Modeling and simulation Elliptical 2nd order linear P.D.E using COMSOL Multiphysics

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Abstract– The Partial Differential Equations (PDEs) are very important in dynamics, aerodynamics, elasticity, heat transfer, waves, electromagnetic theory, transmission lines, quantum mechanics, weather forecasting, prediction of disasters, how universe behave ……. Etc., second order linear PDEs can be classified according to the characteristic equation into 3 types coinciding 3 basic conic sections hyperbolic, parabolic and elliptic; Elliptic equations have none family of (real) characteristic curves. All the three types of equations can be reduced to its first canonical form finding the general solution or the second canonical form similar to 3 basic PDE models; Elliptic equations reduce to a form coinciding with the Laplace's equations Thus, Laplace's equations serve as basic canonical models for all Elliptical second order linear PDEs the reduced canonical form can be modeled by boundary condition with COMSOL Multiphysics and Mathematica elliptical PDEs serve as basic uniform steady state solution for analysis of both parabolic and hyperbolic PDES.

Keywords-- elliptical PDEs – canonical form – constant coefficient PDEs – variable coefficients PDEs – LaPlace equation.

1. Introduction

A PDE is an equation that contains one or more partial derivatives of an unknown function that depends on at least two variables. usually, one of these deals with time t and the remaining with space. PDEs are very important in dynamics, elasticity, heat transfer, electromagnetic theory, and quantum mechanics.

The theory of partial differential equations of the second order is more complicated than the equations of the first order, and it is much more typical of the subject as a whole. Within the context, considerably better results can be achieved for equations of the second order in two independent variables than for equations in space of higher dimensions. Linear equations are the easiest to handle. In general, a second order linear partial differential equation is of the form

$$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} + D(x,y)u_x + E(x,y)u_y + F(x,y)u = G(x,y) \quad (1)$$

where A, B, C, D, E, F and G are in general functions of x and y but they may be constants. The subscripts are defined as partial derivatives where $u_x = \frac{\partial u}{\partial x}$

The classification of partial differential equations is suggested by the classification of the quadratic equation of conic sections in analytic geometry.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + f = 0$$

represents hyperbola, parabola, or ellipse accordingly as B^2−4AC is positive, zero, or negative.

Classifications of PDE are:

(i) Hyperbolic if B^2−4AC > 0
(ii) Parabolic if B^2−4AC = 0
(iii) Elliptic if B^2−4AC < 0

The classification of second-order equations is based upon the possibility of reducing equation by coordinate transformation to canonical or standard form at a point. An equation is said to be hyperbolic, parabolic, or elliptic at a point (x_0, y_0) accordingly as;

$$B^2(x_0,y_0) - 4A(x_0,y_0)C(x_0,y_0) \quad (2)$$

is positive, zero, or negative. If this is true at all points, then the equation is said to be hyperbolic, parabolic, or elliptic. In the case of two independent variables, a transformation can always be found to reduce the given equation to canonical form in a given domain. However, in the case of several independent variables, it is not, in general, possible to find such a transformation

To transform equation (1) to a canonical form we make a change of independent variables. Let the new variables be;

$\varepsilon = \varepsilon(x, y)$, $\eta = \eta(x, y)$

Assuming that ε and η are twice continuously differentiable and that the Jacobian;

$$J = \begin{vmatrix} \varepsilon_x & \varepsilon_y \\ \eta_x & \eta_y \end{vmatrix}$$

is nonzero in the region under consideration, then x and y can be determined uniquely. Let x and y be twice continuously differentiable functions of ε and η Then we have,

$$u_x = \frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\varepsilon \varepsilon_x + u_\eta \eta_x$$

$$u_y = \frac{\partial u}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = u_\varepsilon \varepsilon_y + u_\eta \eta_y$$

