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On pseudo W_2-symmetric manifolds with applications to f(R) gravity

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ABSTRACT **: In this article, we define new kind of manifolds, namely pseudo $\mathcal{W}_2$ −symmetric manifolds. We first focus on studying the geometric properties of pseudo $\mathcal{W}_2$ − symmetric manifold. Pseudo $\mathcal{W}_2$ −symmetric spacetimes and Pseudo $\mathcal{W}_2$ −symmetric perfect fluid space-times in f(R) theory of gravity are invesitgated. We get in this case the form of the isotropic pressure p. Also, the form of energy density σ is obtained. As final point of study, some energy conditions are studied.**

KEYWORDS: ***W_2 −curvature tensor, Modified theories of gravity, energy conditions in $f(R)$ modified gravity theory***

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I. INTRODUCTION

T Pseudo symmetric manifolds, briefly denoted by $(PS)_n$, were introduced in 1987 by Chaki [5]. A manifold M is named $(PS)_n$ manifold if its Riemann tensor $\mathcal{R}_{hijk}$ obeys

$$\nabla_l \mathcal{R}_{hijk} = 2\lambda_l \mathcal{R}_{hijk} + \lambda_h \mathcal{R}_{lijk} + \lambda_i \mathcal{R}_{hljk} + \lambda_j \mathcal{R}_{hilk} + \lambda_k \mathcal{R}_{hijl}, \quad (1.1)$$

bein R_{ij} denotes the Ricci tensor. With λ_l being is a non-zero 1-form, and ∇ is the covariant differentiation with respect to the metric tensor g.

It is known that in a manifold M the $\mathcal{W}_2$ −-curvature tensor, we will use $\mathcal{W}$ instead of $\mathcal{W}_2$ −for simlicity, is defined by[11, 13]

$$\mathcal{W}_{hijk} = R_{hijk} + \frac{1}{n-1}\left[g_{hj}R_{ik} - g_{ij}R_{hk}\right]. \quad (1.2)$$

This article deals with a type of a non-flat semi-Riemannian manifold whose W_2-curvature tensor obeys to the following condition

$$\nabla_l \mathcal{W}_{hijk} = 2\lambda_l \mathcal{W}_{hijk} + \lambda_h \mathcal{W}_{lijk} + \lambda_i \mathcal{W}_{hljk} + \lambda_j \mathcal{W}_{hilk} + \lambda_k \mathcal{W}_{hijl}. \quad (1.3)$$

Such a manifold is named pseudo $\mathcal{W}_2$ −-symmetric manifold. The n −dimensional pseudo $\mathcal{W}_2$ −symmetric manifold is denoted by $(PWS)_n$.

In Einstein's theory of gravity (or, named by Standard gravity theory), it is to be noted that there is a direct relation between the matter and the geometry of any spacetime. This relation is given by Einstein's field equations (EFE)

$$R_{ij} - \frac{R}{2}g_{ij} = \kappa T_{ij}$$

with κ being the Newtonian constant and T_{ij} being the energy-momentum tensor [12]. EFE imply that T_{ij} is of divergence-free, that is $\nabla_l T_i^l = 0$. Such condition is directly fulfilled whenever $\nabla_l T_{ij} = 0$. Many modifications of the Einstein' theory of gravity are introduced by various authors. This is because Einstein's theory has many shortcomings. The famous modification is coined in 1970[10]. It is named by the f(R) modified gravity theory. By replacing R with a generic function f(R) in the Einstein-Hilbert action one easily can get this modified theory. The f(R) field equations are expressed as

$$\kappa T_{ij} = f'(R)R_{ij} - f''(R)f\nabla_i R\nabla_j R - f''(R)\nabla_i\nabla_j R + g_{ij}\left[f'''(R)\nabla_k R\nabla^k R + f''(R)\nabla^2 R - \frac{1}{2}f(R)\right], \quad (1.4)$$

where f(R) is an arbitrary R. The derivative of f(R) which is $f'(R) = \frac{df}{dR}$, is always positive to ensure attractive gravity [7].

Recently, weakly Ricci symmetric spacetimes $(WRS)_4$ are considered in f(R) gravity theory [2]. Also, almost pseudo-Ricci symmetric spacetimes $(APRS)_4$ are recovered in f(R) gravity theory. Further, conformally flat generalized Ricci recurrent spacetimes are investigated in f(R) and f(R, G) gravity theory [3, 4].

