Modeling and simulation hyperbolic 2nd order linear P.D.E using COMSOL Multiphysics

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Abstract– The Partial Differential Equations (PDEs) are very important in dynamics, aerodynamics, elasticity, heat transfer, waves, electromagnetic theory, transmission lines, quantum mechanics, weather forecasting, prediction of disasters, how universe behave ……. Etc., second order linear PDEs can be classified according to the characteristic equation into 3 types coinciding 3 basic conic sections hyperbolic, parabolic and elliptic; Elliptic equations have none family of (real) characteristic curves. All the three types of equations can be reduced to its first canonical form finding the general solution or the second canonical form similar to 3 basic PDE models; Hyperbolic equations have two distinct families of (real) characteristic curves. Hyperbolic type of equations can be reduced to its first canonical form finding the general solution or the second canonical form similar to basic PDE models; Hyperbolic equations reduce to a form coinciding with the wave equation. Thus, the wave equation serves as basic canonical models for all second order hyperbolic linear P.D.E the reduced canonical form can be modeled by initial and boundary condition with COMSOL Multiphysics allowing the analysis of physical phenomena to predict the variance over time for different types of transmission line (RG59, CAT5, PIC, EXL-120, ……) as shown in tables of fig (5,7,8,11) used for different electrical applications data transmission, audio and video transmission, signal transmission…etc..

Keywords-- hyperbolic PDEs – canonical form – constant coefficient PDEs – variable coefficients PDEs – wave equation.

1. Introduction

A PDE is an equation that contains one or more partial derivatives of an unknown function that depends on at least two variables. usually, one of these deals with time t and the remaining with space. PDEs are very important in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics.

The theory of partial differential equations of the second order is more complicated than the equations of the first order, and it is much more typical of the subject as a whole. Within the context, considerably better results can be achieved for equations of the second order in two independent variables than for equations in space of higher dimensions. Linear equations are the easiest to handle. In general, a second order linear partial differential equation is of the form

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y) \quad (1)$$

where A, B, C, D, E, F and G are in general functions of x and y but they may be constants. The subscripts are defined as partial derivatives where $u_x = \frac{\partial u}{\partial x}$

2. Canonical form

The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + f = 0$$

represents hyperbola, parabola, or ellipse accordingly as B^2-4AC is positive, zero, or negative.

Classifications of PDE are:

(i) Hyperbolic if $B^2-4AC > 0$
(ii) Parabolic if $B^2-4AC = 0$
(iii) Elliptic if $B^2-4AC < 0$

The classification of second-order equations is based upon the possibility of reducing equation by coordinate transformation to canonical or standard form at a point. An equation is said to be hyperbolic, parabolic, or elliptic at a point (x_0, y_0) accordingly as;

$$B^2(x_0,y_0)-4A(x_0,y_0)C(x_0,y_0) \quad (2)$$

is positive, zero, or negative. If this is true at all points, then the equation is said to be hyperbolic, parabolic, or elliptic. In the case of two independent variables, a transformation can always be found to reduce the given equation to canonical form in a given domain. However, in the case of several independent variables, it is not, in general, possible to find such a transformation

To transform equation (1) to a canonical form we make a change of independent variables. Let the new variables be;

$\varepsilon = \varepsilon\,(x, y)$, $\eta = \eta\,(x, y)$

Assuming that ε and η are twice continuously differentiable and that the Jacobian;

$$J = \begin{vmatrix} \varepsilon_x & \varepsilon_y \\ \eta_x & \eta_y \end{vmatrix}$$

is nonzero in the region under consideration, then x and y can be determined uniquely. Let x and y be twice continuously differentiable functions of ε and η Then we have,

$u_x = \frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\varepsilon \varepsilon_x + u_\eta \eta_x$

$u_y = \frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\varepsilon \varepsilon_y + u_\eta \eta_y$

$u_{xx} = \frac{\partial u_x}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x} + \frac{\delta u_x}{\delta \eta}\frac{\delta \eta}{\delta x} = u_{\varepsilon\varepsilon}{\varepsilon_x}^2 + 2u_{\varepsilon\eta}\varepsilon_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\varepsilon\varepsilon_{xx} + u_\eta\eta_{xx}$

$u_{yy} = \frac{\partial u_y}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\varepsilon\varepsilon}{\varepsilon_y}^2 + 2u_{\varepsilon\eta}\varepsilon_y\eta_y + u_{\eta\eta}{\eta_y}^2 + u_\varepsilon\varepsilon_{yy} + u_\eta\eta_{yy}$

$u_{xy} = \frac{\partial u_x}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\varepsilon\varepsilon}\varepsilon_x\varepsilon_y + u_{\eta\eta}\eta_x\eta_y + u_\varepsilon\varepsilon_{xy} + u_\eta\eta_{xy} + u_{\varepsilon\eta}(\varepsilon_x\eta_y + \varepsilon_y\eta_x)$

substituting in (1)

$A^*(x,y)u_{xx} + B^*(x,y)u_{xy} + C^*(x,y)u_{yy} + D^*(x,y)u_x + E^*(x,y)u_y + F^*(x,y)u = G^*(x,y)$ (3)

Where;

$A^* = A{\varepsilon_x}^2 + B\varepsilon_x\varepsilon_y + c{\varepsilon_y}^2$

$B^* = 2A\varepsilon_x\eta_x + B(\varepsilon_x\eta_y + \varepsilon_y\eta_x) + 2C\varepsilon_y\eta_y$

$C^* = A{\eta_x}^2 + B\eta_x\eta_y + C{\eta_y}^2$

$D^* = A\varepsilon_{xx} + B\varepsilon_{xy} + C\varepsilon_{yy} + D\varepsilon_x + E\varepsilon_y$

$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$

$F^* = F$, $G^* = G$

The resulting equation (3) is in the same form as the original equation (1) under the general transformation. The nature of the equation remains constant if the Jacobian does not vanish.

${B^*}^2 - 4A^*C^* = J^2(B^2 - 4AC)$ and $J^2 \neq 0$, We shall assume that the equation under consideration is of the single type in a given domain. The classification of equation (1) depends on the coefficients A (x, y), B (x, y), and C (x, y) at a given point (x, y) so equation (1) rewritten as;

$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = H(x,y,u,u_x,u_y)$ (4)

Where; A, B, C≠0

And equation (3) rewritten as;

$A^*(x,y)u_{\varepsilon\varepsilon} + B^*(x,y)u_{\varepsilon\eta} + C^*(x,y)u_{\eta\eta} = H(\varepsilon,\eta,u,u_\varepsilon,u_\eta)$

Where A^*, C^* =0

$A{\varepsilon_x}^2 + B\varepsilon_x\varepsilon_y + c{\varepsilon_y}^2 = 0$

$A{\eta_x}^2 + B\eta_x\eta_y + C{\eta_y}^2 = 0$

Since the 2 equations from the same type, we can rewrite them;

$A{\varepsilon_x}^2 + B\varepsilon_x\varepsilon_y + c{\varepsilon_y}^2 = 0$ where ε stands for the 2 functions ε, η

Dividing by ${\varepsilon_y}^2$ $A(\frac{\varepsilon_x}{\varepsilon_y})^2 + B\frac{\varepsilon_x}{\varepsilon_y} + C = 0$

$\frac{dy}{dx} = -\frac{\varepsilon_x}{\varepsilon_y}$ $A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$

therefore, two roots are $\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$

These equations, which are known as the characteristic equations, are ordinary differential equations for families of curves in the by-plane along which ε = constant and η = constant. The integrals of equation are called the characteristic curves. Since the equations are first order ordinary differential equations, the solutions may be written as;

$\Phi_1(x, y) = c_1$ $\Phi_2(x, y) = c_2$ with c_1 and c_2 as constants.

Hence the transformations

$\varepsilon = \Phi_1(x, y)$, $\eta = \Phi_2(x, y)$

will transform equation (4) to a canonical form.

We show that the characteristic of any elliptical PDE can be transformed as;

$*B^2 - 4AC > 0$ so, we have 2 real different characteristic integration yields reduced into first canonical form $u_{\varepsilon\eta} = H(\varepsilon,\eta,u,u_\varepsilon,u_\eta)$, $B^* \neq 0$ to find general solution.

let we have new independent variable α, β

where $\alpha = \varepsilon + \eta$, $\beta = \varepsilon - \eta$ since ε and η are twice continuously differentiable functions then α, β are the same.