2. Canonical form

$u_{xx} = \frac{\partial u_x}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial x} + \frac{\delta u_x}{\delta \eta}\frac{\delta \eta}{\delta x} = u_{\varepsilon\varepsilon}{\varepsilon_x}^2 + 2u_{\varepsilon\eta}\varepsilon_x\eta_x +$
$u_{\eta\eta}{\eta_x}^2 + u_\varepsilon\varepsilon_{xx} + u_\eta\eta_{xx}$
$u_{yy} = \frac{\partial u_y}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\varepsilon\varepsilon}{\varepsilon_y}^2 + 2u_{\varepsilon\eta}\varepsilon_y\eta_y +$
$u_{\eta\eta}{\eta_y}^2 + u_\varepsilon\varepsilon_{yy} + u_\eta\eta_{yy}$
$u_{xy} = \frac{\partial u_x}{\partial \varepsilon}\frac{\partial \varepsilon}{\partial y} + \frac{\partial u_x}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\varepsilon\varepsilon}\varepsilon_x\varepsilon_y + u_{\eta\eta}\eta_x\eta_y +$
$u_\varepsilon\varepsilon_{xy} + u_\eta\eta_{xy} + u_{\varepsilon\eta}(\varepsilon_x\eta_y + \varepsilon_y\eta_x)$
substituting in (1)
$A^*(x,y)u_{xx} + B^*(x,y)u_{xy} + C^*(x,y)u_{yy} +$
$D^*(x,y)u_x + E^*(x,y)u_y + F^*(x,y)u = G^*(x,y)$ (3)
Where;
$A^* = A{\varepsilon_x}^2 + B\varepsilon_x\varepsilon_y + c{\varepsilon_y}^2$
$B^* = 2A\varepsilon_x\eta_x + B(\varepsilon_x\eta_y + \varepsilon_y\eta_x) + 2C\varepsilon_y\eta_y$
$C^* = A{\eta_x}^2 + B\eta_x\eta_y + C{\eta_y}^2$
$D^* = A\varepsilon_{xx} + B\varepsilon_{xy} + C\varepsilon_{yy} + D\varepsilon_x + E\varepsilon_y$
$E^* = A\eta_{xx} + B\eta_{xy} + C\eta_{yy} + D\eta_x + E\eta_y$
$F^* = F \quad , \quad G^* = G$
The resulting equation (3) is in the same form as the original equation (1) under the general transformation. The nature of the equation remains constant if the Jacobian does not vanish.
${B^*}^2 - 4A^*C^* = J^2(B^2 - 4AC)$ and $J^2 \neq 0$, We shall assume that the equation under consideration is of the single type in a given domain. The classification of equation (1) depends on the coefficients
A (x, y), B (x, y), and C (x, y) at a given point
(x, y) so equation (1) rewritten as;
$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} =$
$H(x,y,u,u_x,u_y)$ (4)
Where; A, B, $C \neq 0$
And equation (3) rewritten as;
$A^*(x,y)u_{\varepsilon\varepsilon} + B^*(x,y)u_{\varepsilon\eta} + C^*(x,y)u_{\eta\eta} =$
$H(\varepsilon,\eta,u,u_\varepsilon,u_\eta)$
Where A^*, $C^* = 0$
$A{\varepsilon_x}^2 + B\varepsilon_x\varepsilon_y + c{\varepsilon_y}^2 = 0$
$A{\eta_x}^2 + B\eta_x\eta_y + C{\eta_y}^2 = 0$
Since the 2 equations from the same type, we can rewrite them;
$A{\varepsilon_x}^2 + B\varepsilon_x\varepsilon_y + c{\varepsilon_y}^2 = 0$
where ε stands for the 2 functions ε, η
Dividing by ${\varepsilon_y}^2$ $\quad A(\frac{\varepsilon_x}{\varepsilon_y})^2 + B\frac{\varepsilon_x}{\varepsilon_y} + C = 0$
$\frac{dy}{dx} = -\frac{\varepsilon_x}{\varepsilon_y} \quad A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C = 0$;
therefore, two roots are $\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$
These equations, which are known as the characteristic equations, are ordinary differential equations for families of curves in the by-plane along which
ε = constant and η = constant. The integrals of equation are called the characteristic curves. Since the equations are first order ordinary differential equations, the solutions may be written as;
$\Phi_1(x, y) = c_1$ $\quad \Phi_2(x, y) = c_2$ with c_1 and c_2 as constants.
Hence the transformations
$\varepsilon = \Phi_1(x, y)$, $\eta = \Phi_2(x, y)$
will transform equation (4) to a canonical form.
We show that the characteristic of any elliptical PDE can be transformed as;
$*B^2 - 4AC < 0$ so, we have no real characteristic but it has complex solution which is analytic along some neighborhood domain can be reduced into first canonical form
$u_{\varepsilon\eta} = H(\varepsilon,\eta,u,u_\varepsilon,u_\eta)$ where $\varepsilon = \alpha + i\beta$
$\eta = \alpha - i\beta$ are two conjugate functions where;
$\alpha = \frac{1}{2}(\varepsilon + \eta)$, $\beta = \frac{1}{2i}(\varepsilon - \eta)$
$u_x = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = u_\alpha\alpha_x + u_\beta\beta_x$
$u_y = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial y} = u_\alpha\alpha_y + u_\beta\beta_y$
$u_{xx} = \frac{\partial u_x}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u_x}{\partial \beta}\frac{\partial \beta}{\partial x} = u_{\alpha\alpha}{\alpha_x}^2 + 2u_{\alpha\beta}\alpha_x\beta_x + u_{\beta\beta}{\beta_x}^2 +$
$u_\alpha\alpha_{xx} + u_\beta\beta_{xx}$
$u_{yy} = \frac{\partial u_y}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial u_y}{\partial \eta}\frac{\partial \eta}{\partial y} = u_{\alpha\alpha}{\alpha_y}^2 + 2u_{\alpha\eta}\alpha_y\beta_y + u_{\beta\beta}{\beta_y}^2 +$
$u_\alpha\alpha_{yy} + u_\beta\beta_{yy}$
$u_{xy} = \frac{\partial u_x}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial u_x}{\partial \beta}\frac{\partial \beta}{\partial y} = u_{\alpha\alpha}\alpha_x\alpha_y + u_{\beta\beta}\beta_x\beta_y + u_\alpha\alpha_{xy} +$
$u_\beta\beta_{xy} + u_{\alpha\beta}(\alpha_x\beta_y + \alpha_y\beta_x)$
substituting in (1)
$A^*(x,y)u_{xx} + B^*(x,y)u_{xy} + C^*(x,y)u_{yy} + D^*(x,y)u_x +$
$E^*(x,y)u_y + F^*(x,y)u = G^*(x,y)$ (3)
Where;
$A^* = A{\alpha_x}^2 + B\alpha_x\alpha_y + c{\alpha_y}^2$
$B^* = 2A\alpha_x\beta_x + B(\alpha_x\beta_y + \alpha_y\beta_x) + 2C\alpha_y\beta_y$
$C^* = A{\beta_x}^2 + B\beta_x\beta_y + C{\beta_y}^2$
$D^* = A\alpha_{xx} + B\alpha_{xy} + C\alpha_{yy} + D\alpha_x + E\alpha_y$
$E^* = A\beta_{xx} + B\beta_{xy} + C\beta_{yy} + D\beta_x + E\beta_y$
$F^* = F \quad , \quad G^* = G$
The resulting equation (3) is in the same form as the original equation (1) under the general transformation. The nature of the equation remains constant if the Jacobian does not vanish.
${B^*}^2 - 4A^*C^* = J^2(B^2 - 4AC)$ and $J^2 \neq 0$, We shall assume that the equation under consideration is of the single type in a given domain. The classification of equation (1) depends on the coefficients
A (x, y), B (x, y), and C (x, y) at a given point
(x, y) so equation (1) rewritten as;
$A(x,y)u_{xx} + B(x,y)u_{xy} + C(x,y)u_{yy} = H(x,y,u,u_x,u_y)$
(4)
Where; A, B, $C \neq 0$
And equation (3) rewritten as;
$A^*(x,y)u_{\alpha\alpha} + B^*(x,y)u_{\alpha\beta} + C^*(x,y)u_{\beta\beta} = H(\varepsilon,\eta,u,u_\alpha,u_\beta)$
where $B^*(x,y)u_{\varepsilon\eta} = 0$
$u_{\varepsilon\eta} \neq 0$ *so* $B^* = 0$ $\quad A^* = C^*$
$B^* = 2A\varepsilon_x\eta_x + B(\varepsilon_x\eta_y + \varepsilon_y\eta_x) + 2C\varepsilon_y\eta_y = 0$
which is transformed into second canonical form

