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Non-redundant Fractional Fourier transform domain of band-limited Signals

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ABSTRACT

Sampling plays a vital role in digital signal processing, as the continuous-time signal should be sampled and then rebuilt from its samples, this approach complies with the traditional sampling theorem. The sampling theorem for the conventional Fourier Transform should be generalized to the continuous fractional Fourier transform case since the continuous fractional Fourier transform (FRFT) has developed into a highly effective tool in signal processing, optics, and other Engineering and scientific applications. The continuous fractional Fourier transform of a specific rotation angle is proposed in this publication, along with a method for sampling continuous band-limited signals to derive their discrete-time versions without aliasing. The method is developed from the sampling theorem for the conventional Fourier transform and the integral formulation of the continuous fractional Fourier transform. The lemmas and corollary put forward in this work are generalizations of the Fourier transform's traditional form. Finally, the numerical outcomes convincingly support our study.

1. Introduction

Any digital application, such as in signal processing, relies on the sampling theorem to sample a continuous signal and reconstruct it from its samples.

According to Shannon sampling theory [1], which is frequently applied to band-limited continuous-time signals in the FT domain, the sampling rate—also referred to as the Nyquist rate [2]—must be at least twice the maximum frequency in the signal in order to reconstruct the band-limited signal from its samples.

A powerful new time-frequency analysis tool with numerous applications in optics and other engineering fields [3–12], the FRFT can also be seen as a generalization of the traditional FT.

Recent work [13–19] has resulted in sampling theorems related to the FRFT, which should be considered as a generalization of the sampling theorem for the FT. The frequency domain application of the conventional Shannon theorem was developed by Xia and Zayed [13, 20], The FT was expanded by Erseghe et al. [21] to include variations for continuous and discrete-time signals, which is equivalent to the FRFT. Shannon's interpolation theorem was generalized for the FRFT [14]. In order to reconstruct a band-limited signal, Zayed [22] developed two sampling formulas that use samples from the signal and its Hilbert transformation collected at half the Nyquist rate. Tao et al.'s [17] generalization of the conventional sample rate conversion theory led to the derivation of the sampling theorem for band-limited signals in the FRFT domain. A necessary and sufficient condition for function space uniform sampling was created using FRFT by Liu et al. [23] and Ma et al. new estimates of the fractional power spectral density and fractional correlation function are proposed using non-uniform sampling of random signals with non-stationarity and limited bandwidths in the fractional Fourier domain.

This paper's primary goal is to present the sampling theorem in the FRFT domain for continuous band-limited signals with constrained bandwidth for a specific angle of rotation in the Time-Frequency plane. The proof is based on the sampling theorem in the conventional FT and the integral definition of the FRFT. Section II will discuss the Shannon theory in conventional FT as well as preliminary information on the definition and fundamental characteristics of FRFT and Time-Frequency plane. The fundamental contribution of this study is the sampling theorem in the FRFT domain, which will be presented in Section

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III. The simulation results are provided in Section IV. Conclusion is drawn in Section V.

2. Preliminaries

2.1. The FRFT

The first attempt to define the FRFT was undertaken in 1939 by Kober [25]. Certain kinds of ordinary and partial differential equations that emerge in quantum mechanics need to be solved, Namias [26] rediscovered the FRFT in 1980. Later, his findings were improved [22], and the FRFT was created as a more inclusive version of the FT. The FRFT has numerous uses in the fields of optics and signal processing [7,10].

The integral definition of the a^{th} order FRFT of a function (signal) $x(t)$ has been defined as [3]:

$$\mathcal{F}^{\alpha}\{x(t)\}(t_{\alpha}) = X(u) = \int_{-\infty}^{\infty} x(t)\ K_{\alpha}(t,u)dt \#(1a)$$

Where:

$$K_{\alpha}(t,u) = \begin{cases} \sqrt{\frac{1-i\,cot(\alpha)}{2\pi}}\ exp\left[i\pi\left((t^2+u^2)cot(\alpha) - 2tu\ csc(\alpha)\right)\right] & , \quad if\ \alpha \neq n\pi \\ \delta(u-t) & , \quad if\ \alpha = 2n\pi. \\ \delta(u-t) & , \quad if\ \alpha = (2n+1)\pi \end{cases} \#(1b)$$

and

$$\alpha = \frac{a\pi}{2}\ . \#(1c)$$

The FRFT generalizes the FT to an arbitrary fractional order a and reduces to the FT for $a = 1$ [27].