$u_x = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = u_\alpha\alpha_x + u_\beta\beta_x$

$u_y = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial y} = u_\alpha\alpha_y + u_\beta\beta_y$

$u_{xx} = \frac{\partial u_x}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u_x}{\partial \beta}\frac{\partial \beta}{\partial x} = u_{\alpha\alpha}{\alpha_x}^2 + 2u_{\alpha\beta}\alpha_x\beta_x + u_{\beta\beta}{\beta_x}^2 + u_\alpha\alpha_{xx} + u_\beta\beta_{xx}$

$u_{yy} = \frac{\partial u_y}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\alpha\alpha}{\alpha_y}^2 + 2u_{\alpha\eta}\alpha_y\beta_y + u_{\beta\beta}{\beta_y}^2 + u_\alpha\alpha_{yy} + u_\beta\beta_{yy}$

$u_{xy} = \frac{\partial u_x}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial u_x}{\partial \beta}\frac{\partial \beta}{\partial y} = u_{\alpha\alpha}\alpha_x\alpha_y + u_{\beta\beta}\beta_x\beta_y + u_\alpha\alpha_{xy} + u_\beta\beta_{xy} + u_{\alpha\beta}(\alpha_x\beta_y + \alpha_y\beta_x)$

substituting in (1)

$A^*(x,y)u_{xx} + B^*(x,y)u_{xy} + C^*(x,y)u_{yy} + D^*(x,y)u_x + E^*(x,y)u_y + F^*(x,y)u = G^*(x,y)$ (3)

Where;

$A^* = A{\alpha_x}^2 + B\alpha_x\alpha_y + c{\alpha_y}^2$

$B^* = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y$

$C^* = A{\beta_x}^2 + B\beta_x\beta_y + C{\beta_y}^2$

$D^* = A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y$

$E^* = A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y$

$F^* = F$, $G^* = G$

The resulting equation (3) is in the same form as the original equation (1) under the general transformation. The nature of the equation remains constant if the Jacobian does not vanish.

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(x, y) so equation (1) rewritten as;
$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} =$
$H(x, y, u, u_x, u_y)$ (4)
Where; A, B, C≠0
And equation (3) rewritten as;
$A^*(x,y)u_{\alpha\alpha} + B^*(x,y)u_{\alpha\beta} + C^*(x,y)u_{\beta\beta} =$
$H(\varepsilon, \eta, u, u_\alpha, u_\beta)$ where $B^*(x,y)u_{\varepsilon\eta} = 0$
$u_{\varepsilon\eta} \neq 0 \quad so \quad B^* = 0 \quad A^* = -C^*$
$B^* = 2A\varepsilon_x\eta_x + B(\varepsilon_x\eta_y + \varepsilon_y\eta_x) + 2C\varepsilon_y\eta_y = 0$
which is transformed into second canonical form $u_{\alpha\alpha} - u_{\beta\beta} = H(\alpha, \beta, u, u_\alpha, u_\beta)$
similar to wave equation to be modeled.

3. Hyperbolic equations

As we can see the coefficient form P.D.E mainly depend on the second canonical form so in order to model and simulate the P.D.E we need to reduce the equation to its second canonical form the plugging the coefficient as shown blow

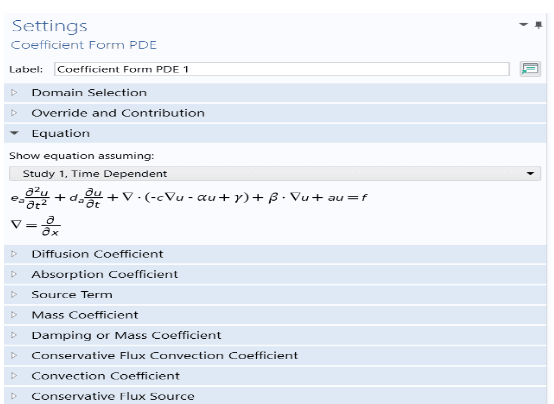

Fig 1. COMSOL interface.

3.1. Fundamental wave equation

$u_{tt} - c^2u_{xx} = 0 \ , \ B^2 - 4AC = 4c^2 > 0$
$\frac{dt}{dx} = \frac{\pm 1}{c}$ separation of variables and integrate we get
$\varepsilon = x + ct \ , \ \eta = x - ct$
$u_{xx} = u_{\varepsilon\varepsilon}{\varepsilon_x}^2 + 2u_{\varepsilon\eta}\varepsilon_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\varepsilon\varepsilon_{xx} + u_\eta\eta_{xx}$
$u_{tt} = u_{\varepsilon\varepsilon}{\varepsilon_t}^2 + 2u_{\varepsilon\eta}\varepsilon_t\eta_t + u_{\eta\eta}{\eta_t}^2 + u_\varepsilon\varepsilon_{tt} + u_\eta\eta_{tt}$
Then substitute in original P.D.E
$-4c^2u_{\varepsilon\eta} = 0 \ , \ c \neq 0$
$u_{\varepsilon\eta} = 0 \ \ then\ integrate\ w.r.t\ \eta$
$u_\varepsilon = f(\varepsilon) \quad then\ integrate\ w.r.t\ \varepsilon$
$u = f(\varepsilon) + g(\eta) = f(x + ct) + g(x - ct)$
Using initial and boundary conditions
$f(x + ct)\ , \ g(x - ct)$ can be determined
$u_{(x,0)} = p(x) \quad , \quad u_{t_{(x,0)}} = V(x)$

3.2. Variable coefficient equation

$x^2u_{xx} - y^2u_{yy} - 2yu_y = 0$
$B^2 - 4AC = 4x^2y^2 > 0 \qquad x, y \neq 0$
$A\lambda^2 - B\lambda + c = 0 \ , \ \frac{dy}{dx} = \frac{\pm y}{x}$
$\frac{dy}{dx} = \frac{y}{x} \quad , \ \frac{dy}{dx} = -\frac{y}{x}$
separation of variables and integrate we get
$\varepsilon = \frac{y}{x} \quad , \qquad \eta = xy$
$\varepsilon_x = -\frac{y}{x^2}\ , \varepsilon_{xx} = \frac{2y}{x^3}\ , \ \varepsilon_{xy} = -\frac{1}{x^2}\ , \varepsilon_y = \frac{1}{x}\ , \varepsilon_{yy} = 0$
$\eta_x = y \ , \ \eta_{xx} = 0\ , \ \eta_{xy} = 1\ , \ \eta_y = x \ , \ \eta_{yy} = 0$
$u_y = \frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{u_\varepsilon}{x} + yu_\eta$
$u_{xx} = u_{\varepsilon\varepsilon}{\varepsilon_x}^2 + 2u_{\varepsilon\eta}\varepsilon_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\varepsilon\varepsilon_{xx} + \cdots$
$\ldots + u_\eta\eta_{xx}$
$u_{xx} = \frac{y^2}{x^4}u_{\varepsilon\varepsilon} - \frac{2y^2}{x^2}u_{\varepsilon\eta} + y^2u_{\eta\eta} + \frac{2y}{x^3}u_\varepsilon$
$u_{yy} = u_{\varepsilon\varepsilon}{\varepsilon_y}^2 + 2u_{\varepsilon\eta}\varepsilon_y\eta_y + u_{\eta\eta}{\eta_y}^2 + u_\varepsilon\varepsilon_{yy} + \cdots$
$\ldots + u_\eta\eta_{yy}$
$u_{yy} = \frac{1}{x^2}u_{\varepsilon\varepsilon} + 2u_{\varepsilon\eta} + x^2u_{\eta\eta}$
$x^2u_{xx} - y^2u_{yy} - 2yu_y = 4y^2u_{\varepsilon\eta} + 2y^2u_\eta = 0$
$y \neq 0\ , \ \ 2u_{\varepsilon\eta} + u_\eta = 0$
The first canonical form is $2u_{\varepsilon\eta} + u_\eta = 0$
$u_\eta = v\ , \ 2v_\varepsilon + v = 0\ , \frac{\partial v}{v} = \frac{\partial \varepsilon}{-2}$
By integration we get $v = e^{\frac{-\varepsilon}{2}} + f(\eta)$
$\partial u = \left(e^{\frac{-\varepsilon}{2}} + f(\eta)\right)\partial\eta$
By integration we get $u = e^{\frac{-\varepsilon}{2}}\eta + g(\eta) + f(\varepsilon)$
$u = xye^{\frac{-y}{2x}} + g(xy) + f\left(\frac{y}{x}\right)$
Using initial and boundary conditions
$f\left(\frac{y}{x}\right) \ , \ g(xy)$ can be determined
$u_{(x,0)} = p(x) \quad , \quad u_{t_{(x,0)}} = V(x)$
Apply second canonical form where;
$\alpha = \varepsilon + \eta = \frac{y}{x} + xy \quad , \ \beta = \varepsilon - \eta = \frac{y}{x} - xy$
$\alpha_x = \frac{-y}{x^2} + y \ , \ \alpha_{xx} = \frac{2y}{x^3} \quad , \ \alpha_y = \frac{1}{x} + x$
$\alpha_{yy} = 0 \ , \ \alpha_{xy} = -\frac{1}{x^2} + 1$
$\beta_x = \frac{-y}{x^2} - y \ , \ \beta_{xx} = \frac{2y}{x^3} \ , \ \beta_y = \frac{1}{x} - x$
$\beta_{yy} = 0\ , \ \beta_{xy} = -\frac{1}{x^2} - 1$
$u_{xx} = u_{\alpha\alpha}{\alpha_x}^2 + 2u_{\alpha\beta}\alpha_x\beta_x + u_{\beta\beta}{\beta_x}^2 + u_\alpha\alpha_{xx} + u_\beta\beta_{xx}$
$x^2u_{xx} = y^2\left(x^2 + \frac{1}{x^2} - 2\right)u_{\alpha\alpha} - 2y^2\left(x^2 - \frac{1}{x^2}\right)u_{\alpha\beta}.$
$\ldots\ldots + y^2\left(x^2 + \frac{1}{x^2} + 2\right)u_{\beta\beta} + \frac{2y}{x}(u_\alpha + u_\beta)$
$u_{yy} = u_{\alpha\alpha}{\alpha_y}^2 + 2u_{\alpha\beta}\alpha_y\beta_y + u_{\beta\beta}{\beta_y}^2 + u_\alpha\alpha_{yy} + u_\beta\beta_{yy}$
$-y^2u_{yy} = -y^2\left(x^2 + \frac{1}{x^2} + 2\right)u_{\alpha\alpha} + \cdots$
$\ldots - 2y^2\left(\frac{1}{x^2} - x^2\right)u_{\alpha\beta} - y^2\left(x^2 + \frac{1}{x^2} - 2\right)u_{\beta\beta}{\beta_y}^2$