The above mentioned studies and many others give us a great motivation to study the geometric properties of pseudo $\mathcal{W}_2-$ symmetric manifold and consider Pseudo $\mathcal{W}_2-$ symmetric spacetimes and Pseudo $\mathcal{W}_2$ −symmetric perfect fluid space-times in f(R) theory of gravity.

We organized this article as follows. In the following section, general properties of pseudo $\mathcal{W}_2$ −symmetric manifold are considered. Next, pseudo $\mathcal{W}_2$ −symmetric spacetime and pseudo $\mathcal{W}_2$ −symmetric perfect fluid spacetime in f(R) gravity are investigated

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II. Geometric properties of $(PWS)_n$

This section is focused on studying certain geometric properties of $(PWS)_n$ manifolds. Transvecting (1.2) with g^{hk}, one have

$$\mathcal{W}_{ij} = \frac{1}{n-1}\left[nR_{ij} - g_{ij}R\right]. \tag{2.1}$$

where $\mathcal{W}_{ij} = g^{hk}\mathcal{W}_{hijk}$. A contraction of (1.3) with g^{hk} implies

$$\nabla_l\mathcal{W}_{ij} = 2\lambda_l\mathcal{W}_{ij} + \lambda^k\mathcal{W}_{lijk} + \lambda_i\mathcal{W}_{lj} + \lambda_j\mathcal{W}_{il} + \lambda^h\mathcal{W}_{hijl}. \tag{2.2}$$

Utilizing (1.2) and (2.1) in (2.2), one infers

$$\nabla_l R_{ij} = \frac{2\lambda_l}{n}\left[nR_{ij} - g_{ij}R\right] + \frac{\lambda_i}{n}\left[nR_{lj} - g_{lj}R\right] + \frac{\lambda_j}{n}\left[nR_{il} - g_{il}R\right] + \frac{\lambda^k}{n}\left[g_{lj}R_{ik} - g_{ij}R_{lk}\right]$$
$$+\frac{1}{n}\left[\lambda_j R_{il} - g_{ij}\lambda^h R_{hl}\right] + \frac{(n-1)}{n}\left(\lambda^k R_{lijk} + \lambda^h R_{hijl}\right) + \frac{1}{n}g_{ij}\nabla_l R. \tag{2.3}$$

A contraction with g^{ij} gives

$$\lambda^j R_{lj} = \frac{R}{n}\lambda_l, \tag{2.4}$$

which simply says that the vector field λ^j is an eigenvector of R_{lj}.

Executing a contraction of (2.3) with g^{li}, one deduces

$$\nabla_j R = 0, \tag{2.5}$$

Now inserting (2.4) and (2.5) in (2.3), we obtain

$$\nabla_l R_{ij} = 2\lambda_l R_{ij} + \lambda_i R_{lj} + \frac{n+1}{n}\lambda_j R_{il} + \left(\frac{-2n-2}{n^2}\right)g_{ij}\lambda_l R + \left(\frac{1-n}{n^2}\right)\lambda_i R g_{lj}$$
$$-\frac{R}{n}g_{il}\lambda_j + \frac{(n-1)}{n}\lambda^k\left(R_{lijk} + R_{kijl}\right). \tag{2.6}$$

Interchanging i and j, one infers

$$\nabla_l R_{ji} = 2\lambda_l R_{ij} + \lambda_j R_{li} + \frac{n+1}{n}\lambda_i R_{jl} + \left(\frac{-2n-2}{n^2}\right)g_{ij}\lambda_l R + \left(\frac{1-n}{n^2}\right)\lambda_j R g_{li}$$
$$-\frac{1}{n}g_{jl}\lambda_i R + \frac{(n-1)}{n}\lambda^k\left(R_{ljik} + R_{kjil}\right). \tag{2.7}$$

Substracting the last two equation, we get

$$\frac{\lambda_j}{n}R_{il} - \frac{\lambda_i}{n}R_{lj} + \frac{\lambda_i}{n^2}Rg_{lj} - \frac{\lambda_j}{n^2}Rg_{li} + \frac{(n-1)}{n}\lambda^k\left(R_{lijk} + R_{kijl} - R_{ljik} - R_{kjil}\right) = 0.$$

Contracting with λ^j and using (2.4), one finds

$$R_{il} = \frac{R}{n}g_{il}, \tag{2.8}$$

which demonstrates that a pseudo $\mathcal{W}_2$ −symmetric manifold is Einstein.