2.2. Time-Frequency plane and its relation to the FRFT

The continuous function (signal) is rotated in the Time-Frequency plane by an arbitrary angle α by the FRFT. Thus, the FRFT is also known as the angular Fourier transform.

The function (signal) x(t) along the u-axis in the Time-Frequency plane with an angle α of is labelled as the fractional Fourier representation as $X_{\alpha}(u)$, as seen in Fig. 1.

For $\alpha = 0$; $X_{\alpha}(u)$ is the original function, and $\mathcal{F}^{\alpha}$ is the identity operator.

For $\alpha = \frac{\pi}{2}$; $X_{\alpha}(u)$ is the function represented on the frequency axis, and $\mathcal{F}^{\alpha}$ is the Fourier transform operator.

For $\alpha = \pi$; $X_{\alpha}(u)$ is the function represented on the negative time axis, and $\mathcal{F}^{\alpha}$ has the same effect as flipping the original function.

For $\alpha = \frac{3\pi}{2}$; $X_{\alpha}(u)$ depicted on the minus frequency axis. In this case, $\mathcal{F}^{\alpha}$ has the same result as inverting the function's Fourier transform.

In general:

$$X_{\alpha}(u) = X_{\alpha+2n\pi}(u)\ \text{, n is an integer.} \#(2)$$

The function represented by $X_{\alpha}(u)$ for $0 < \alpha < \frac{\pi}{2}$ is shown along an axis that is in the first quadrant of the Time-Frequency plane and is related to other Time-Frequency representations like the Wigner transform (WT) and short-time Fourier Transform (STFT).

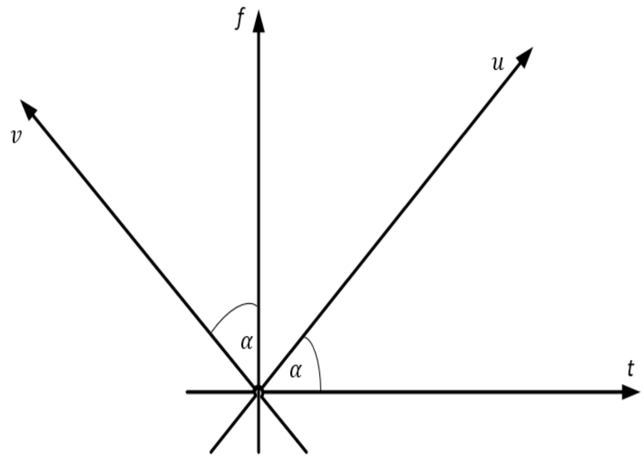

Fig. 1. The time-frequency plane

2.3. Sampling theory

In Engineering applications and digital signal processing, the analog signal needs to be converted to a digital form to be processed by the computer, and then reconstructing the processing signal from its samples.

Analogue signal values at distinct time instants are represented by the sampling process.

$$t = nT_s; n = 0, 1, 2, \dots \#(3)$$

Where T_s is the sampling period. This results in creating discrete-time signal.

As FT is a powerful tool for frequency domain analysis of continuous band-limited signal

$$x(t); t_{min} < t < t_{max} \#(4)$$

But to implement the FT integral of $x(t)$ by a computer program numerically, the discrete time signal

$$x[n] = \mathrm{x}(nT_s); n = 0, 1, 2, \dots \#(5)$$

Corresponding to $x(t)$ should be obtained. This can be done by applying the sampling process to (t) , the FT output of $x[n]$ is $X_s(f)$, which is a collection of repeated versions of $X(f)$ (the FT output of $x(t)$) scaled by the sampling frequency $f_s = \frac{1}{T_s}$, and repeated every f_s. These repeated versions are called images.

The sampling frequency should be larger than twice the signal bandwidth, or more than twice the maximum frequency (f_m), to perfectly reconstruct the analogue signal from its spectrum. which is the conventional Nyquist rate [1,2], i.e.

$$f_s > 2\ f_m \#(6)$$

The periodicity of $X_s(f)$ results in redundant frequency information (aliasing frequencies) in the spectrum of $x[n]$.

3. Proposed technique

In this section, the FRFT domain of Shannon sampling theory will be proved in a simple way by using the main definition of FRFT and conventional Shannon sampling theorem for the FT.

Lemma 1: The FT of the sampled continuous band-limited signal $x(t)$ contains non-redundant information in the interval $f \in \left[0, \frac{f_s}{2}\right]$

where

f_s is the sampling frequency of the signal complied with conventional sampling theory.