$u_y = u_\alpha \alpha_y + u_\beta \beta_y$

$-2yu_y = -2y\left(x + \frac{1}{x}\right)u_\alpha - 2y\left(\frac{1}{x} - x\right)u_\beta$

$x^2u_{xx} - y^2u_{yy} - 2yu_y = u_{\alpha\alpha} - u_{\beta\beta} - \dfrac{(u_\beta - u_\alpha)}{\alpha + \beta}$

$u_{\alpha\alpha} - u_{\beta\beta} = \frac{1}{\alpha+\beta}(u_\beta - u_\alpha)$ which is similar to wave equation that can be modeled.

3.3. *Constant coefficient equation*

$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = 2$

$B^2 - 4AC = 9 > 0$, $4\lambda^2 - 5\lambda + 1 = 0$

$\frac{dy}{dx} = 1$, $\frac{dy}{dx} = \frac{1}{4}$

separation of variables and integrate we get

$\varepsilon = y - x$, $\eta = 4y - x$

$\varepsilon_x = -1$, $\varepsilon_{xx} = 0$, $\varepsilon_{xy} = 0$, $\varepsilon_y = 1$, $\varepsilon_{yy} = 0$

$\eta_x = -1$, $\eta_{xx} = 0$, $\eta_{xy} = 0$, $\eta_y = 4$, $\eta_{yy} = 0$

$u_x = \frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = -u_\varepsilon - u_\eta$

$u_y = \frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\varepsilon + 4u_\eta$

$u_{xx} = u_{\varepsilon\varepsilon}{\varepsilon_x}^2 + 2u_{\varepsilon\eta}\varepsilon_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\varepsilon\varepsilon_{xx} + u_\eta\eta_{xx}$

$u_{xx} = u_{\varepsilon\varepsilon} + 2u_{\varepsilon\eta} + u_{\eta\eta}$

$u_{yy} = u_{\varepsilon\varepsilon}{\varepsilon_y}^2 + 2u_{\varepsilon\eta}\varepsilon_y\eta_y + u_{\eta\eta}{\eta_y}^2 + u_\varepsilon\varepsilon_{yy} + u_\eta\eta_{yy}$

$u_{yy} = u_{\varepsilon\varepsilon} + 8u_{\varepsilon\eta} + 16u_{\eta\eta}$

$u_{xy} = u_{\varepsilon\varepsilon}\varepsilon_x\varepsilon_y + u_{\eta\eta}\eta_x\eta_y + u_\varepsilon\varepsilon_{xy} + u_\eta\eta_{xy} + u_{\varepsilon\eta}(\varepsilon_x\eta_y + \varepsilon_y\eta_x)$

$u_{xy} = -u_{\varepsilon\varepsilon} - 4u_{\eta\eta} - 5u_{\varepsilon\eta}$

$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y = -9u_{\varepsilon\eta} + 3u_\eta = 2$

The first canonical form is $u_{\varepsilon\eta} = u_{\eta\varepsilon} = -\frac{2}{9} + \frac{u_\eta}{3}$

put $u_\eta = v$

$v_\varepsilon = -\frac{2}{9} + \frac{v}{3}$ *the integrating factor* $e^{\int -\frac{1}{3}d\varepsilon}$

$ve^{-\frac{\varepsilon}{3}} = \int -\frac{2}{9}e^{-\frac{\varepsilon}{3}}d\varepsilon$, $ve^{-\frac{\varepsilon}{3}} = \frac{2}{3}e^{-\frac{\varepsilon}{3}} + f(\eta)$

$v = \frac{2}{3} + f(\eta)\, e^{\frac{\varepsilon}{3}}$

$u_\eta = \frac{2}{3} + f(\eta)\, e^{\frac{\varepsilon}{3}}$

Another ODE can be integrated w.r.t η

$u = \frac{2\eta}{3} + g(\eta)\, e^{\frac{\varepsilon}{3}} + f(\varepsilon)$

$u = \frac{2}{3}(4y - x) + g(4y - x)e^{\frac{y-x}{3}} + f(y - x)$

Using initial and boundary conditions

$f(y - x)$, $g(4y - x)$ can be determined

$u_{(x,0)} = p(x)$, $u_{t_{(x,0)}} = V(x)$

Apply second canonical form where;

$\alpha = \varepsilon + \eta = 5y - 2x$, $\beta = \varepsilon - \eta = -3y$

$\alpha_x = -2$, $\alpha_y = 5$, $\alpha_{yy} = \alpha_{xx} = \alpha_{xy} = 0$

$\beta_x = \beta_{xx} = \beta_{yy} = \beta_{xy} = 0$, $\beta_y = -3$

$u_{xx} = u_{\alpha\alpha}{\alpha_x}^2 + 2u_{\alpha\eta}\alpha_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\alpha\alpha_{xx} + \cdots$

$\ldots + u_\eta\eta_{xx}$

$4u_{xx} = 16u_{\alpha\alpha}$

$u_{yy} = u_{\alpha\alpha}{\alpha_y}^2 + 2u_{\alpha\eta}\alpha_y\eta_y + u_{\eta\eta}{\eta_y}^2 + u_\alpha\alpha_{yy} + \cdots$

$\ldots + u_\eta\eta_{yy}$

$u_{yy} = 25u_{\alpha\alpha} - 30u_{\alpha\beta} + 9u_{\beta\beta}$

$u_{xy} = u_{\alpha\alpha}\alpha_x\alpha_y + u_{\eta\eta}\eta_x\eta_y + u_\alpha\alpha_{xy} + u_\eta\eta_{xy} + \cdots$

$\ldots + u_{\alpha\eta}(\alpha_x\eta_y + \alpha_y\eta_x)$

$5u_{xy} = -50u_{\alpha\alpha} + 30u_{\alpha\beta}$

$u_y = u_\alpha\alpha_y + u_\beta\beta_y$

$u_y = 5u_\alpha - 3u_\beta$

$u_x = u_\alpha\alpha_x + u_\beta\beta_x$

$u_x = -2u_\alpha$

$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y - 2 = u_{\alpha\alpha} - u_{\beta\beta} + \ldots$

$\ldots - \frac{1}{3}(u_\alpha - u_\beta) - \frac{2}{9}$ which is similar to wave equation that can be modeled.