Theorem 1: A pseudo $\mathcal{W}_2$ −symmetric manifold is Einstein.

III. pseudo $\mathcal{W}_2$ −symmetric spacetimes in f(R) gravity

As mentioned above, the scalar curvature R of $(PWS)_n$ is constant. Thus, f(R) gravity field equations are in the following form

$$R_{ij} - \frac{f}{2f'}g_{ij} = \frac{\kappa}{f'}T_{ij}. \tag{3.1}$$

In vacuum case, that is $T_{ij} = 0$, we have

$$R_{ij} - \frac{n}{2}\frac{f}{f'} = 0.$$

Contracting with g^{ij} after that integrating to obtain the following result

$$f = \lambda R^{\frac{n}{2}}, \tag{3.2}$$

where λ is a constant.

Conversely, if equation (3.2) fulifilled, thus

$$T_{ij} = 0.$$

Hence, we have the following result:

Theorem 2: A $(PWS)_n$ spacetime in f(R) modified theories gravity is vacuum iff $f = \lambda R^{\frac{n}{2}}$.

Applying the covariant derivative ∇ of equation (3.8) gives

$$\nabla_k T_{ij} = 0. \tag{3.3}$$

Thus, we have:

Theorem 3: The energy-momentume tensor of a $(PWS)_n$ spacetime satisfying f(R) gravity is covariantly constant.

Definition 1: The vector ξ is called a Killing vector filed if

$$\mathcal{L}_\xi g_{ij} = 0. \tag{3.4}$$

On the other hand, ξ is called conformal Killing if

$$\mathcal{L}_\xi g_{ij} = 2\varphi g_{ij}, \tag{3.5}$$

where $\mathcal{L}_\xi$ is the Lie derivative w.r.t ξ. The function φ is a scalar function[1, 11, 16].

Definition 2 : Let M be a spacetime. Then M called admit a matter collineation w.r.t. ξ if $\mathcal{L}_\xi$ of T_{ij} satisfies

$$\mathcal{L}_\xi T_{ij} = 0. \tag{3.6}$$

On the other side, it is said that T_{ij} has the Lie inheritance property along the flow lines ξ if $\mathcal{L}_\xi T_{ij}$ obeys the following condition [1, 11, 16]

$$\mathcal{L}_\xi T_{ij} = 2\varphi T_{ij}. \tag{3.7}$$

Now using (2.33) in (3.1), one gets

$$\left(\frac{R}{n} - \frac{f}{2f'}\right) g_{ij} = \frac{\kappa}{f'} T_{ij}. \tag{3.8}$$

In a pseudo $\mathcal{W}_2$ −symmetric flat spacetime R is constant. Consequently, f and f' are constants. Here, we assume a non-vacuum pseudo $\mathcal{W}_2$ −symmetric flat spacetime M. Thus, $\mathcal{L}_\xi$ of equation (3.8) gives

$$\left(\frac{R}{n} - \frac{f}{2f'}\right) \mathcal{L}_\xi g_{ij} = \frac{\kappa}{f'} \mathcal{L}_\xi T_{ij}. \tag{3.9}$$

Now, consider ξ is Killing on the spacetime M. That is, equation (3.4) satisfied, hence one gets

$$\mathcal{L}_\xi T_{ij} = 0.$$

Conversely, if equation (3.6) holds, then form (3.9) it follows that

$$\mathcal{L}_\xi g_{ij} = 0.$$

We thus have the following theorem:

Theorem 4: Let M be a $(PWS)_n$ spacetime obeying f(R) theories of gravity. Then ξ is Killing iff M admits a matter collineation w.r.t ξ.

Now, assume that the vector field ξ is a conformal Killing. Tthat is, equation (3.5) holds. Equation (3.9) implies

$$\mathcal{L}_\xi T_{ij} = 2\varphi T_{ij}.$$

Conversely, assume that (3.7) satisfied. Thus, (3.9) implies that

$$\mathcal{L}_\xi g = 2\varphi g_{ij}.$$

We hence can state the following theorem:

Theorem 5: Let M be a $(PWS)_n$ spacetime obeying f(R) modified theories of gravity. Then M has a conformal Killing ξ iff T_{ij} has the Lie inheritance property along ξ.