Proof

Consider Sinusoidal signal with single frequency f_0

$$x(t) = A\,cos(2\pi f_0 t + \emptyset) \quad (7)$$

Sample $x(t)$ using sampling frequency $f_s > 2\,f_0$, to produce sampled version of $x(t)$ is $x[n]$ which is defined as:

$$x[n] = A\,cos(2\pi f_0 n T_s + \emptyset) = A\,cos\left(2\pi(lf_s \pm f_0)\frac{n}{f_s} + \emptyset\right) \#(8)$$

implies that, the spectrum of the signal will be repeated at frequencies:

$$lf_s \pm f_0\,; l = 0, 1, 2, \dots \#(9)$$

To get the basic spectrum and avoid aliasing, then

$$f_s - f_0 > f_0 \#(10a)$$

$$\therefore f_s > 2f_0 \#(10b)$$

And the basic spectrum of the signal should be viewed only in the period:

$$f_s - f_0 > f \geq f_0 \#(11)$$

Which can be achieved for:

$$f = \frac{f_s}{2} \#(12)$$

As f in that case will be greater than or equal to f_0 and in the same time less than $f_s - f_0$.

So

$$f_{max} = \frac{f_s}{2} \#(13)$$

The result above can be generalized for any expandable signal with maximum frequency f_m as a linear combination of sinusoidal signals can be used to express the signal.

Q.E.D.

Lemma 2: The continuous-time band-limited signal in the FRFT, $X(t_\alpha)$, contains non-redundant information in the interval $t_\alpha \in \left[0, \frac{f_s}{2}\sin\alpha\right]$

where

f_s is the sampling frequency of the signal complied with conventional sampling theory.

Proof

The FRFT definition for signal $x(t)$ given by (1) can be expressed as [14]:

$$\mathcal{F}^\alpha\{x(t)\}(t_\alpha) = X(t_\alpha) = \sqrt{\frac{1 - i\cot\alpha}{2\pi}}\; e^{i\pi t_\alpha^2 \cot\alpha} \int_{-\infty}^{\infty} e^{i\pi t^2 \cot\alpha} x(t)\; e^{-i2\pi t t_\alpha \csc\alpha} dt. \#(14)$$

Let

$$t_\alpha \csc\alpha = \hat{f} \#(15)$$

$$\mathcal{F}^\alpha\{x(t)\}(t_\alpha) = X(t_\alpha) = \sqrt{1 - i\cot\alpha}\; e^{i0.5\pi \hat{f}^2 \sin 2\alpha} \mathcal{F}^{\frac{\pi}{2}}\{e^{i\pi t^2 \cot\alpha} x(t)\}(\hat{f}) \#(16)$$

where $\mathcal{F}^{\frac{\pi}{2}}$ is the FT operator defined in terms of frequency as

$$\mathcal{F}^{\frac{\pi}{2}}\{x(t)\}(f) = X(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\; e^{-i2\pi t f} dt \#(17)$$

This means that $X(t_\alpha)$ can be defined by the conventional FT representation of $e^{i\pi t^2 \cot\alpha} x(t)$, with frequency $\hat{f}$.

Lemma 1 implies that

$$\hat{f}_{max} = \frac{f_s}{2}\,. \#(18)$$

Hence

$$(t_\alpha)_{max} = \frac{f_s}{2} \sin\alpha \#(19)$$

Q.E.D.

From lemma 2; there are two approaches for the suggested procedure of time-frequency analysis of continuous band-limited signal $x(t)$; both approaches will give the same result.

Approach (1):

Get $x[n] = x(nT_s);\ n = 0,1,2, \dots$ from $x(t)$

where

T_s is the sampling period $\left(T_s = \frac{1}{sampling\ frequency} = \frac{1}{f_s}\right)$

and

$f_s > 2\,f_M$ (The Nyquist rate for conventional FT)

f_M represents the value of the maximum frequency in $x(t)$.

Get $X(t_\alpha)$ by evaluating the FRFT integral numerically for angle α of $x[n]$.

Plot $X(t_\alpha)$ versus t_α from $t_\alpha = 0$ to $t_\alpha = \frac{f_s}{2}\sin\alpha$.

Approach (2):

Get $x[n] = x(nT_s);\ n = 0,1,2, \dots$ from $x(t)$

where

T_s is the sampling period $\left(T_s = \frac{1}{sampling\ frequency} = \frac{1}{f_s}\right)$

and

$f_s > 2\, f_M \csc\alpha$ - the Nyquist rate for FRFT-

f_M represents the value of the maximum frequency in $x(t)$.

Evaluate the FRFT integral numerically at angle α of $x[n]$ to get (t_α) .