4. Modelling using COMSOL

1- wave equation $u_{tt} - c^2u_{xx} = 0$ put $c = 1$

$u_{tt} - u_{xx} = 0$ as shown in fig.2

Initial conditions

$u_{(x,0)} = \sin(4\pi x)$, $u_{t_{(x,0)}} = 0$

Boundary conditions

$u_{(0,t)} = u_{(l,t)} = 0$, $t > 0$

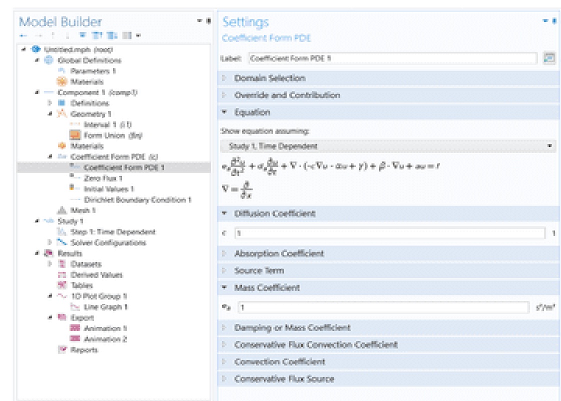

Fig. 2.a. COMSOL coefficient interface.

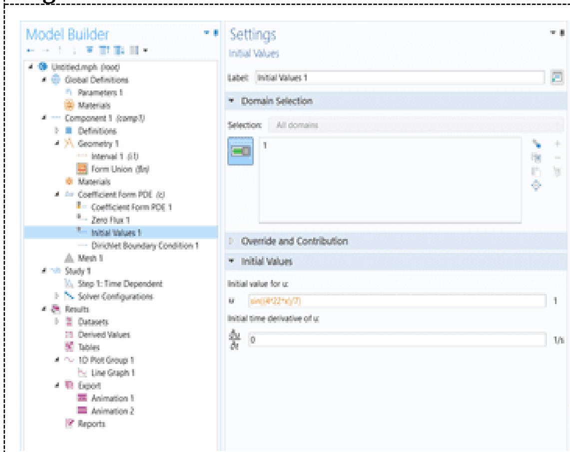

fig 2.b. COMSOL initial condition interface.

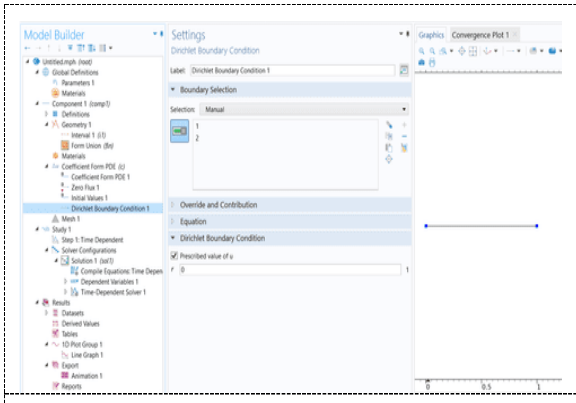

Fig 2.c. COMSOL boundary condition interface.

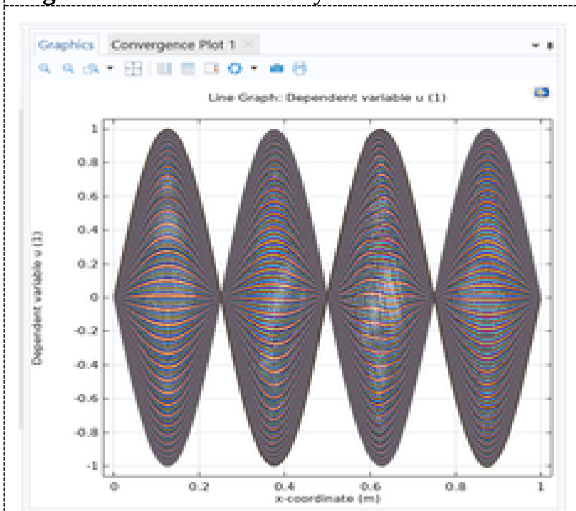

Fig 2.d. *2D Real animated plot for u and x.*

2- constant coefficient hyperbolic equation

$$4u_{xx} + 5u_{xy} + u_{yy} + u_x + u_y - 2 = 0$$

$u_{\alpha\alpha} - u_{\beta\beta} - \frac{1}{3}\left(u_\alpha - u_\beta\right) = -\frac{2}{9}$ as shown in fig.3

Initial conditions

$$u_{(\beta,0)} = \sin(4\pi\beta) \quad , \quad u_{t(\beta,0)} = 0$$

Boundary conditions

$$u_{(0,t)} = u_{(l,t)} = 0 \quad , \quad t > 0$$

Let $\alpha = t\ ,\ \beta = x$

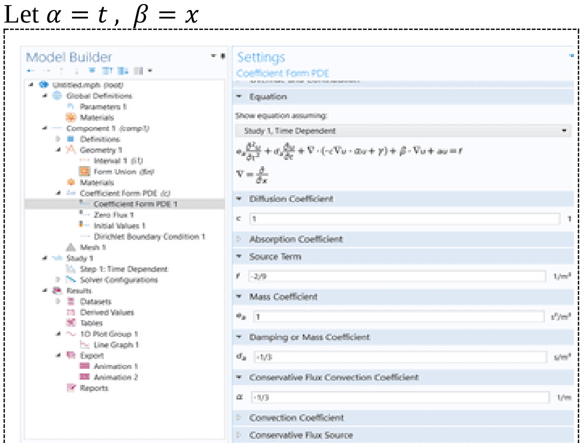

Fig 3.a. COMSOL coefficient interface.

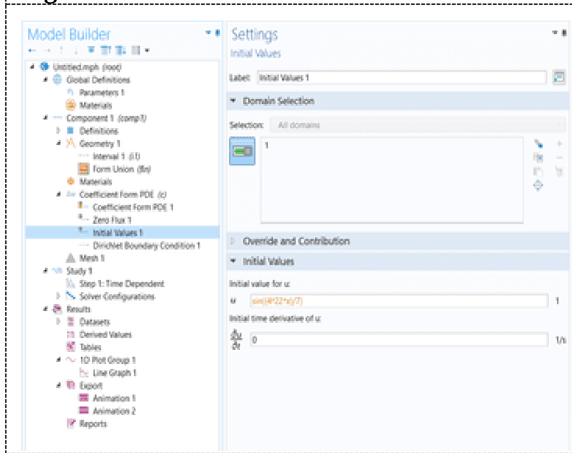

Fig 3.b. COMSOL initial condition interface.

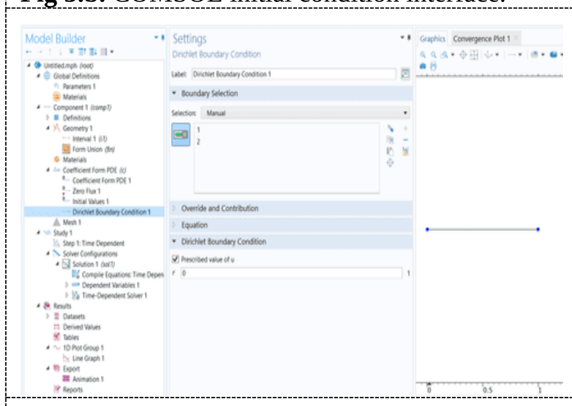

Fig 3.c. COMSOL boundary condition interface.