IV –$(PWS)_4$ perfect fluid spacetimes in f(R) gravity

Here, we will consider that the four velocity vector field $u^i = \lambda^i$. For $n = 4$, equation (2.4) becomes

$$\lambda^i R_{ij} = \frac{R}{4}\lambda_j. \tag{4.1}$$

As known, T_{ij} in perfect fluid spacetimes has the following form

$$T_{ij} = (p + \sigma)\lambda_i\lambda_j + p g_{ij}, \tag{4.2}$$

where p is the isotropic pressure. With σ being the energy density [12].

Combining (3.1) and (4.2), one gets

$$R_{ij} = \frac{\kappa}{f'}\left[(p + \sigma)\lambda_i\lambda_j + p g_{ij}\right] + \frac{f}{2f'} g_{ij}. \tag{4.3}$$

Contracting with λ^i and using (4.1), we obtain

$$\sigma = \frac{2f - Rf'}{4k}. \tag{4.4}$$

Again transvecting equation (4.3) with g^{ij} and using (4.4), one infers

$$p = -\frac{2f - Rf'}{4\kappa}. \tag{4.5}$$

Thus, we get the following theorem:

Theorem 6: The isotropic pressure p and the energy-density σ are constants in a 4-dimensional pseudo $\mathcal{W}_2$ $-$symmetric perfect fluid spacetime. Moreover, they have the following forms $\sigma = \frac{2f-Rf'}{4k}$ and $p = -\frac{2f-Rf'}{4\kappa}$.

Adding (4.4) to (4.5), we acquire that

$$p + \sigma = 0, \tag{4.6}$$

This equation says that the spacetime represents dark matter. Alternatively, era the fluid behaves as a cosmological constant[15].

Theorem 7: A 4-dimensional pseudo $\mathcal{W}_2$ $-$symmetric (PWS) $_4$ perfect fluid spacetime denotes dark matter era.

The use of (2.33) in (3.1) implies

$$\left(\frac{R}{4} - \frac{f}{2f'}\right) g_{hk} = \frac{\kappa}{f'} T_{hk}.$$

Applying ∇_l, one infers

$$\nabla_l T_{hk} = 0.$$

Corollary 1: T_{hk} of a 4-dimensional pseudo $\mathcal{W}_2$ $-$symmetric (PWS) $_4$ perfect fluid spacetime obeying f(R) gravity is constant.

In general relativity, that is, $f = R$, equations (4.4) and (4.5) take the following forms

$$\sigma = \frac{R}{4k},$$

$$p = -\frac{R}{4\kappa}.$$

The last two equations are combined to give

$$p + \sigma = 0,$$

Thus, we have:

Theorem 8: In general relativity, p and the σ of a (PWS) $_4$ perfect fluid spacetime have the following forms $p = -\frac{R}{4\kappa}$ and $\sigma = \frac{R}{4k}$. Furthermore, spacetime denotes dark matter era.

Equation (3.1) may be rewritten as

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{\kappa}{f'} T_{ij}^{eff}, \tag{4.7}$$

where

$$T_{ij}^{eff} = T_{ij} + \frac{f-Rf'}{2\kappa} g_{ij}.$$

Hence, equation (4.2) is in the following form

$$T_{ij}^{eff} = \left(p^{eff} + \sigma^{eff}\right)\lambda_i\lambda_j + p^{eff} g_{ij},$$

where

$$p^{eff} = p + \frac{f-Rf'}{2\kappa} \quad \text{and} \quad \sigma^{eff} = \sigma - \frac{f-Rf'}{2\kappa}.$$

The use of values of p and σ entails that

$$p^{eff} = -\frac{Rf'}{4\kappa},$$

$$\sigma^{eff} = \frac{Rf'}{4\kappa}.$$

Theorem 9: In a 4-dimensional pseudo $\mathcal{W}_2$ $-$symmetric (PWS) $_4$ perfect fluid spacetime p^{eff} and σ^{eff} have the following forms $p^{eff} = -\frac{Rf'}{4\kappa}$ and $\sigma^{eff} = \frac{Rf'}{4\kappa}$.

V – Energy conditions in 4-dimensional pseudo $\mathcal{W}_2$ $-$symmetric spacetimes

In this part of our study,energy conditions EC in 4-dimensional pseudo $\mathcal{W}_2$ $-$symmetric spacetimes will be studied. Actually, these EC conditions behaves as a filtration system of Tij in standard and modified modified theories of gravity[4, 2, 3]. Let us begin to measure p^{eff} and σ^{eff} to discuss some of these EC.