Plot $X(t_\alpha)$ versus t_α from $t_\alpha = 0$ to $t_\alpha = \frac{f_s}{2}$.

Corollary 1: A continuous band-limited sinusoidal signal with frequency f_0 sampled using sampling frequency $f_s > 2\, f_0$, then its $X(t_\alpha)$ in the FRFT domain will contain information at values:

$$t_\alpha = (kf_s \pm f_0)\ \sin\alpha\ ; k = 0, 1, 2, \dots \#(20)$$

Corollary 1 concluded from the FT case of the continuous band-limited sinusoidal signal as in Eq. (9) and lemma 2.

4. Simulation Results

In preceding section, it has been proven that $X(t_\alpha)$ is the FRFT of continuous band-limited signal $x(t)$ with aliasing frequencies located at $(lf_s \pm f_0)\, sin(\alpha)\, ; l = 0, 1, 2, \dots$ and the principal alias frequency will be in the region of

$t_\alpha \in \left[0, \frac{f_s}{2} \sin\alpha\right]$.

The following simulation example will consolidate the idea.

Example 1:

Consider a continuous band-limited signal $x(t)$ consisting of three frequency components such that:

$$x(t) = \sin 2f_1\pi t + \sin 2f_2\pi t + \sin 2f_3\pi t\,;\ \ 0\ \mathrm{s} \le t \le 3\mathrm{s}\#(21)$$

with $f_1 = 500\ Hz\ ; f_2 = 1\ kHz\ ; f_3 = 1.5\ kHz.$

The Recursive Adaptive Lobatto Quadrature (RALQ) method [27] is used to evaluate $\left|X\left(t_{\frac{\pi}{6}}\right)\right|$, $\left|X\left(t_{\frac{\pi}{4}}\right)\right|$, $\left|X\left(t_{\frac{\pi}{3}}\right)\right|$, $\left|X\left(t_{\frac{\pi}{2}}\right)\right|$ numerically using sampling rate

$f_s = 2.5 * 1500\ Hz = 3.75\ kHz$.

And $\left|X\left(t_{\frac{\pi}{6}}\right)\right|$, $\left|X\left(t_{\frac{\pi}{4}}\right)\right|$, $\left|X\left(t_{\frac{\pi}{3}}\right)\right|$, $\left|X\left(t_{\frac{\pi}{2}}\right)\right|$ are plotted versus t_α for the following two cases:

Case (1): ($t_\alpha = 0$ to $t_\alpha = \frac{f_s}{2}\, sin(\alpha)$, as in Fig. 2.), there is only three components (principal aliasing component) appeared related to three frequency components f_1, f_2, f_3, for t_α exactly equal to $f_1\ \sin\alpha$, $f_2\ \sin\alpha$, $f_3\ \sin\alpha$;

$\forall\ \alpha = \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$.

Case (2): ($t_\alpha = 0$ to $t_\alpha = \frac{f_s}{2}$, as in Fig. 3.), there exists other aliasing components rather than the three principal aliasing components are appearing for case of $\alpha = \frac{\pi}{6}, \frac{\pi}{4}$ and their location are exactly calculated by Eq. (20), which confirm lemma 2 and corollary 1.

5. Conclusions

This work deals with the sampling theorem of a continuous band-limited signal for FRFT at a specific rotation angle. This study derives the sampling theorem for the FRFT of signals from the FRFT integral definition and the sampling theorem for conventional FT. Two techniques for sampling the signal before determining its FRFT for a particular rotation angle are proven to be effective in simulations.