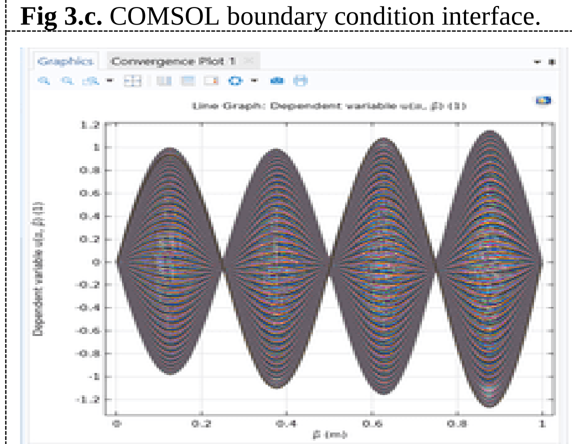

Fig 3.d. *2D Real animated plot for u and β.*

3- variable coefficient hyperbolic equation

$$x^2u_{xx} - y^2u_{yy} - 2yu_y = 0$$

$$u_{\alpha\alpha} - u_{\beta\beta} = \frac{(u_\beta - u_\alpha)}{\alpha+\beta}$$

$u_{\alpha\alpha} - u_{\beta\beta} + \frac{1}{\alpha+\beta}\left(u_\alpha - u_\beta\right) = 0$ as shown in fig.4

Initial conditions

$$u_{(\beta,0)} = \sin(4\pi\beta) \quad , \quad u_{t(\beta,0)} = 0$$

Boundary conditions

$u_{(0,t)} = u_{(l,t)} = 0 \quad , \quad t > 0$

Let $\alpha = t \; , \; \beta = x$

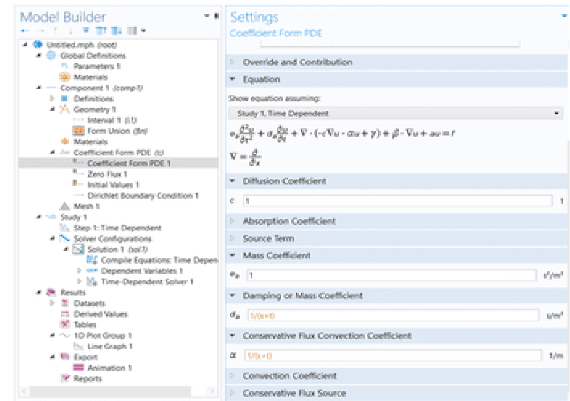

Fig 4.a. COMSOL coefficient interface.

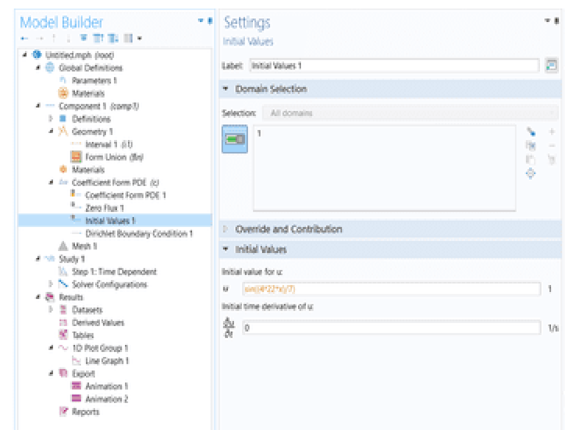

Fig 4.b. COMSOL initial condition interface.

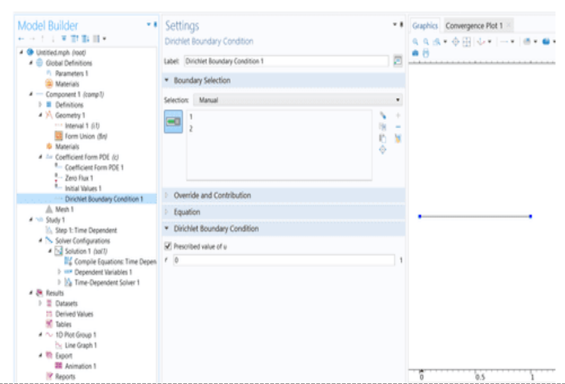

Fig 4.c. COMSOL boundary condition interface.

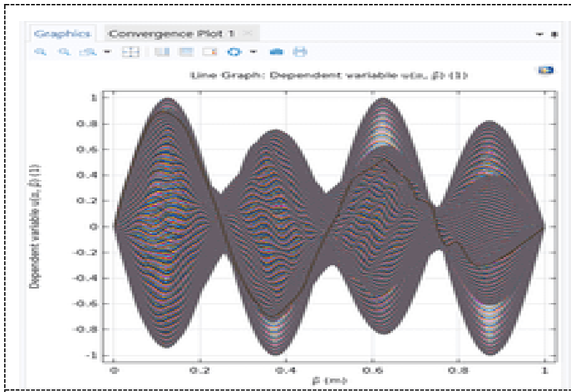

Fig 4.d. *2D Real animated plot for u and β.*

5. Physical applications

1-Motion of stretched string in musical instruments such as guitar, piano described by

$u_{tt} - c^2 u_{xx} = 0$

where $c^2 = \frac{T}{\mu}$ T horizontal component of tension force, μ mass per unit length Suppose a such string placed on x-axis

I. Damping forces are neglected such as air resistance
II. Weight of string is also neglected
III. Tension force is tangential to string curve

Initial position function

$u_{(x,0)} = p(x) = \sin\left(\frac{nx\pi}{l}\right)$

l *is length of string*

Initial velocity function

$u_{t_{(x,0)}} = V(x) = 0$ (Initially at rest)

Boundaries $u_{(0,t)} = u_{(l,t)} = 0 \quad , \quad t > 0$

D'Addario EXL-120 manufacturer specs

String no.	Thickness [in.] (d)	Recommended tension [lbs.] (T)	ρ [g/cm^3]
1	0.00899	13.1	7.726 (steel alloy)
2	0.0110	11.0	"
3	0.0160	14.7	"
4	0.0241	15.8	6.533 (nickel-wound steel alloy)
5	0.0322	15.8	"
6	0.0416	14.8	"

Fig 5. table of electrical string specs.

Where $c^2 = \frac{T}{\mu} = \frac{4T}{\pi d^2 \rho}$ from the table of different electrical guitar strings we may form many equations with the same boundary

Let *length* $l = 1m \; , \; n = 4 \, , \; c^2 = 4$ as shown in fig.6

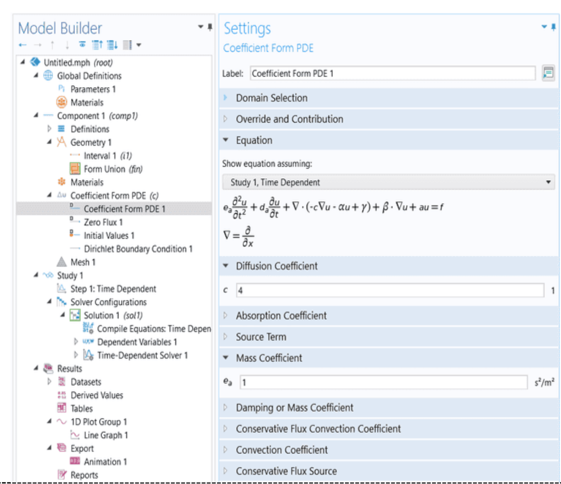

Fig 6.a. COMSOL coefficient interface.

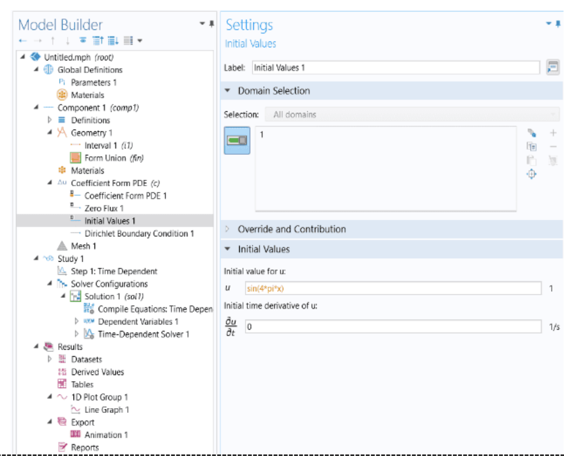

Fig 6.b. COMSOL initial condition interface.