$u_{\alpha\alpha} + u_{\beta\beta} = H(\alpha, \beta, u, u_\alpha, u_\beta)$
similar to LaPlace equation to be modeled.

3. Elliptical equations

3.1. Fundamental Laplace equation

$u_{xx} + u_{yy} = 0 \qquad A = 1 \quad B = 0 \quad C = 1$

$B^2 - 4AC = -4 < 0$

$A\lambda^2 - B\lambda + c = 0 \qquad \lambda^2 + 1 = 0$

$\lambda = \pm i \qquad \frac{dy}{dx} = i \qquad \frac{dy}{dx} = -i$

$\int dy = \int i\,dx \qquad iy = -x + c$

$x + iy = c1 \quad let\ \varepsilon = x + iy$

$\int dy = \int -i\,dx \qquad iy = x + c$

$x - iy = c2 \quad let\ \eta = x - iy$

$\varepsilon_{xx} = \varepsilon_{xy} = \varepsilon_{yy} = 0 \qquad \varepsilon_y = i \qquad \varepsilon_x = 1$

$\eta_{xx} = \eta_{yy} = \eta_{xy} = 0 \qquad \eta_x = 1 \qquad \eta_y = -i$

$u_{xx} = u_{\varepsilon\varepsilon}{\varepsilon_x}^2 + 2u_{\varepsilon\eta}\varepsilon_x\eta_x + u_{\eta\eta}{\eta_x}^2 + u_\varepsilon\varepsilon_{xx} + u_\eta\eta_{xx} =$
$-u_{\varepsilon\varepsilon} + 2u_{\varepsilon\eta} - u_{\eta\eta}$

$u_{yy} = u_{\varepsilon\varepsilon}{\varepsilon_y}^2 + 2u_{\varepsilon\eta}\varepsilon_y\eta_y + u_{\eta\eta}{\eta_y}^2 + u_\varepsilon\varepsilon_{yy} + u_\eta\eta_{yy} =$
$u_{\varepsilon\varepsilon} + 2u_{\varepsilon\eta} + u_{\eta\eta}$

$u_{xx} + u_{yy} = 4u_{\varepsilon\eta}$

$so\ the\ canonical\ form\ is \qquad 4u_{\varepsilon\eta} = 0$

$u_\varepsilon = g(\varepsilon) \ \ ,\ u = f(\varepsilon) + g(\eta)$

$u = f(x + iy) + g(x - iy) = f(z) + g(\bar{z})$

$u = u_{(x,y)} + iv_{(x,y)}$

General method for particular solution

$u = u_{(x,y)} + iv_{(x,y)}$ u, v should be analytic and harmonic function $u_x = v_y\ ,\ u_y = -v_x$

$where\ u, v$ are Real function of Real variables then the Real and Imaginary part of u each represents a solution for Laplace P.D.E or any combination of them

As Laplace equation is symmetric so the solution should be radial so we can set $u = v(r)$