Conflict of Interest

The authors declare no conflict of interest

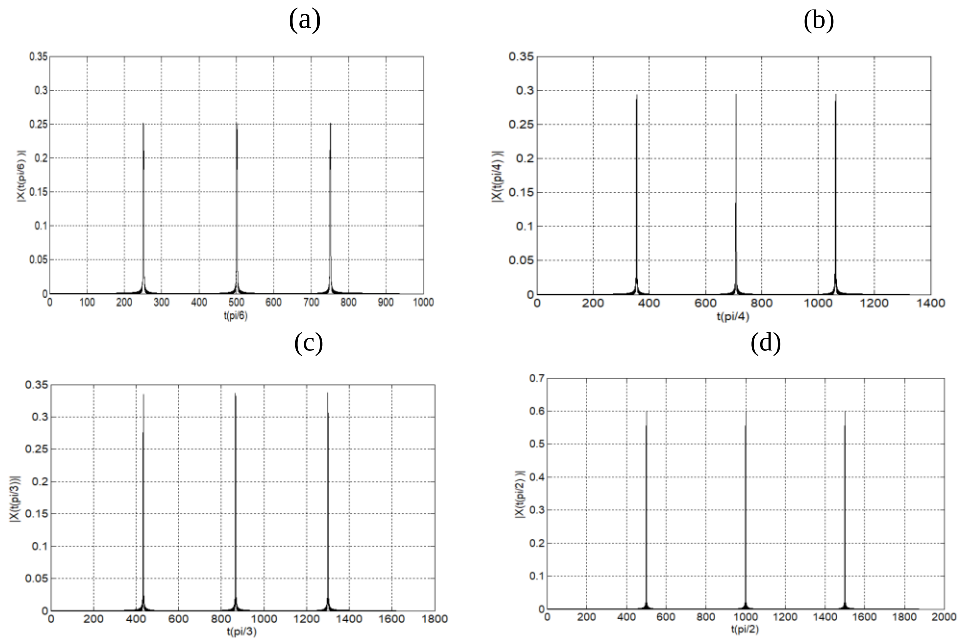

Fig. 2: (a) $\left|X\left(t_{\frac{\pi}{6}}\right)\right|$ for case1; (b) $\left|X\left(t_{\frac{\pi}{4}}\right)\right|$ for case 1;
(c) $\left|X\left(t_{\frac{\pi}{3}}\right)\right|$ for case 1; (d) $\left|X\left(t_{\frac{\pi}{2}}\right)\right|$ for case 1.

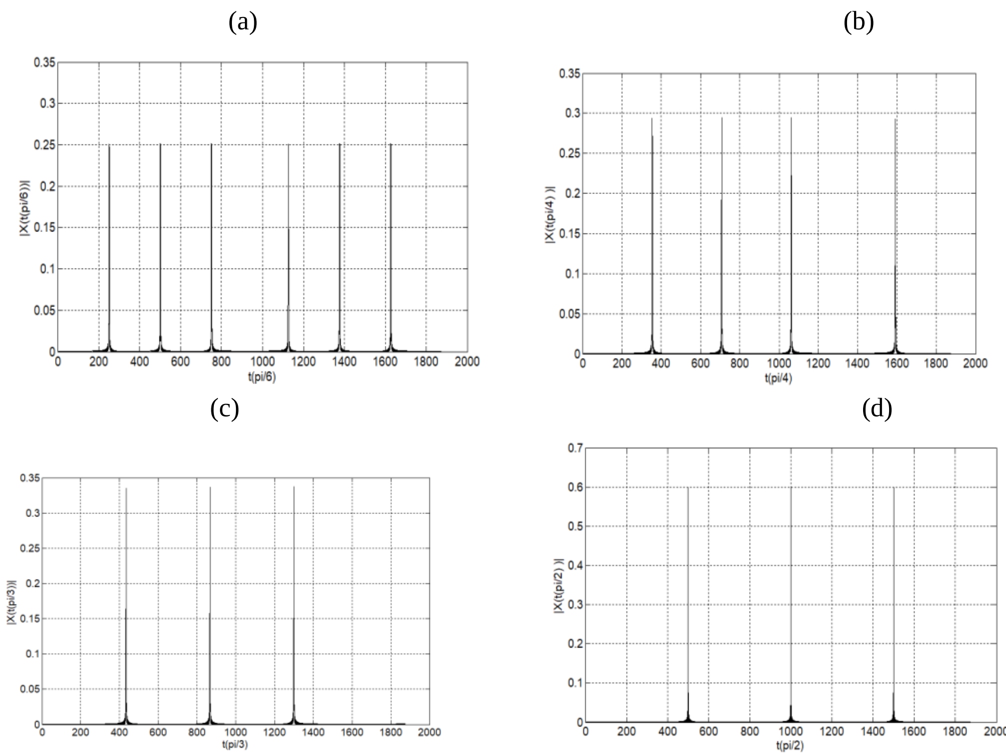

Fig. 3: (a) $\left|X\left(t_{\frac{\pi}{6}}\right)\right|$ for case 2; (b) $\left|X\left(t_{\frac{\pi}{4}}\right)\right|$ for case 2;
(c) $\left|X\left(t_{\frac{\pi}{3}}\right)\right|$ for case 2; (d) $\left|X\left(t_{\frac{\pi}{2}}\right)\right|$ for case 2

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Abbreviation and symbols

FT	**Fourier transform**
FRFT	**Continuous fractional Fourier transform.**
STFT	**Short time Fourier transform.**
T	**Wigner transform**
RALQ	**Recursive adaptive Lobatto quadrature.**