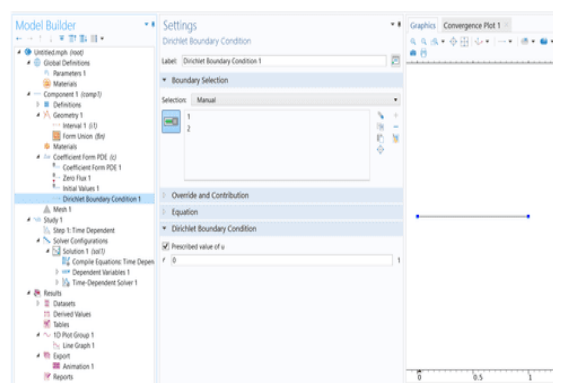

Fig 6.c. COMSOL boundary condition interface.

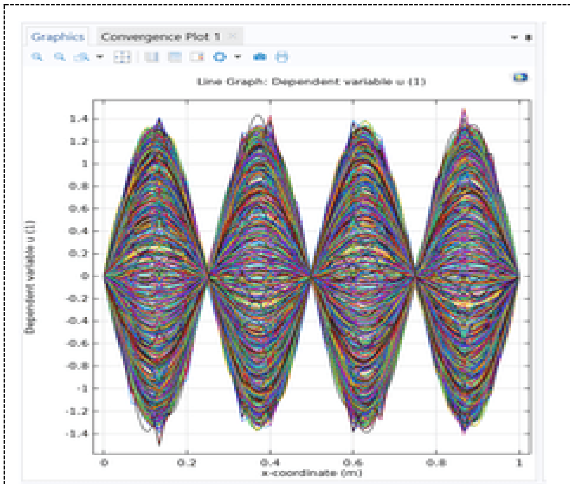

Fig 6.d. *2D Real animated plot for u and* β.

2- longitudinal waves travelling along thin Rod with Youngs Y modulus and mass density ρ where the constant $c^2 = \frac{Y}{\rho}$ is phase velocity where c is specific for each material same as before inserting the coefficient, initial conditions and boundary condition

Table: Calculated and measured longitudinal wave speeds in thin rods made up of common metals. Sources: Haynes and Lide 2011c, Wikipedia contributors 2012.

Metal	$Y\,(\mathrm{N\,m^{-2}})$	$\rho\,(\mathrm{kg\,m^{-3}})$	$\sqrt{Y/\rho}\,(\mathrm{m\,s^{-1}})$	$v\,(\mathrm{m\,s^{-1}})$
Aluminium	7.0×10^{10}	2.7×10^3	5100	5000
Copper	1.2×10^{11}	8.9×10^3	3600	3800
Lead	1.6×10^{10}	1.1×10^4	1100	1100
Nickel	2.0×10^{11}	8.9×10^3	4700	4900
Silver	8.3×10^{10}	1.1×10^4	2800	2700
Tin	5.0×10^{10}	7.4×10^3	2600	2700
Zinc	1.1×10^{11}	7.1×10^3	3900	3900

Fig 7. table of longitudinal wave specs in thin rods of different metals.

3-high frequency AC lossless cable (optical fiber, submarine cable, transmission lines) where;
the cable is made such that resistance R and leakage of conductance G is also neglected as $\omega L \gg R$, $\omega C \gg G$

the general telegraph equation

$i_{xx} = LCi_{tt} + (RC + GL)i_t + RGi$, $R = G = 0$, L inductance, C capacitance, R resistance

$i_{tt} - \frac{1}{LC}\, i_{xx} = 0$ high freq. AC similar to wave equation

$$i(x,t) = f\left(x + \frac{t}{\sqrt{LC}}\right) + g\left(x - \frac{t}{\sqrt{LC}}\right)$$

$$V(x,t) = f\left(x + \frac{t}{\sqrt{LC}}\right) + g\left(x - \frac{t}{\sqrt{LC}}\right)$$

Designation	Cable form	Application	R	$L^{\dagger}$	G	C	Z_0
			Ω/km	μH/km	nS/km	nF/km	Ω
CAT5[5]	Twisted pair	Data transmission	176	490	<2	49	100
CAT5e[6]	Twisted pair	Data transmission	176		<2		100
CW1308[7]	Twisted pair	Telephony	98		<20		
RG59[8]	Coaxial	Video	36	430		69	75
RG59[9]	Coaxial (foam dielectric)	Video	17	303		54	75
RG58[10][11]	Coaxial	Radio frequency	48	253	<0.01	101	50
Low loss[12]	Coaxial (Foam dielectric)	Radio frequency transmitter feed	2.86	188		75	50
DIN VDE 0816[13]	Star quad	Telephony (trunk lines)	31.8		<0.1	35	

Fig 8. table of longitudinal wave specs in thin rods of different metals.

For example, RG59 coaxial cable in our home for tv operating at frequency 3GHZ we note that $\omega L = 8105309 \frac{H.HZ}{km} \gg 36 \frac{\Omega}{km}$

$8105309 \gg 36$, $\omega L \gg R$ so, R and G are neglected let length of cable $1km$ so $L = 430\ \mu H$, $C = 69\ nF$

$$i_{tt} - \frac{1}{LC} i_{xx} = i_{tt} - \frac{1}{(430*0.069)*10^{-6}} i_{xx} = i_{tt} + \cdots$$

$\ldots - 33704\ i_{xx} = 0$ as shown in fig.9

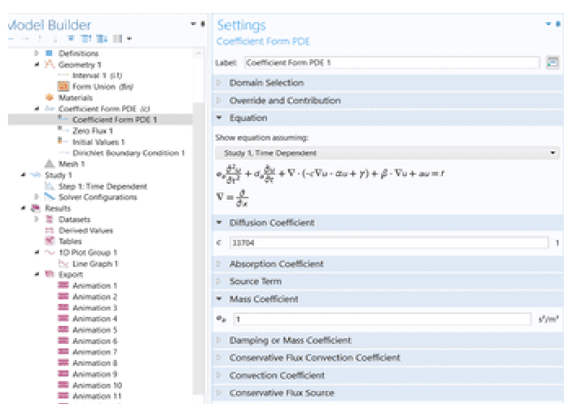

Fig 9.a. COMSOL coefficient interface.

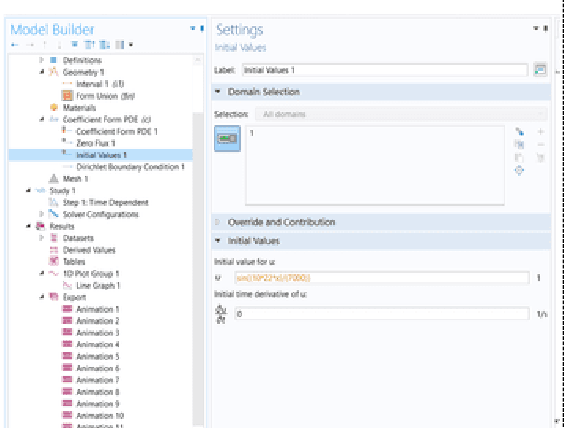

Fig 9.b. COMSOL initial condition interface.

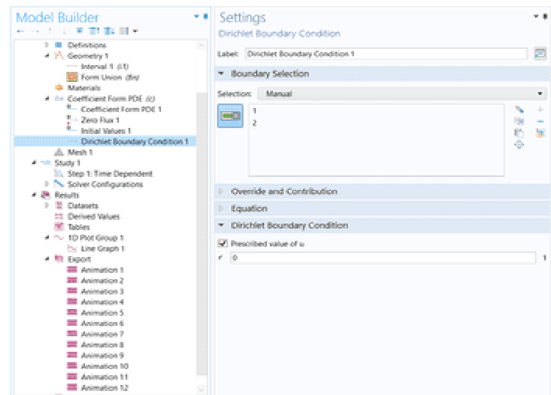

Fig 9.c. COMSOL boundary condition interface.

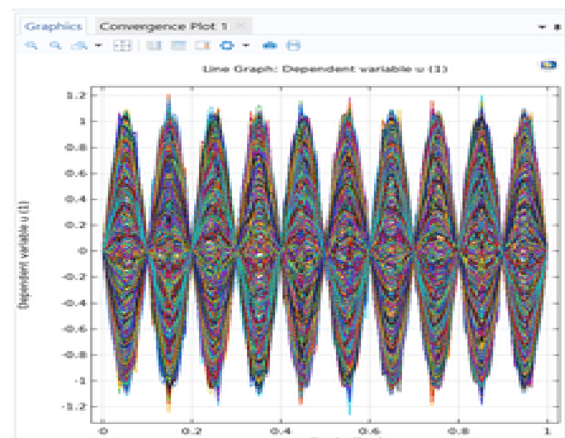

Fig 9.d. *2D Real animated plot for u and x*.