$r = \sqrt{x^2 + y^2} \quad , \qquad u_x = v_r r_x = \frac{x}{r}v_r$

$u_{xx} = \frac{v_r}{r} + \frac{x^2}{r^2}v_{rr} - \frac{x^2}{r^3}v_r$

$u_y = v_r r_y = \frac{y}{r}v_r$

$u_{yy} = \frac{v_r}{r} + \frac{y^2}{r^2}v_{rr} - \frac{y^2}{r^3}v_r$

$u_{xx} + u_{yy} = \frac{1}{r}v_r + v_{rr}$

By this method the P.D.E reduced into ode where;

$\frac{1}{r}v_r + v_{rr} = 0$ solving for v by integration

$v = c\ln\left(\frac{1}{r}\right) + c^* \quad which\ is\ the\ fundmental\ sol.$

$where \quad c = \frac{1}{\pi} \quad , \quad c^* = 0$

$u = v = \frac{-1}{2\pi}\ln(x^2 + y^2)$

$u = v\frac{-1}{2\pi}(\ln(x + iy) + \ln(x - iy))$

$from\ \ general\ solution$ shown in fig. (1)

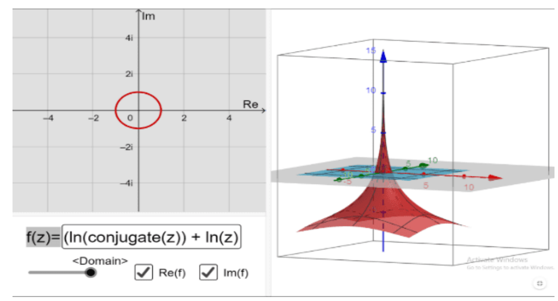

Fig. 1.a. complex 3D plot for u, x and y.

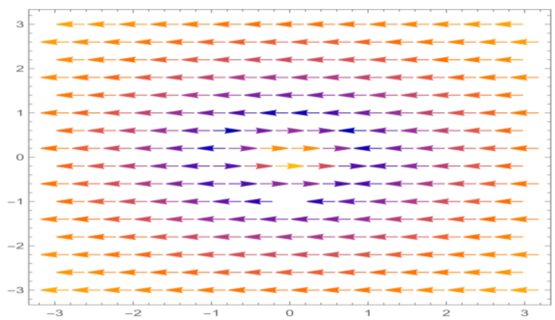

Fig. 1.b. complex vector plot for x and y.

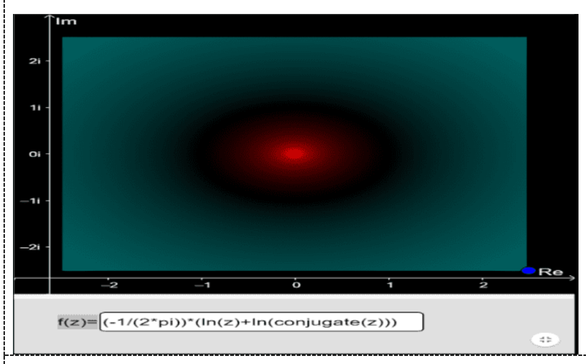

Fig. 1.c. complex contour plot for x and y.

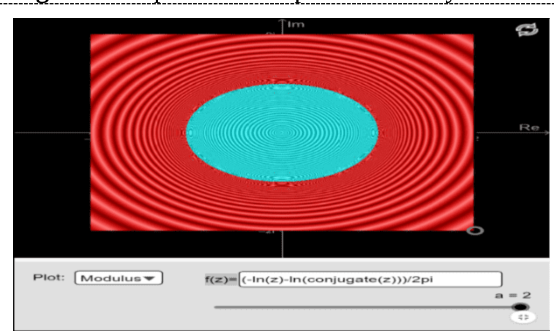

Fig. 1.d. complex modulus plot for x and y.

3.2. Variable coefficient equation

$u_{xx} + x^2 u_{yy} = 0$

$B^2 - 4AC = -4x^2 < 0 \qquad x \neq 0$

$A\lambda^2 - B\lambda + c = 0$

$\frac{dy}{dx} = \pm jx$

$\varepsilon = 2y - jx^2 \quad , \quad \eta = 2y + jx^2$

$u_{xx} = -4x^2 u_{\varepsilon\varepsilon} + 8u_{\varepsilon\eta} - 4x^2 u_{\eta\eta}$

$x^2 u_{yy} = 4x^2 u_{\varepsilon\varepsilon} + 8u_{\varepsilon\eta} + 4x^2 u_{\eta\eta}$

by adding

$16u_{\varepsilon\eta} = 0 \ , \quad u_{\varepsilon\eta} = 0 \quad , u_\varepsilon = g(\varepsilon)$

$u = f(\varepsilon) + g(\eta)$

$u = f(2y - jx^2) + g(2y + jx^2)$

Apply second canonical form

$\alpha = \frac{1}{2}(\varepsilon + \eta) = 2y \ , \beta = \frac{1}{2i}(\varepsilon - \eta) = -x^2$

$u_{xx} = 4x^2 u_{\beta\beta} - 2u_\beta \ , x^2 u_{yy} = 4x^2 u_{\alpha\alpha}$

by adding and simplify we get

$u_{\alpha\alpha} + u_{\beta\beta} = \frac{-1}{2\beta} u_\beta$ which is similar to la place equation to be modeled