Equation (3.1) can be easily given as

$$R_{ij} - \frac{1}{2} R g_{ij} = \frac{\kappa}{f'} T_{ij}^{eff}, \tag{5.1}$$

where

$$T_{ij}^{eff} = T_{ij} + \frac{f-Rf'}{2\kappa} g_{ij}.$$

Thus (4.2) may be rewritten in as

$$T_{ij}^{eff} = \left(p^{eff} + \sigma^{eff}\right) u_i u_j + p^{eff} g_{ij},$$

where

$$p^{eff} = p + \frac{f-Rf'}{2\kappa} \quad \text{and} \quad \sigma^{eff} = \sigma - \frac{f-Rf'}{2\kappa}.$$

We can easily get

$$p^{eff} = -\frac{Rf'}{4\kappa},$$

$$\sigma^{eff} = \frac{Rf'}{4\kappa}.$$

Now, we will state certain EC in f(R) modified theory of gravity [4]. NEC denotes null energy condition, WEC refers to weak energy condition, DEC denotes dominant energy condition, and SEC refers to Strong energy condition.

1. **NEC**: it says that $p^{eff} + \sigma^{eff} \geq 0$.
2. **WEC**: it states that $\sigma^{eff} \geq 0$ and $p^{eff} + \sigma^{eff} \geq 0$.
3. **DEC**: it states that $\sigma^{eff} \geq 0$ and $p^{eff} \pm \sigma^{eff} \geq 0$.
4. **SEC**: it states that $\sigma^{eff} + 3p^{eff} \geq 0$ and $p^{eff} + \sigma^{eff} \geq 0$.

In this setting, all previous EC are always fulfilled if $Rf' \geq 0$. As known previously, f' is can not be negative (must be positive) to ensure attractive gravity. Thus, the discussed EC are consistently fulifilled if $R \geq 0$.

REFERENCES

[1] H. M Abu-Donia, S. Shenawy, and A. A. Syied, The W*-curvature Tensor on Relativistic Space-times, Kyungpook Mathematical Journal, 60 (2020): 185-195.

[2] A. De, T. Loo, S. Arora, and P. K. Sahoo, Energy conditions for a (WRS) $_4$ spacetime in F (R)-gravity, The European Physical Journal Plus, 136 (2021): 1-10.

[3] A. De, T. Loo, R. Solanki, and P. K. Sahoo, A conformally flat generalized Ricci recurrent spacetime in F (R)-gravity, Physica Scripta, 96 (2021): 085001.

[4] A. De, Tee-How Loo, Raja Solanki, and P. K. Sahoo, How a Conformally Flat Impacts Gauss-Bonnet Gravity?, Fortschritte der Physik, 69 (2021): 2100088.

[5] M. C. Chaki, On pseudo symmetric manifolds, An. Stiint. Univ. AL.I. Cuza din Iasi Sect. I-a Math. N.S. 33(1) (1987), 53–58.

[6] M. C. Chaki, and S. K. Saha, On pseudo-projective Ricci symmetric manifolds, Bulg. J. Phys, 21(1-2) (1994): 1-7.

[7] S. Capozziello, C. A. Mantica, and L. G. Molinari, General properties of f (R) gravity vacuum solutions, International Journal of Modern Physics D, 29(13) (2020): 2050089.

[8] Luther Pfahler Eisenhart Riemannian geometry. Princeton university press, 1997.

[9] A. Gray, Einstein-like manifolds which are not Einstein, Geom. Dedicata, 7(3) (1978): 259-280.

[10] H. A. Buchdahl, Non-linear Lagrangians and cosmological theory, Monthly Notices of the Royal Astronomical Society, 150 (1970): 1-8.

[11] S. Mallickand Uday Chand De, Spacetimes admitting W2-curvature tensor, International Journal of Geometric Methods in Modern Physics, 11 (2014): 1450030.

[12] B. O'Neill, Semi-semi-Riemannian Geometry with Applications to Relativity, Academic Press, New York-London, 1983.

[13] G.P. Pokhariyal and R.S. Mishra, The curvature tensor and their relativistic significance, Yokohoma Math. J. 18 (1970), 105–108.

[14] M. M. Postnikov, Geometry VI: Riemannian Geometry, Springer Science & Business Media, 91(2013).

[15] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, Exact solutions of Einstein's field equations, Cambridge university press, 2009.

[16] F. O. Zengin, M-projectively flat Spacetimes, Math. Rep, 14 (2012): 363-