For another example, CAT5 twisted pair cable in data transmission for different network OSI model (CCNA) operating at frequency 100MHZ we note that $\omega L = 307876.1 \frac{H.HZ}{km} \gg 176 \frac{\Omega}{km}$

$307876.1 \gg 176$, $\omega L \gg R$

$\omega C = 30.79 \frac{F.HZ}{km} \gg 2 * 10^{-9} \frac{S}{km}$

$30.79 \gg 2 * 10^{-9}$, $\omega C \gg G$

so, R and G are neglected let length of cable $1km$ so $L = 490\ \mu H$, $C = 49\ nF$

$$i_{tt} - \frac{1}{LC} i_{xx} = i_{tt} - \frac{1}{(490*0.049)*10^{-6}} i_{xx} = i_{tt} + \cdots$$

$\ldots - 41649.33\ i_{xx} = 0$ as shown in fig.10

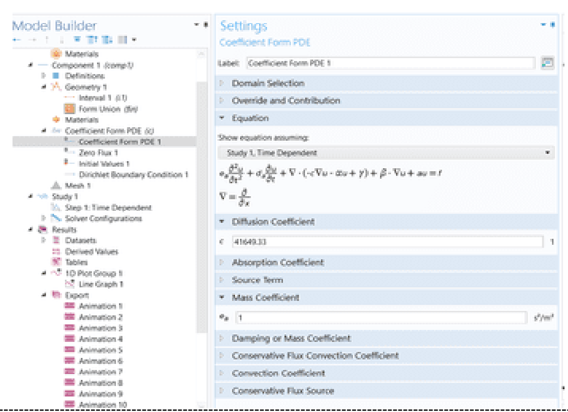

Fig 10.a. COMSOL coefficient interface.

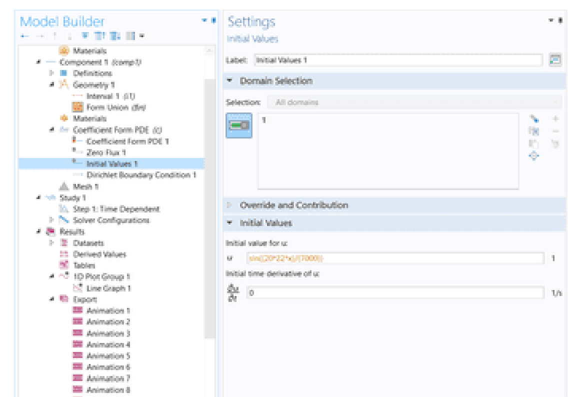

Fig 10.b. COMSOL initial condition interface.

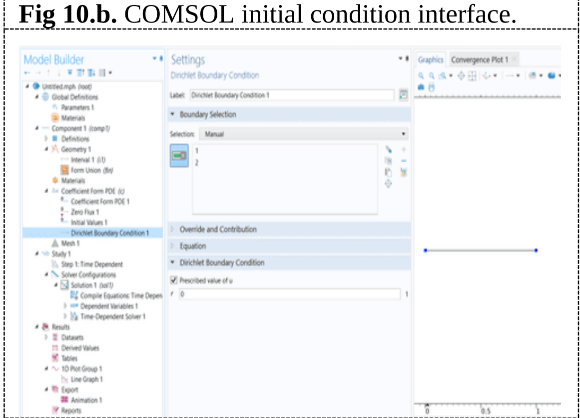

Fig 10.c. COMSOL boundary condition interface.

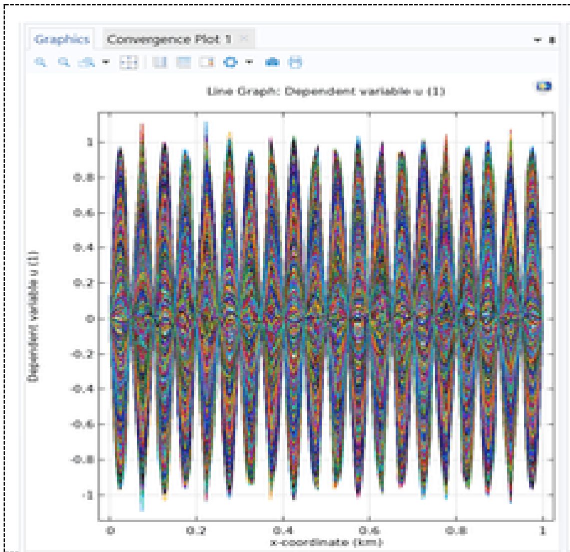

Fig 10.d. *2D Real animated plot for u and x*.

4- The telegrapher's equations are a pair of linear differential equations which describe the voltage and current on an electrical transmission line with distance and time.

the general telegraph equation

$i_{xx} = LCi_{tt} + (RC + GL)i_t + RGi$, $R = G = 0$, L inductance, C capacitance, R resistance

$\frac{1}{LC} i_{xx} = i_{tt} + \frac{(RC+GL)}{LC} i_t + \frac{RG}{LC} i$

$c^2 i_{xx} = i_{tt} + a^* i_t + b^* i$

Put $u = e^{\frac{a^* t}{2}} i$

$c^2 u_{xx} = u_{tt} + \left(b^* - \frac{{a^*}^2}{4}\right) u$ apply first canonical form to find general solution where;

$\varepsilon = x + ct$, $\eta = x - ct$

$$u_{\varepsilon\eta} + \frac{{a^*}^2 - 4b^*}{16c^2} u = 0$$

Apply second canonical form where;

$\alpha = \varepsilon + \eta = 2x$, $\beta = \varepsilon - \eta = 2ct$

$c^2 u_{xx} - u_{tt} = \left(b^* - \frac{{a^*}^2}{4}\right) u$

$u_{\alpha\alpha} - u_{\beta\beta} = \left(\frac{4b^* - {a^*}^2}{16c^2}\right) u$

Representative parameter data for 24-gauge telephone polyethylene insulated cable (PIC) at 70 °F (294 K)

Frequency	*R*		*L*		*G*		*C*	
Hz	Ω/km	Ω/1000 ft	μH/km	μH/1000 ft	μS/km	μS/1000 ft	nF/km	nF/1000 ft
1 Hz	172.24	52.50	612.9	186.8	0.000	0.000	51.57	15.72
1 kHz	172.28	52.51	612.5	186.7	0.072	0.022	51.57	15.72
10 kHz	172.70	52.64	609.9	185.9	0.531	0.162	51.57	15.72
100 kHz	191.63	58.41	580.7	177.0	3.327	1.197	51.57	15.72
1 MHz	463.59	141.30	506.2	154.3	29.111	8.873	51.57	15.72
2 MHz	643.14	196.03	486.2	148.2	53.205	16.217	51.57	15.72
5 MHz	999.41	304.62	467.5	142.5	118.074	35.989	51.57	15.72

Fig 11. table of longitudinal wave specs in thin rods of different metals.

Operating maximum frequency 5 MHZ let length of cable 1Km.

$R = 999.41\Omega\,, L = 467.5\mu H\,, C = 51.57nF$

$G = 118.074\mu S$

Note that the previous condition is not satisfied $\omega L \gg R\,, \omega C \gg G$ so, we cannot neglect R, G to reduce to wave equation thus the use of telegraph equation is a must and more general.

$$\frac{1}{LC} i_{xx} = i_{tt} + \frac{(RC+GL)}{LC} i_t + \frac{RG}{LC} i$$

$$4.146 * 10^{10}\, i_{xx} = i_{tt} + 2.14 * 10^{6} i_t + \cdots$$

$\ldots+4.8946 * 10^{9} i$ as shown in fig. 12

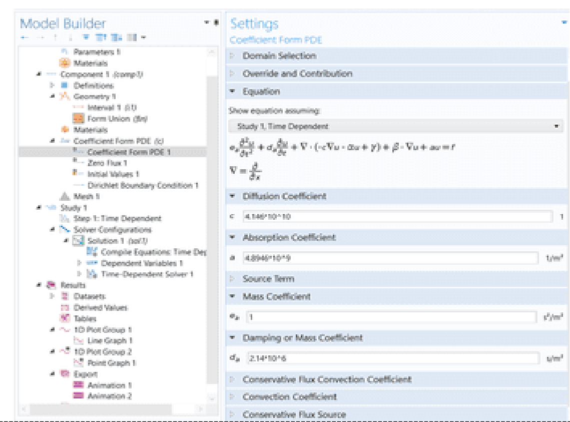

Fig 12.a. COMSOL coefficient interface.