3.3. Constant coefficient equation

$u_{xx} + 2u_{xy} + 5u_{yy} + u_x = 0$

$B^2 - 4AC = -16 < 0 \quad , \frac{dy}{dx} = 1 \pm 2j$

Separate variables and integrate to get

$\varepsilon = y - (1 + 2j)x \quad , \quad \eta = y - (1 - 2j)x$

$16u_{\varepsilon\eta} - (u_\varepsilon + u_\eta) - 2j(u_\varepsilon - u_\eta) = 0$

equating real and imaginary parts

From imaginary equating we get $u_\varepsilon = u_\eta$

From real equating we get

$16u_{\varepsilon\eta} - (u_\varepsilon + u_\eta) = 0$

from imaginary by subistuting

$16u_{\varepsilon\eta} - 2u_\varepsilon = 0$

using ODE by integrating factor we get

$u = e^{\frac{\eta}{8}} f(\varepsilon) + g(\eta)$

$u = e^{\frac{y-(1-2j)x}{8}} f(y - (1 + 2j)x) + g(y - (1 - 2j)x)$

Which is general solution

Apply second canonical form

$\alpha = \frac{1}{2}(\varepsilon + \eta) = y - x \quad , \quad \beta = \frac{1}{2i}(\varepsilon - \eta) = 2x$

and substitute

$u_{\alpha\alpha} + u_{\beta\beta} = \frac{1}{4}(u_\alpha - 2u_\beta)$ which is similar to la place equation that can be modeled

4. Physical application

1-Electrostatic potential charge in free region where the potential in the rectangle whose upper side is kept at potential 110 V and whose other sides are grounded.

$0 \le x \le 40 \ , \qquad 0 \le y \le 20$

la place equ. $u_{xx} + u_{yy} = 0$ (*cartesian*)

where u is the potential as shown fig. (2)

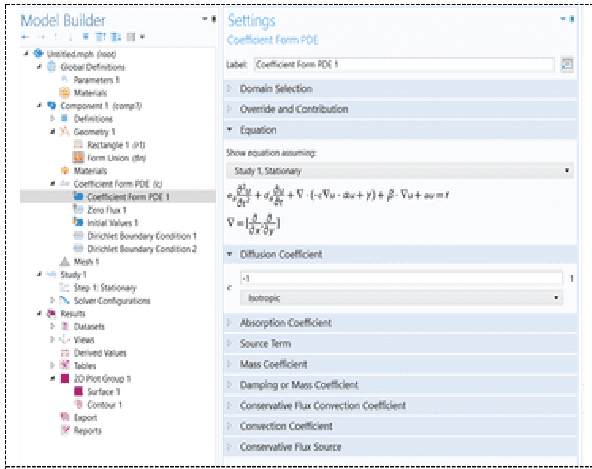

Fig. 2.a. COMSOL Coefficient interface.

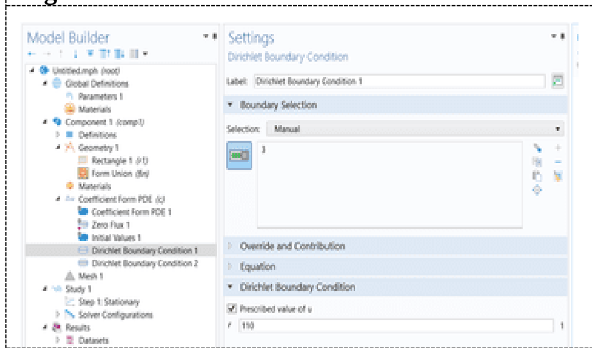

Fig. 2.b. Boundary condition.

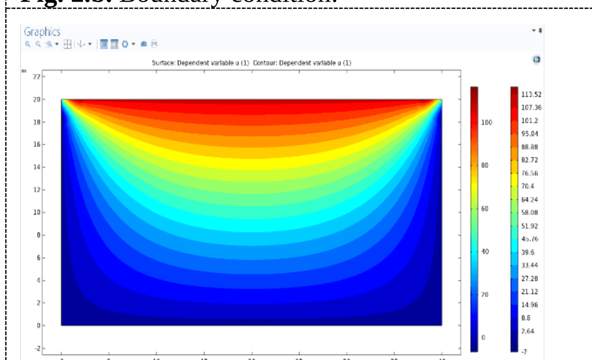

Fig. 2.c. Real 2D plot for u and x.

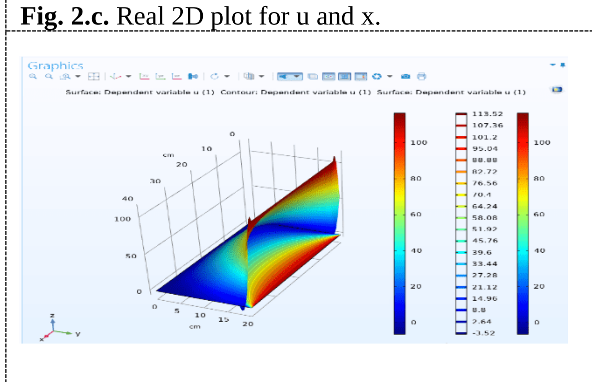

Fig. 2.d. Real 3D plot for u, y and x.

2- Electrostatic potential charge in free region where the potential in the rectangle whose upper side is kept at potential 110 V and whose other sides are grounded.

