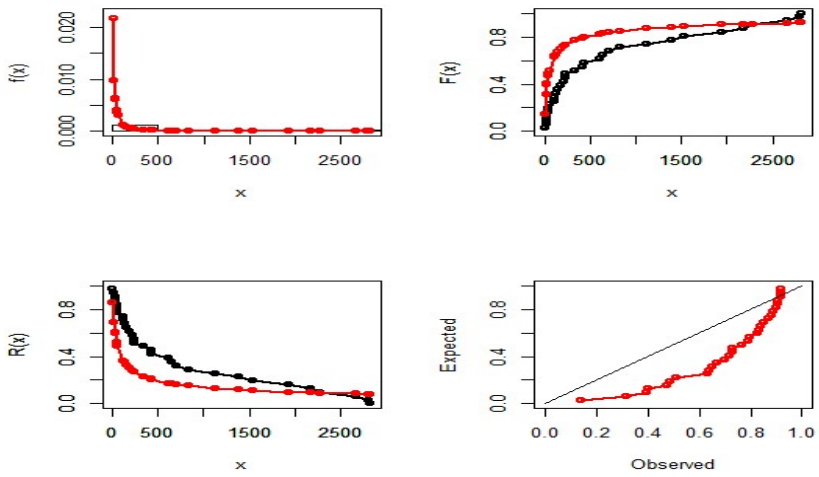

Figure 5: The histogram, pdf, empirical cdf, empirical sf and the P-P plots of the DNAPTE model for number of deaths in Europe

6. Conclusion

The DNAPTE distribution is proposed in this article as a new discrete probability distribution. It can be used as an alternative to some well-known discrete distributions. The discrete novel alpha power transformed exponential distribution's mathematical properties are presented. The model parameters are estimated using the ML estimation method with Type II censoring. Comprehensive simulation results are obtained to validate the theoretical results. The DNAPTE distribution's utility is demonstrated empirically through two applications to the number of deaths caused by COVID-19 in China and Europe.

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