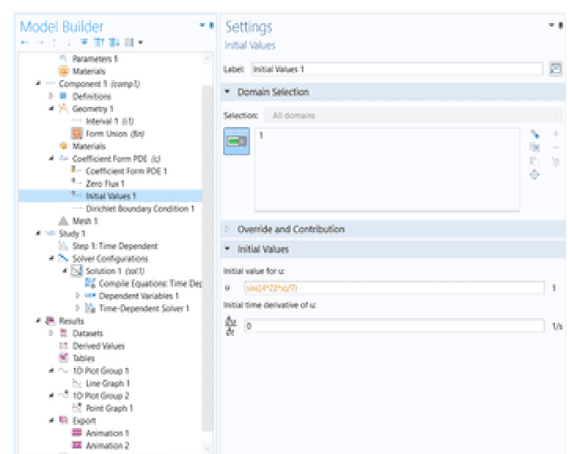

Fig 12.b. COMSOL initial condition interface.

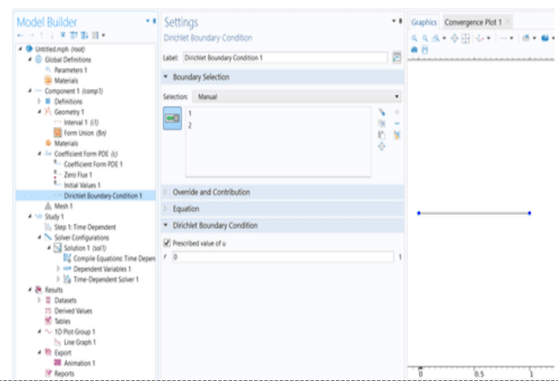

Fig 12.c. COMSOL boundary condition interface.

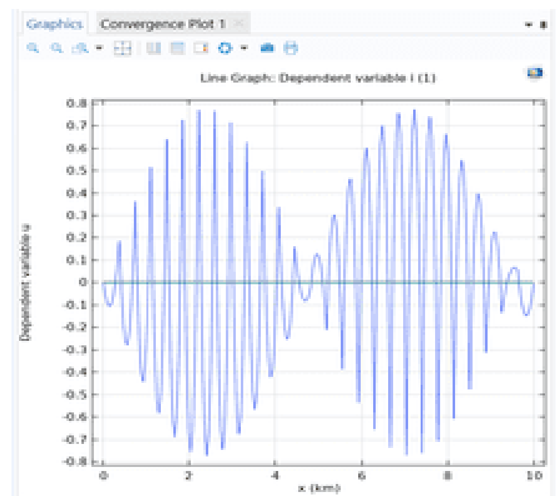

Fig 12.d. *2D Real animated plot for u and x*.

$$c^2 i_{xx} = i_{tt} + a^* i_t + b^* i$$

Put $u = e^{\frac{a^* t}{2}}\, i$

$$c^2 u_{xx} = u_{tt} + \left(b^* - \frac{{a^*}^2}{4}\right) u$$

Apply second canonical form where;

$$u_{\alpha\alpha} - u_{\beta\beta} = \left(\frac{4b^* - {a^*}^2}{16c^2}\right) u$$

$u_{\alpha\alpha} - u_{\beta\beta} = -6.8786\, u$ as shown in fig. 13

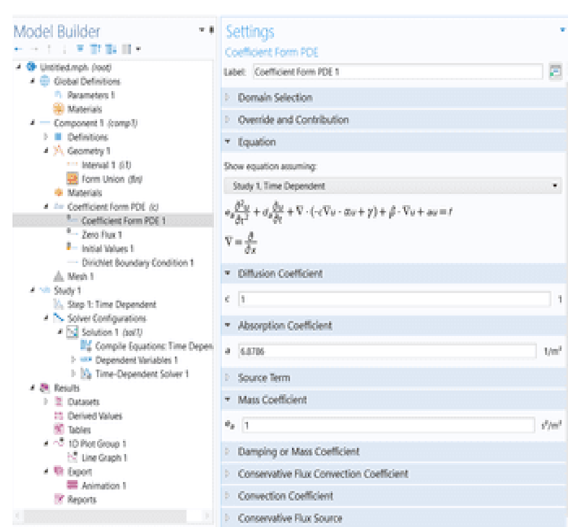

Fig 13.a. COMSOL coefficient interface.

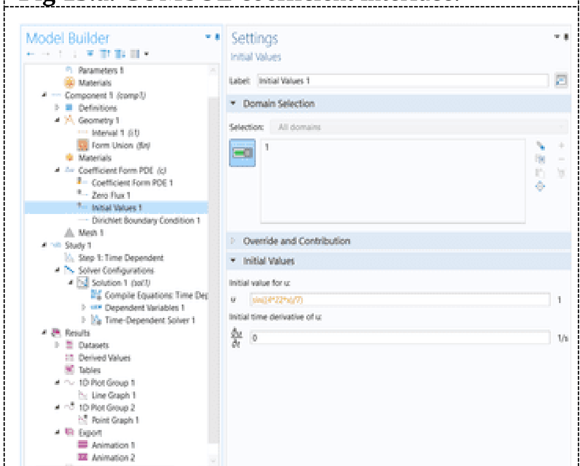

Fig 13.b. COMSOL initial condition interface.

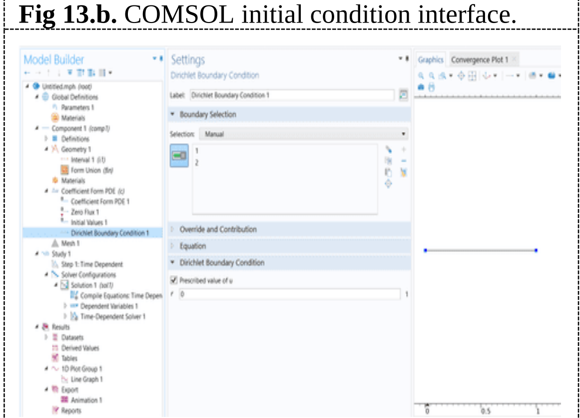

Fig 13.c. COMSOL boundary condition interface.

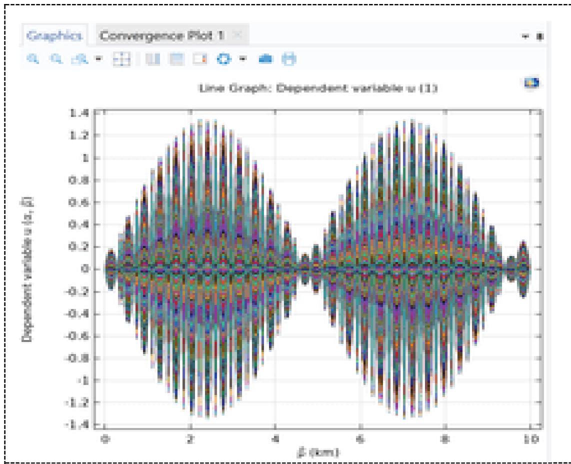

Fig 13.d. *2D Real animated plot for u and* $\boldsymbol{\beta}$**.**

6. Conclusion

The second-order linear PDEs can be classified into three types, which are invariant under changes of variables. The types are determined by discriminant. This exactly corresponds to the different cases for the quadratic equation satisfied by the slope of the characteristic curves. Hyperbolic equations have two family of (real) characteristic curves. All the second order hyperbolic PDE of equations can be reduced to second canonical form similar to basic wave equation using initial and boundary conditions for COMSOL Multiphysics to be simulated and modeled allowing the analysis of physical phenomena to predict the variance over time for different types of transmission line (RG59, CAT5, PIC, EXL-120, ……) as shown in tables of fig (5,7,8,11) used for different electrical applications data transmission, audio and video transmission, signal transmission…etc.

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