$0 \le x \le 40$, $0 \le y \le 20$, $u_{xx} + 2u_{xy} + 5u_{yy} + u_x = 0$ where u is the potential as shown fig. (3)

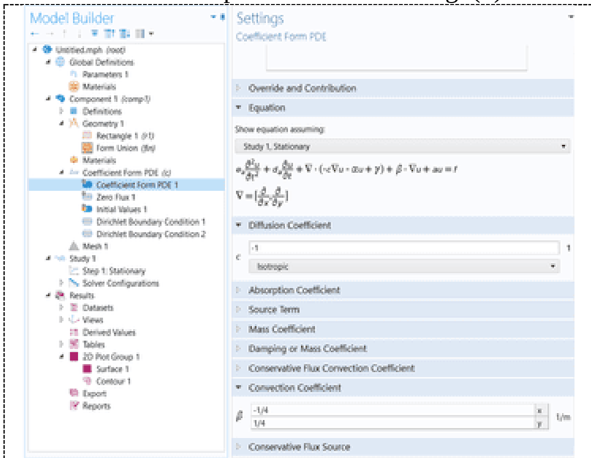

Fig. 3.a. COMSOL Coefficient interface.

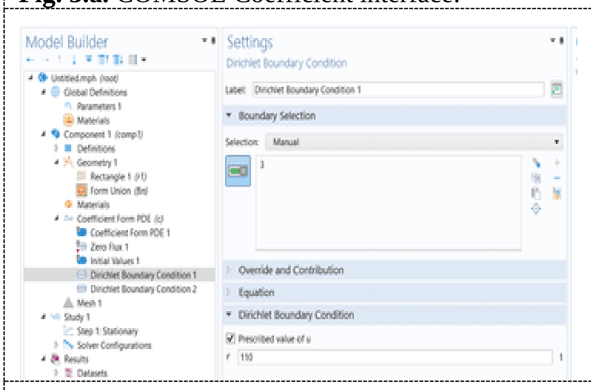

Fig. 3.b. Boundary condition.

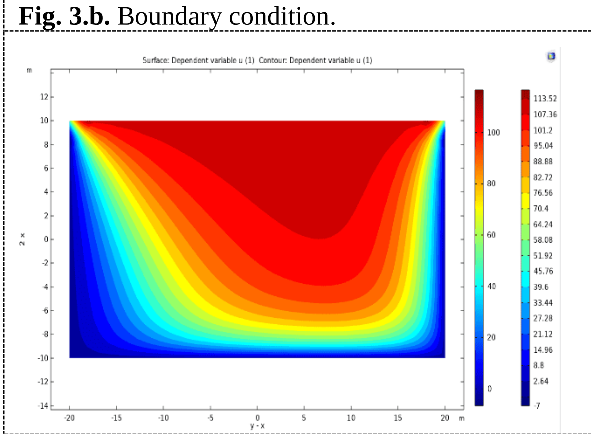

Fig. 3.c. Real 2D plot for u and x.

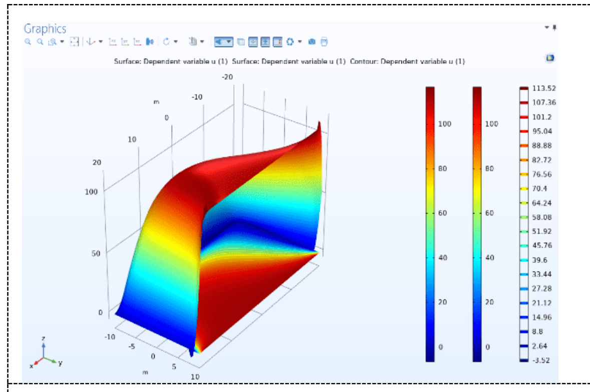

Fig. 3.d. Real 3D plot for u, y and x.

3- Electrostatic potential charge in free region where the potential in the rectangle whose upper side is kept at potential 110 V and whose other sides are grounded.

$0 \le x \le 40$, $0 \le y \le 20$, $u_{xx} + x^2 u_{yy} = 0$ where u is the potential as shown fig. (4)

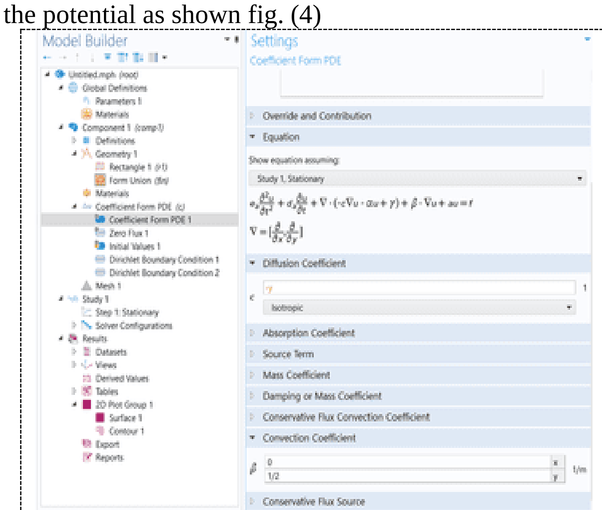

Fig. 4.a. COMSOL Coefficient interface.

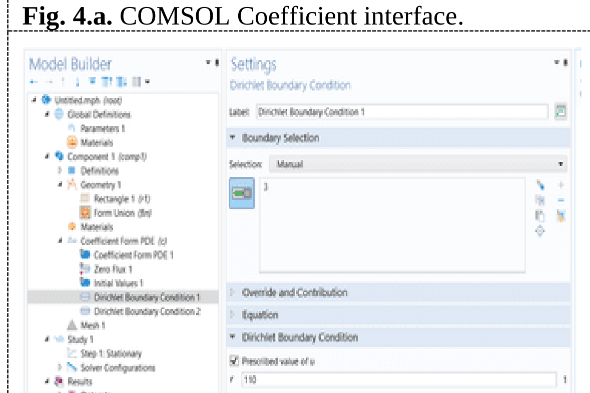

Fig. 4.b. Boundary condition.

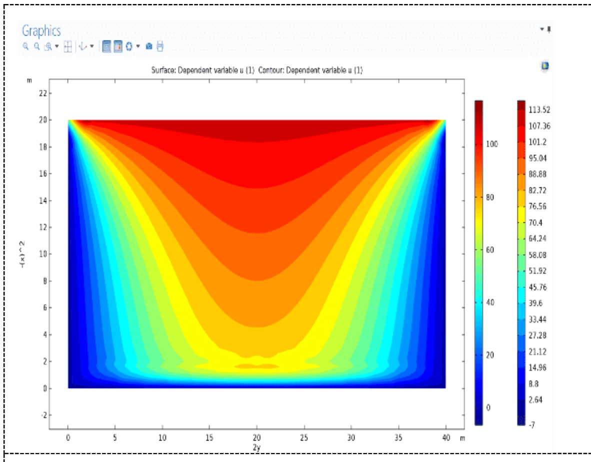

Fig. 4.c. Real 2D plot for u and x.

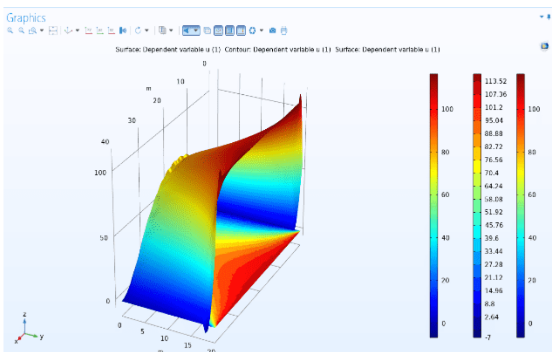

Fig. 4.d. Real 3D plot for u, y and x.

4-The potential flow of an ideal incompressible fluid about a circular cylinder of radius R with a constant incident velocity v *la place equation*

$$u_{rr}+\frac{1}{r}\,u_r+\frac{1}{r^2}\,u_{\theta\theta}=0\ \ (cylinderical)$$

$$u=f(\varepsilon)+g(\eta)$$

$$u=f(\ln(r)-i\theta)+g(\ln(r)+i\theta)$$

$$u=Ar^ne^{in\theta}+\frac{B}{r^n}e^{-in\theta}$$

$$Re=\left(Ar^n+\frac{B}{r^n}\right)\cos(n\theta)$$

$$Im=\left(Ar^n-\frac{B}{r^n}\right)sin(n\theta)$$

by multiplying Re, Im *we get*

$$u=\left(Ar^n+\frac{B}{r^n}\right)(\mathrm{C}\cos(n\theta)+\mathrm{D}\sin(n\theta))$$

$$n=1,2,3,4,5,\ldots$$

We are going to solve this P.D.E twice with different initial and boundaries once for stream lines

Then for velocity potential.

$$u_{(R,\theta)}=0\quad,\quad r=R\quad,\quad r\to\infty$$

$$u\to vrsin(\theta)\quad from\ IC, BC$$

$$Ar^n(\mathrm{C}\cos(n\theta)+\mathrm{D}\sin(n\theta))=vrsin(\theta)$$

comparing coffecient

$$n=1\,, c=0\,,\ AD=v$$

$$\left(AR+\frac{B}{R}\right)(\mathrm{D}\sin(\theta))=0\ ,\ \ B=-AR^2$$

substitute *in original so*

$$u=\left(Ar^n+\frac{B}{r^n}\right)(\mathrm{C}\cos(n\theta)+\mathrm{D}\sin(n\theta))$$

$$u=\left(Ar-\frac{AR^2}{r}\right)Dsin\theta$$

$$u=v\left(r-\frac{R^2}{r}\right)\sin(\theta)$$

*u is stream function*as shown in fig. (5)

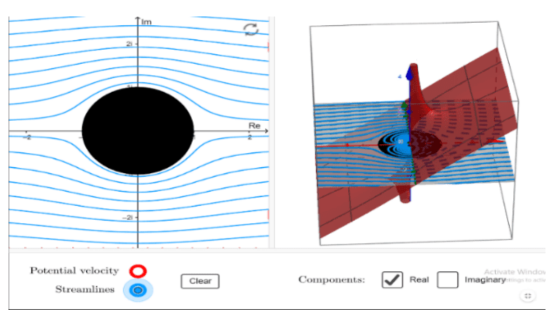

Fig. 5.a. complex 3d plot for Real part u and 2d plot for stream lines.

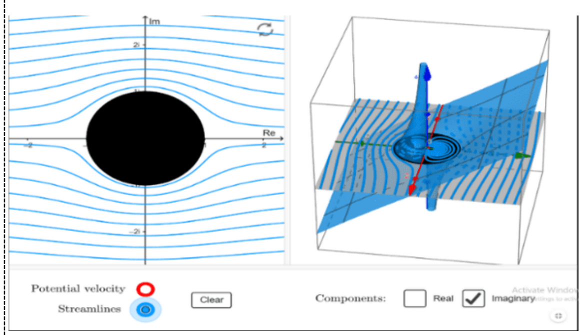

Fig. 5.b. complex 3d plot for Imaginary part of u and 2d plot for stream lines.

Solving the same P.D.E again for velocity potential where;

$$u_{(R,\theta)}=2vRcos(\theta)\quad,\quad r=R$$

$$r\to\infty\quad,\quad u\to vrcos(\theta)$$

comparing coffecients to get $D=0$

$n=1\,,\ AC=v\,,\ B=AR^2$ *then substitute*

$$u_{(r,\theta)}=v\left(r+\frac{R^2}{r}\right)\cos(\theta)$$

where u is velocity potential as shown in fig. (6)

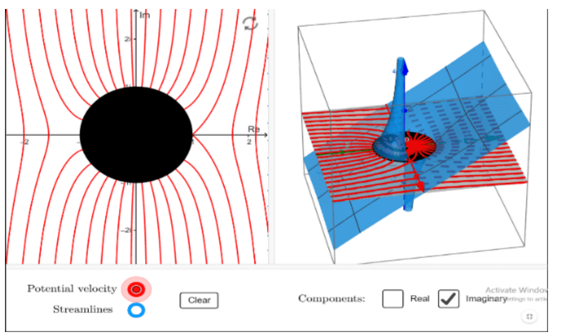

Fig. 6.a. complex 3d plot for Real part u and 2d plot for velocity potential.

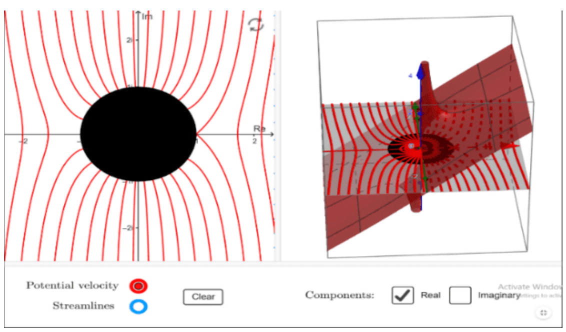

Fig. 6.b. complex 3d plot for Imaginary part of u and 2d plot for velocity potential.

By adding stream lines and velocity potential to get the potential flow

$$U = v\left(r - \frac{R^2}{r}\right)\sin(\theta) + v\left(r + \frac{R^2}{r}\right)\cos(\theta)$$

as shown in fig. (7)

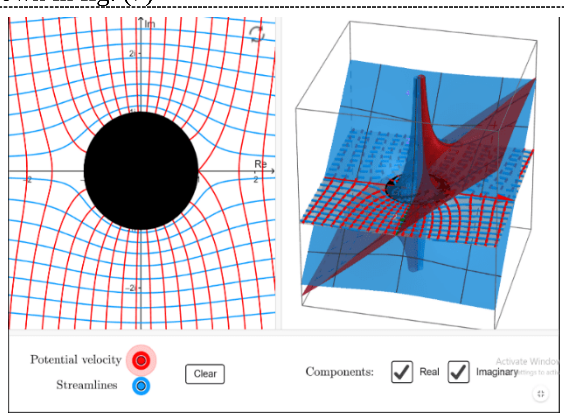

Fig. 7.a. complex 3d plot for u and 2d plot for stream lines and velocity potential.

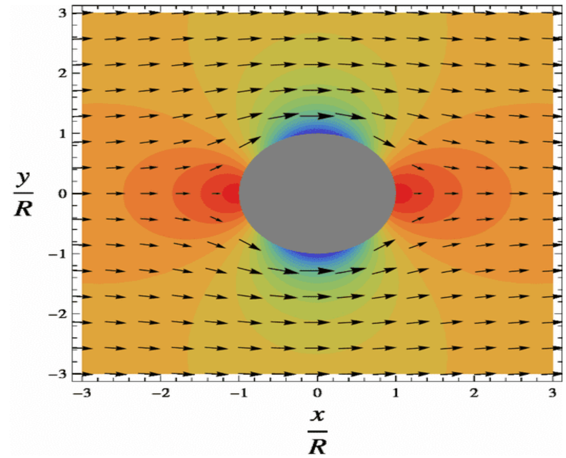

Fig. 7.b. animated vector field plot for circular cylinder

5. Conclusion

The second-order linear PDEs can be classified into three types, which are invariant under changes of variables. The types are determined by discriminant. This exactly corresponds to the different cases for the quadratic equation satisfied by the slope of the characteristic curves. Elliptical equations have none family of (real) characteristic curves. All the second order elliptical PDE of equations can be reduced to canonical forms to be simulated and modeled allowing the analysis of physical phenomena to predict the variance over time as it serves the steady state solution for both hyperbolic and parabolic linear PDES which act as basic steady simplified solution for hyperbolic and parabolic equation.

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